CLASS NOTES
MATH 654 (FALL 2015)

BERTRAND GUILLOU

1. Wed, Aug. 26

The first algebraic tool that you learned about for distinguishing spaces is the fundamental group $\pi_1(X)$. As you saw, this is already sufficient for distinguishing surfaces. But this tool has several drawbacks:

(1) It fails to distinguish many spaces. For example, S^2 and S^3 are both simply-connected but are not homotopy equivalent.
(2) It is in practice very difficult to calculate! You may be able to compute the group in terms of giving a presentation (listing generators and relations), but this does not mean you understand the group. Recall that in general given a group G with a given presentation, there is no algorithm to determine whether a given word represents the trivial element. Of course, for many *particular* group presentations there are perfectly good algorithms.

One remedy for (2) is to consider instead the *abelianized fundamental group*. As you saw before, this also suffices for the classification of surfaces. This is great since abelian groups are much easier to work with. For instance, we know that every finitely generated abelian group is a direct sum of cyclic groups. On the other hand, this is a coarser invariant and therefore fails even harder to distinguish spaces. With this tool, the torus $S^1 \times S^1$ and the figure eight $S^1 \vee S^1$ look the same.

One approach is to consider higher analogues of the fundamental group. Recall that the fundamental group is defined as

$$\pi_1(X, x) \cong [S^1, (X, x)]_*,$$

where the brackets denote based homotopy classes of based maps. From this definition, it seems reasonable to define

$$\pi_n(X, x) \cong [S^n, (X, x)]_*.$$

Note that in the case $n = 0$, based homotopy classes of maps from $S^0 = \{-1, 1\}$ correspond precisely to unbased homotopy classes of maps from $\{-1\}$ to X, so that $\pi_0(X, x)$ corresponds precisely to the path-components of X.

When $n = 1$, we know we get a group, and we can ask what we get for $n \geq 2$. Recall that the group structure on $\pi_1(X, x)$ can be defined using the pinch map $S^1 \longrightarrow S^1 \vee S^1$ via

$$\begin{aligned} [S^1, (X,x)]_* \times [S^1, (X,x)]_* &\longrightarrow [S^1, (X,x)]_* \\ (S^1 \xrightarrow{\alpha} X, S^1 \xrightarrow{\beta} X) &\longmapsto \left(S^1 \xrightarrow{p} S^1 \vee S^1 \xrightarrow{(\alpha,\beta)} X\right) \end{aligned}$$

We can try to do the same for the $\pi_n(X)$, starting from a pinch map for S^n. If we recall that $S^n \cong (S^1)^{\wedge n}$, then we see that pinching in each of the n coordinates leads to n different choices of pinch maps. In fact, these all provide the same multiplication by the following result

Proposition 1.1 (Eckmann-Hilton Argument). *Let X be a set with two binary operations, denoted $*_1$ and $*_2$, and a distinguished element $e \in X$, such that*

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(1) *e is a unit element for both $*_1$ and $*_2$*
(2) *$*_1$ and $*_2$ satisfy the "interchange law": for all w, x, y, z in X,*

$$(w *_1 x) *_2 (y *_1 z) = (w *_2 y) *_1 (x *_2 z).$$

Then in fact $*_1 = *_2$ and this operation is both associative and commutative.

Proof. We show that the operations agree and are commutative.

$$x *_2 y = (x *_1 e) *_2 (e *_1 y) = (x *_2 e) *_1 (e *_2 y) = x *_1 y$$

and

$$y *_2 x = (e *_1 y) *_2 (x *_1 e) = (e *_2 x) *_1 (y *_2 e) = x *_1 y.$$

These arguments are best visualized by thinking of $*_1$ as a "horizontal" multiplication and $*_2$ as a "vertical" multiplication. Then the interchange law says that you can either first multiply horizontally and then vertically or in the other order, and you get the same answer. ■

Applying the Eckmann-Hilton argument to the n-choices of pinch maps on $\pi_n(X)$ show that this is an abelian group of $n \geq 2$. The unit element is the constant map at the basepoint. To verify the interchange law holds, for example when $n = 2$, it suffices to see that the diagram

$$\begin{tikzcd}
& (S^1 \vee S^1) \wedge S^1 \arrow[dr, "\mathrm{id}\wedge p"] & & \\
S^2 \cong S^1 \wedge S^1 \arrow[ur, "p\wedge\mathrm{id}"] \arrow[dr, "\mathrm{id}\wedge p"'] & & (S^1 \vee S^1) \wedge (S^1 \vee S^1) \arrow[r, "\cong"] & \bigvee_4 S^2 \\
& S^1 \wedge (S^1 \vee S^1) \arrow[ur, "p\wedge\mathrm{id}"'] & &
\end{tikzcd}$$

commutes. But both composites along the sides of the diamond give $p \wedge p$, so we are done.

