Math 654 - Algebraic Topology
Homework 9
Fall 2015

1. (Adjoint functors) A pair of functors $L : \mathscr{C} \rightleftarrows \mathscr{D} : R$ is said to be an **adjoint pair** if there is a bijection

$$\mathrm{Hom}_{\mathscr{D}}(L(X), Y) \cong \mathrm{Hom}_{\mathscr{C}}(X, R(Y))$$

that is natural in both $X \in \mathscr{C}$ and $Y \in \mathscr{D}$. The functor L is called the **left adjoint**, while R is the **right adjoint**.

   (a) Show that for any abelian group A, the functors

$$(-) \otimes A : \mathbf{Ab} \rightleftarrows \mathbf{Ab} : \mathrm{Hom}(A, -)$$

   are an adjoint pair.

   (b) Show that any left adjoint automatically preserves coproducts. That is, show that there is an isomorphism $L\big(\coprod_i X_i\big) \cong \coprod_i L(X_i)$.

   (c) Show that any left adjoint preserves pushouts. That is, the diagram

$$\begin{tikzcd} L(A) \arrow[r] \arrow[d] & L(Y) \arrow[d] \\ L(X) \arrow[r] & L(X \amalg_A Y) \end{tikzcd}$$

   is automatically a pushout diagram.

2. (a) Show that abelianization $(-)_{ab} : \mathbf{Gp} \longrightarrow \mathbf{Ab}$ is left adjoint to the inclusion $\mathbf{Ab} \hookrightarrow \mathbf{Gp}$.

   (b) Use this to show that if $N \trianglelefteq G$, then $(G/N)_{ab} \cong G_{ab}/N_{ab}$.

3. (a) Show that the free abelian group construction $\mathbb{Z}\{-\} : \mathrm{Set} \longrightarrow \mathbf{Ab}$ is left adjoint to the "forgetful functor" $U : \mathbf{Ab} \longrightarrow \mathrm{Set}$ that forgets the group structure and remembers only the underlying set.

   (b) Conclude that $\mathbb{Z}\{A \coprod B\} \cong \mathbb{Z}\{A\} \oplus \mathbb{Z}\{B\}$.

   (c) Show that $\mathbb{Z}\{A \times B\} \cong \mathbb{Z}\{A\} \otimes \mathbb{Z}\{B\}$. (Hint: $U\big(\mathrm{Hom}(\mathbb{Z}(Y), A)\big) \cong A^Y$.)