2. Fri, Aug. 28

Ok, great! We have a bunch of nice abelian groups $\pi_n(X)$. Can we compute these?

Back in Math 651, the first interesting example of a fundamental group was $\pi_1(S^1) \cong \mathbb{Z}$. In fact, this generalizes to the statement that $\pi_n(S^n) \cong \mathbb{Z}$ (we may prove this later). You also saw that $\pi_1(S^n) = 0$ if $n > 1$, and this also generalizes to the statement $\pi_k(S^n) = 0$ if $n > k$. So the "interesting" cases are $\pi_{n+k}(S^n)$.

When $n = 1$, there turns out to be nothing here. In fact, covering space theory can be used to show

Proposition 2.1. *Let $p : E \longrightarrow B$ be a covering map. Then p induces an isomorphism*

$$p_* : \pi_n(E) \longrightarrow \pi_n(B)$$

for all $n \geq 2$.

We conclude that $\pi_n(S^1) \cong \pi_n(\mathbb{R}) = 0$, since $\mathbb{R}$ is contractible.

The next example to try is $\pi_{2+k}(S^2)$.

Example 2.2. For $X = S^2$, we know

$$\pi_1(S^2) = 0, \quad \pi_2(S^2) \cong \mathbb{Z}, \quad \pi_3(S^2) \cong \mathbb{Z} \quad \pi_4(S^2) \cong \pi_5(S^2) \cong \mathbb{Z}/2\mathbb{Z}, \quad \pi_6(S^2) \cong \mathbb{Z}/12\mathbb{Z}.$$

But these homotopy groups $\pi_n(S^2)$ are only known up to $n = 64$, although it is known that (1) they are all finite, except for $\pi_2(S^2)$ and $\pi_3(S^2)$, and (2) infinitely many are nonzero. This was proved by J. P. Serre.

The situation is similar for the homotopy groups $\pi_n(S^k)$ in general. The homotopy groups of spheres are in some sense the "holy grail" of algebraic topology. They are a major driving force behind a great amount of research, though we know that we will never know all of the homotopy groups.

What this suggests is that if we try to use the homotopy groups $\pi_n(X)$ to distinguish spaces, we are not likely to get very far. Calculating homotopy groups is hard!!

Instead, we want a simpler invariant, from the point of view of computation. This is where homology enters the story.

Categories and Functors

Before we delve into homology, we pause to introduce some convenient language that will appear many times throughout this course (and throughout your mathematical careers!). This is the language of categories, functors, and natural transformations.

Definition 2.3. A **category** $\mathscr{C}$ is a collection of "objects", denoted $Ob(\mathscr{C})$, together with, for each pair of objects $X, Y \in Ob(\mathscr{C})$, a set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ of "morphisms" which satisfies the following:

- For each $X, Y, Z \in Ob(\mathscr{C})$, there is a "composition" function
$$\circ : \mathrm{Hom}_{\mathscr{C}}(Y, Z) \times \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}}(X, Z).$$
We write $g \circ f$ or gf for $\circ(g, f)$.
- For each $X \in Ob(\mathscr{C})$ there exists an "identity morphism" $\mathrm{id}_X \in \mathrm{Hom}_{\mathscr{C}}(X, X)$ such that for any $Y, Z \in Ob(\mathscr{C})$ and $f \in \mathrm{Hom}_{\mathscr{C}}(Y, X)$, $g \in \mathrm{Hom}_{\mathscr{C}}(X, Z)$ we have
$$\mathrm{id}_X \circ f = f \qquad \text{and} \qquad g \circ \mathrm{id}_X = g.$$
- Composition is associative, i.e., $h(gf) = (hg)f$.

Remark 2.4. We often write $\mathscr{C}(X, Y)$ for $\mathrm{Hom}_{\mathscr{C}}(X, Y)$, and we often write $X \in \mathscr{C}$ for $X \in Ob(\mathscr{C})$.

Remark 2.5. A category $\mathscr{C}$ is called small if the collection $Ob(\mathscr{C})$ of objects forms a set.

Categories abound in mathematics. Here are just a few of the more common examples.

Example 2.6.

(1) **Set**: the objects are sets and the morphisms are functions.
(2) **FinSet**: the objects are finite sets and morphisms are functions.
(3) $\mathbf{Vect_k}$, where k is a field: the objects are vector spaces over k and morphisms are k-linear homomorphisms.
(4) **Gp**: the objects are groups and the morphisms are homomorphisms.
(5) **AbGp**: the objects are abelian groups and the morphisms are homomorphisms.
(6) **Top**: the objects are topological spaces and the morphisms are continuous maps.
(7) $\mathbf{Top}_*$: the objects are based topological spaces (spaces with a distinguished base point) and the morphisms are basepoint-preserving continuous maps.
(8) $\mathbf{Ho}(\mathbf{Top})$: the objects are topological spaces and the morphisms are homotopy classes of maps.
(9) $\mathbf{Ho}(\mathbf{Top}_*)$: the objects are based topological spaces and the morphisms are based homotopy classes of maps.

These are all "large" categories (many objects). Small categories also arise often, though in a different way.

Example 2.7.

(10) $\bullet$ denotes a category with a single object and only an identity morphism.

(11) $\bullet \longrightarrow \bullet$ denotes a category with two objects and one morphism connecting the two objects.
(12) $\bullet \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \bullet$ denotes a category with two objects and three parallel morphisms.

We defined categories so that we could talk about functors.

Definition 2.8. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. A **(covariant) functor** $F : \mathscr{C} \to \mathscr{D}$ is the following data: for each $C \in \mathscr{C}$ we have an object $F(C) \in \mathscr{D}$, and for each arrow $f \in \mathrm{Hom}_{\mathscr{C}}(C, C')$ we have an arrow $F(f) \in \mathrm{Hom}_{\mathscr{D}}(F(C), F(C'))$ such that

$$F(\mathrm{id}_C) = \mathrm{id}_{F(C)} \qquad \text{and} \qquad F(g \circ f) = F(g) \circ F(f).$$

A **contravariant functor** $F : \mathscr{C} \to \mathscr{D}$ is a functor that reverses the directions of the morphisms. If $f : C \longrightarrow C'$ is a morphism, then the contravariant functor F produces a morphism $F(f) : F(C') \longrightarrow F(C)$. We still require compatibility with composition, which now looks like $F(g \circ f) = F(f) \circ F(g)$.

Remark 2.9. If $F : \mathscr{C} \to \mathscr{D}$ is a covariant functor and f is an arrow in $\mathscr{C}$, we often write f_* for $F(f)$. If F is contravariant, we write f^* for $F(f)$.

Definition 2.10. Let $\mathscr{C}$ be any category. We define the **opposite category** $\mathscr{C}^{op}$ to be the category with the same objects as $\mathscr{C}$ and with

$$\mathrm{Hom}_{\mathscr{C}^{op}}(X, Y) := \mathrm{Hom}_{\mathscr{C}}(Y, X).$$

That is, we merely switch all of the directions of the arrows in $\mathscr{C}$. Composition in $\mathscr{C}^{op}$ is induced from composition in $\mathscr{C}$. Note that if morphisms in $\mathscr{C}$ correspond to functions (with possibly extra structure) then morphisms in $\mathscr{C}^{op}$ will not correspond to functions.

The main reason to define opposite categories is that a contravariant functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is precisely the same thing as a covariant functor $F : \mathscr{C}^{op} \longrightarrow \mathscr{D}$.

Example 2.11.

(1) There is a functor $\mathbf{Top} \longrightarrow \mathbf{Ho}(\mathbf{Top})$ (and similarly in the based context) which does nothing on objects and which takes a map to its homotopy class.

(2) Functors $\{\bullet \longrightarrow \bullet\} \longrightarrow \mathbf{Top}$ are given exactly by diagrams of shape $X \xrightarrow{f} Y$ in $\mathbf{Top}$.

3. Mon, Aug. 31

(3) Abelianization defines a functor $(-)_{ab} : \mathbf{Gp} \longrightarrow \mathbf{AbGp}$. On objects, this is $G \mapsto G_{ab}$. On morphisms, suppose that $\varphi : H \longrightarrow G$ is a homomorphism. Then $\varphi_{ab} : H_{ab} \longrightarrow G_{ab}$ is the induced morphism, defined using the universal property of quotients as in the diagram

$$\begin{tikzcd} H \arrow[r, "\varphi"] \arrow[dr] & G \arrow[r, two heads] & G_{ab}. \\ & H_{ab} \arrow[ur, dashed, "\varphi_{ab}"'] & \end{tikzcd}$$

Here the functor axioms are that

$$(\varphi \circ \lambda)_{ab} = \varphi_{ab} \circ \lambda_{ab} \qquad \text{and} \qquad (\mathrm{id}_G)_{ab} = \mathrm{id}_{G_{ab}}.$$

(4) The free abelian group functor $F : \mathbf{Set} \to \mathbf{AbGp}$ is defined on objects by

$$F(X) = \bigoplus_{x \in X} \mathbb{Z}.$$

An element of $F(X)$ is a finite formal $\mathbb{Z}$-linear combination of elements of X, and the group operation is defined by

$$\left(\sum_{x\in X} n_x x\right) + \left(\sum_{x\in X} m_x x\right) := \sum_{x\in X}(n_x + m_x)x.$$

Given a function $f : X \to Y$, $F(f)$ is defined by

$$F(f)\left(\sum_{x\in X} n_x x\right) := \sum_{x\in X} n_x f(x).$$

(5) The fundamental group defines a functor $\pi_1 : \mathbf{Top}_* \to \mathbf{Gp}$ which assigns to a space X with basepoint x the fundamental group $\pi_1(X, x)$. Given a basepoint-preserving map of based spaces $f : X \to Y$, the homomorphism $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is defined by sending the class of a loop α to the class of the loop $f \circ \alpha$. The formulas

$$(g \circ f)_* = g_* \circ f_* \qquad \text{and} \qquad (\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X)}$$

say that $\pi_1(-)$ is a functor. In fact, since the homomorphism f_* only depends on the homotopy class of f, this functor factors as

$$\begin{tikzcd} \mathbf{Top} \arrow[rr, "\pi_1"] \arrow[dr] & & \mathbf{Gp}. \\ & \mathbf{Ho}(\mathbf{Top}) \arrow[ur, "\pi_1"'] & \end{tikzcd}$$

(6) **Represented functors**: Given a category $\mathscr{C}$ and an object $X \in \mathscr{C}$, define a functor

$$F_X = \mathrm{Hom}(X, -) : \mathscr{C} \longrightarrow \mathbf{Set}$$

by

$$Y \mapsto \mathrm{Hom}_{\mathscr{C}}(X, Y).$$

To see what this does on morphisms, given a morphism $f : Y \longrightarrow Z$ in $\mathscr{C}$, we are required to have a function

$$F_X(Y) = \mathrm{Hom}_{\mathscr{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(X, Z) = F_X(Z).$$

We define this to simply be composition with f. You should check for yourself that this really defines a functor.

If we instead put X in the other slot, we get a functor

$$H_X = \mathrm{Hom}(-, X) : \mathscr{C}^{op} \longrightarrow \mathbf{Set}.$$

This functor is contravariant, since if $f : Y \longrightarrow Z$ is morphism in $\mathscr{C}$, then composition with f gives a function

$$H_X(Z) = \mathrm{Hom}_{\mathscr{C}}(Z, X) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(Y, X) = H_X(Y).$$

(7) If we consider the previous construction, taking $\mathscr{C} = \mathbf{Top}$ and $X = *$, then the functor $F_* : \mathbf{Top} \longrightarrow \mathbf{Set}$ is the "underlying set" functor.

(8) If we consider the previous construction, taking $\mathscr{C} = \mathbf{Ho}(\mathbf{Top}_*)$, then F_{S^1} is precisely the fundamental group functor! That this functor takes values in groups rather than just sets stems from the fact that S^1 has extra structure: it is a "cogroup object" in $\mathbf{Ho}(\mathbf{Top}_*)$. Similarly, the represented functor F_{S^n} is the functor $\pi_n(-)$.

Homology

There are several variants of homology, as we will see. Following Hatcher, we will start with "simplicial" homology. The input for this flavor of homology is what Hatcher calls a **Δ-complex**. Δ^n is the usual notation for the standard n-simplex, which can be defined as

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1, \quad t_i \geq 0\}.$$

We will denote by $v_i \in \Delta^n$ the vertex defined by $t_i = 1$ and $t_j = 0$ if $j \neq i$. Note that each "facet" of the simplex, in which we have restricted one of the coordinates to zero, is an $(n-1)$-dimensional simplex. More generally, if we set k of the coordinates equal to zero, we get a face which is an $(n-k)$-dimensional simplex.

Δ-complexes are obtained by gluing together simplices along faces. We will need to keep track of orientations of simplices. In the standard n-simplex, we declare the ordering of vertices $v_0 \leq v_1 \leq \dots \leq v_n$. All gluings performed in constructing a Δ-complex are required to be orientation-preserving identifications. Thus if we want to glue an edge of Δ^2 to an edge of Δ^4, we first note the ordering of the vertices on each of the two edges, and we then glue together along the unique order-preserving linear isomorphism between the two edges.

To match up with the notion of CW-complex that you saw in MA551/651, another way to view Δ-complexes is as a pushout (gluing)

$$\begin{tikzcd} \coprod_i \coprod_{\mathcal{F}_i} \Delta^{n_i} \arrow[r] \arrow[d] & \coprod_\alpha \Delta^{n_\alpha} \arrow[d] \\ \coprod_i \Delta^{n_i} \arrow[r] & X \end{tikzcd}$$

Here, each $\mathcal{F}_i$ is a collection of n_i-dimensional faces (of various simplices) to be glued together.

Remark 4.1. This is a more convenient generalization of simplicial complex. A **simplicial complex** is also obtained by gluing together simplices, but there we require that each n-simplex has $n+1$ distinct vertices and also that an n-simplex is uniquely specified by its vertices.

Example 4.2.

(1) $X = S^1$. This can be built as a Δ-complex by starting with a 1-simplex Δ^1 and then identifying the two faces together. Note that this Δ-complex is not a simplicial complex.
(2) $X = S^1$. Another choice is to start with two simplices Δ^1 and glue them together end-to-end. This is *still* not a simplicial complex, since the two 1-simplices have the same vertex set.
(3) $X = S^1$. To get a simplicial complex, we can start with three 1-simplices and glue together end-to-end.

Let's look at some surfaces.

Example 4.3.

(1) $X = S^2$, the sphere. We can obtain S^2 by glueing together two 2-simplices Δ^2 $\{a, b, c\}$ and $\{x, y, z\}$. We first glue $\{a, c\}$ to $\{x, z\}$ to get a square. We then glue $\{a, b\}$ to $\{x, y\}$ and $\{b, c\}$ to $\{y, z\}$.
(2) $X = S^1 \times S^1$, the torus. We can obtain T^2 by glueing together two 2-simplices Δ^2 $\{a, b, c\}$ and $\{x, y, z\}$. We first glue the edge $\{a, c\}$ to the edge $\{x, z\}$ to get a square. We then glue $\{a, b\}$ to $\{y, z\}$, and finally we glue $\{b, c\}$ to $\{x, z\}$. This is not a simplicial complex, since in the end we are left with a single vertex.

(3) $X = \mathbb{RP}^2$, the projective plane. We can also obtain this by glueing together 2-simplices $\{a, b, c\}$ and $\{x, y, z\}$. We first glue $\{a, b\}$ to $\{x, y\}$. We then glue $\{b, c\}$ to $\{x, z\}$ and $\{a, c\}$ to $\{y, z\}$.
(4) $X = K$, the Klein bottle. We can also obtain this by glueing together 2-simplices $\{a, b, c\}$ and $\{x, y, z\}$. We first glue $\{a, b\}$ to $\{x, z\}$. We then glue $\{a, c\}$ to $\{y, z\}$ and $\{b, c\}$ to $\{x, y\}$.

The Simplicial Chain Complex:

Given a Δ-complex X, let $C_n^\Delta(X)$ be the free abelian group on the set of n-simplices of X. An element of $C_n^\Delta(X)$ is referred to as an (simplicial) n-chain on X. Our goal is to assemble the $C_n^\Delta(X)$, as n varies, into a "chain complex"

$$\ldots \longrightarrow C_3^\Delta(X) \longrightarrow C_2^\Delta(X) \longrightarrow C_1^\Delta(X) \longrightarrow C_0^\Delta(X).$$

To say that this is a chain complex just means that composing two successive maps in the sequence gives 0. We wish to specify a homomorphism

$$\partial_n : C_n^\Delta(X) \longrightarrow C_{n-1}^\Delta(X).$$

Since $C_n^\Delta(X)$ is a free abelian group, the homomorphism ∂_n is completely specified by its value on each generator, namely each n-simplex. Let σ be an n-simplex of X. Note that, since we have a chosen ordering of the vertices of σ, the n-simplex σ determines a unique order-preserving map $\sigma : \Delta^n \longrightarrow X$, which restricts to an embedding of the open simplex.

There are $n+1$ standard inclusions $d^i : \Delta^{n-1} \hookrightarrow \Delta^n$, given by inserting 0 in position i in Δ^n. Since no faces get collapsed down in the glueing performed to assemble X, composing σ with an inclusion d^i gives an $(n-1)$-simplex of X (where the ordering is inherited from that of σ).

Definition 4.4. The **simplicial boundary homomorphism**

$$\partial_n : C_n^\Delta(X) \longrightarrow C_{n-1}^\Delta(X)$$

is defined by

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i [\sigma \circ d^i].$$

Example 4.5.

(1) If σ is a 1-simplex (from v_0 to v_1), then

$$\partial_1(\sigma) = [\sigma \circ d^0] - [\sigma \circ d^1] = [v_1] - [v_0].$$

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(2) If σ is a 2-simplex with vertices v_0, v_1, and v_2, and edges e_{01}, e_{02}, and e_{12}, then

$$\partial_2(\sigma) = [\sigma \circ d^0] - [\sigma \circ d^1] + [\sigma \circ d^2] = [e_{12}] - [e_{02}] + [e_{01}]$$

The claim is that this defines a chain complex. The signs have been inserted into the definition to make this work out.

Proposition 5.1. *The boundary squares to zero, in the sense that $\partial_{n-1} \circ \partial_n = 0$.*

Proof. We will use

Lemma 5.2. *For $i > j$, the composite*

$$\Delta^{n-2} \xrightarrow{d^j} \Delta^{n-1} \xrightarrow{d^i} \Delta^n \quad \textit{is equal to the composite} \quad \Delta^{n-2} \xrightarrow{d^{i-1}} \Delta^{n-1} \xrightarrow{d^j} \Delta^n.$$

Consider the case $i = 3$, $j = 1$, $n = 4$. We have

$$d^3(d^1(t_1, t_2, t_3)) = d^3(t_1, 0, t_2, t_3) = (t_1, 0, t_2, 0, t_3) = d^1(t_1, t_2, 0, t_3) = d^1(d^2(t_1, t_2, t_3)).$$

This argument generalizes.

For the proposition,

$$\begin{aligned}
\partial_{n-1}\Big(\partial_n(\sigma)\Big) &= \partial_{n-1}\left(\sum_{i=0}^{n}(-1)^i[\sigma \circ d^i]\right) \\
&= \sum_{i=0}^{n}(-1)^i\,\partial_{n-1}([\sigma \circ d^i]) \\
&= \sum_{i=0}^{n}(-1)^i\sum_{j=0}^{n-1}(-1)^j[\sigma \circ d^i \circ d^j] \\
&= \sum_{i=0}^{n}\sum_{j<i}(-1)^i(-1)^j[\sigma \circ d^i \circ d^j] + \sum_{i=0}^{n}\sum_{j\geq i}(-1)^i(-1)^j[\sigma \circ d^i \circ d^j] \\
\text{(changing bounds)}\quad &= \sum_{i=1}^{n}\sum_{j<i}(-1)^i(-1)^j[\sigma \circ d^i \circ d^j] + \sum_{i=0}^{n-1}\sum_{j\geq i}(-1)^i(-1)^j[\sigma \circ d^i \circ d^j] \\
\text{(Lemma)}\quad &= \sum_{i=1}^{n}\sum_{j<i}(-1)^i(-1)^j[\sigma \circ d^j \circ d^{i-1}] + \sum_{i=0}^{n-1}\sum_{j\geq i}(-1)^i(-1)^j[\sigma \circ d^i \circ d^j] \\
&= -\sum_{j=0}^{n-1}\sum_{i-1\geq j}(-1)^j(-1)^{i-1}[\sigma \circ d^i \circ d^j] + \sum_{i=0}^{n-1}\sum_{j\geq i}(-1)^i(-1)^j[\sigma \circ d^i \circ d^j] \\
&= 0.
\end{aligned}$$

■

We have shown that any two successive simplicial boundary homomorphisms compose to zero, so that we have a chain complex. What do we do with a chain complex? Take homology!

Definition 5.3. If

$$\ldots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \ldots$$

is a chain complex, then we define the ***n*th homology group** $\mathrm{H}_n(C_*, \partial_*)$ to be

$$\mathrm{H}_n(C_*, \partial_*) := \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

Note that the fact that $\partial_n \circ \partial_{n+1} = 0$ implies that $\operatorname{im} \partial_{n+1}$ is a subgroup of $\ker \partial_n$, so that the definition makes sense. Recall that a complex (C_*, ∂_*) is said to be **exact** at C_n if we have equality $\ker \partial_n = \operatorname{im} \partial_{n+1}$. Thus the homology group $\mathrm{H}_n(C_*, \partial_*)$ "measures the failure of C_* to be exact at C_n."

Definition 5.4. Given a Δ-complex X, we define the **simplicial homology groups** of X to be

$$\mathrm{H}_n^{\Delta}(X; \mathbb{Z}) := \mathrm{H}_n(C_*^{\Delta}(X), \partial_*).$$

Note that we only defined the groups $C_n^{\Delta}(X)$ for $n \geq 0$. For some purposes, it is convenient to allow chain groups C_n for negative values of n, so we declare that $C_n^{\Delta}(X) = 0$ for $n < 0$. This means that $\ker \partial_0 = C_0^{\Delta}(X)$, so that $\mathrm{H}_0^{\Delta} = C_0^{\Delta}(X)/\operatorname{im} \partial_1 = \operatorname{coker}(\partial_1)$. Similarly, if X has no simplices above dimension n, then we see $C_k^{\Delta}(X) = 0$ for $k > n$, which implies that $\mathrm{H}_k^{\Delta}(X) = 0$. Also, $\partial_{n+1} = 0$, so that $\mathrm{H}_n^{\Delta}(X) = \ker \partial_n$.

Remark 5.5. It is worth noting that since each $C_n^\Delta(X)$ is free abelian and $\ker \partial_n$ and $\operatorname{im} \partial_{n+1}$ are both subgroups, they are necessarily also free abelian.

Example 5.6.

(1) Consider $X = S^1$, built as a simplicial complex with a single 1-simplex e, whose two vertices have been glued together. Thus we have a single 0-simplex. Our chain complex looks like

$$\begin{tikzcd} C_1^\Delta(S^1) \arrow[r, "\partial_1"] \arrow[d, equal] & C_0^\Delta(S^1) \arrow[d, equal] \\ \mathbb{Z}\{e\} & \mathbb{Z}\{v\} \end{tikzcd}$$

The differential is given by $\partial_1(e) = [v] - [v] = 0$. It follows that $\mathrm{H}_1^\Delta(S^1) = \mathbb{Z}$ and $\mathrm{H}_0^\Delta(S^1) = \mathbb{Z}$. Since all of the higher chain groups are zero, the same holds for the higher homology groups $\mathrm{H}_n^\Delta(S^1)$.

(2) We had other constructions of S^1 as a simplicial complex. Our second construction had two 1-simplices e and f and two vertices x and y, with $\partial(e) = [y] - [x]$ and $\partial(f) = [x] - [y]$. Now our chain complex looks like

$$\begin{tikzcd} C_1^\Delta(S^1) \arrow[r, "\partial_1"] \arrow[d, equal] & C_0^\Delta(S^1) \arrow[d, equal] \\ \mathbb{Z}\{e,f\} \arrow[r, "\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}"'] & \mathbb{Z}\{x,y\} \end{tikzcd}$$

Thus $\ker \partial_1 = \mathbb{Z}\{e+f\}$ and $\operatorname{im} \partial_1 = \mathbb{Z}\{y-x\}$. It follows that $\mathrm{H}_1^\Delta(S^1) = \mathbb{Z}$ and $\mathrm{H}_0^\Delta(S^1) = \mathbb{Z}$.

(3) $X = S^2$. We built this as a Δ-complex by gluing together two 2-simplices z_1 and z_2 along their boundaries. Our chain complex is

$$\begin{tikzcd} C_2^\Delta(S^2) \arrow[r, "\partial_2"] \arrow[d, equal] & C_1^\Delta(S^2) \arrow[r, "\partial_1"] \arrow[d, equal] & C_0^\Delta(S^2) \arrow[d, equal] \\ \mathbb{Z}\{z_1,z_2\} \arrow[r, "\begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 1 \end{pmatrix}"'] & \mathbb{Z}\{y_1,y_2,y_3\} \arrow[r, "\begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}"'] & \mathbb{Z}\{x_1,x_2,x_3\} \end{tikzcd}$$

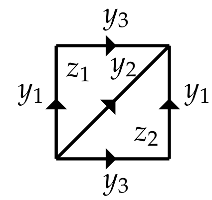
We see that the kernel of ∂_2 is $\mathbb{Z}\{z_1 - z_2\}$, so that $\mathrm{H}_2^\Delta(S^2) \cong \mathbb{Z}$.

The image of ∂_2 is $\mathbb{Z}\{y_1 - y_2 + y_3\}$, which is also seen to be the kernel of ∂_1. Thus $\mathrm{H}_1^\Delta(S^2) = 0$.

The third column of ∂_1 is the difference of the first two, so that the image of ∂_1 is $\mathbb{Z}\{x_2 - x_1, x_3 - x_1\}$. It follows that

$$\mathrm{H}_0^\Delta(S^2) = \mathbb{Z}\{x_1,x_2,x_3\}/\langle x_2 - x_1, x_3 - x_1\rangle \cong \mathbb{Z}\{x_1\}.$$

(4) $X = T^2$. The torus was similarly built by gluing two 2-simplices. The chain complex we obtain from our gluing data pictured to the right is



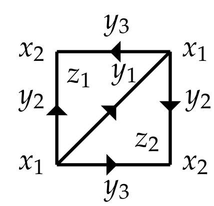
$$\begin{array}{ccccc} C_2^\Delta(T^2) & \xrightarrow{\partial_2} & C_1^\Delta(T^2) & \xrightarrow{\partial_1} & C_0^\Delta(T^2) \\ \| & & \| & & \| \\ \mathbb{Z}\{z_1, z_2\} & \xrightarrow[\begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 1 \end{pmatrix}]{} & \mathbb{Z}\{y_1, y_2, y_3\} & \xrightarrow[\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}]{} & \mathbb{Z}\{x\} \end{array}$$

The ∂_2 is the same as for S^2, so we again find $\mathrm{H}_2(T^2) \cong \mathbb{Z}$. But now $\ker \partial_1 = \mathbb{Z}\{y_1, y_2, y_3\}$, so that

$$\mathrm{H}_1^\Delta(T^2) = \mathbb{Z}\{y_1, y_2, y_3\} / \langle y_1 - y_2 + y_3 \rangle \cong Z\{y_1, y_3\}.$$

Since $\operatorname{im} \partial_1 = 0$, we see that $\mathrm{H}_0^\Delta(T^2) \cong \mathbb{Z}$.

(5) $X = \mathbb{RP}^2$. The projective plane was built from two simplices as in the picture to the right. This produces the chain complex



$$\begin{array}{ccccc} C_2^\Delta(\mathbb{RP}^2) & \xrightarrow{\partial_2} & C_1^\Delta(\mathbb{RP}^2) & \xrightarrow{\partial_1} & C_0^\Delta(\mathbb{RP}^2) \\ \| & & \| & & \| \\ \mathbb{Z}\{z_1, z_2\} & \xrightarrow[\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}]{} & \mathbb{Z}\{y_1, y_2, y_3\} & \xrightarrow[\begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix}]{} & \mathbb{Z}\{x_1, x_2\} \end{array}$$

In this case, $\ker \partial_2 = 0$, so that $\mathrm{H}_2^\Delta(\mathbb{RP}^2) = 0$.

For H_1^Δ, we see that $\ker \partial_1 = \mathbb{Z}\{y_1, y_2 - y_3\}$. The image of ∂_2 is $\mathbb{Z}\{y_1 - y_2 + y_3, y_1 + y_2 - y + 3\}$. Thus the quotient is

$$\begin{aligned} \mathrm{H}_1^\Delta(\mathbb{RP}^2) &= \mathbb{Z}\{y_1, y_2 - y_3\} / \langle y_1 - y_2 + y_3, y_1 + y_2 - y_3 \rangle \\ &\cong \mathbb{Z}\{y_1\} / \langle 2y_1 \rangle \cong \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

Finally, the image of ∂_1 is $\mathbb{Z}\{x_2 - x_1\}$, so that

$$\mathrm{H}_0^\Delta(\mathbb{RP}^2) \cong \mathbb{Z}\{x_1, x_2\} / \langle x_2 - x_1 \rangle \cong \mathbb{Z}\{x_1\}.$$

Remark 6.1. In general, homology groups can be computed by finding the **Smith normal form** for the differentials. For example, in the $X = T^2$ case, the SNF for ∂_2 is $\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, from which we read off that the kernel is 1-dimensional. The differential ∂_1 is simply zero, and up to a change of basis, the differential ∂_2 hits a generator. It follows that a rank two group survives to give H_1^Δ.

Similarly, for $X = \mathbb{RP}^2$, the SNF for ∂_2 is $\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$, which shows that $\mathrm{H}_2^\Delta = 0$. The SNF for ∂_1 is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so that the kernel is rank 2. Up to change of basis, ∂_1 hits one generator and twice the other, so that H_1^Δ is $\mathbb{Z}/2\mathbb{Z}$.

Now that we have computed some examples, we want to develop the machine some more, so that we don't need to compute by hand every time. The first question we will address is how homology behaves with respect to disjoint unions.

Proposition 7.1. *Let X and Y be Δ-complexes. There is then a canonical Δ-complex structure on $X \sqcup Y$, and we have*

$$\mathrm{H}_n^{\Delta}(X \sqcup Y) \cong \mathrm{H}_n^{\Delta}(X) \oplus \mathrm{H}_n^{\Delta}(Y)$$

for all n.

Proof. The point is that we already have a direct sum decomposition on the level of chain complexes. Namely, if we write $\Delta_n(X)$ for the set of n-simplices of X, then

$$\Delta_n(X \sqcup Y) = \Delta_n(X) \sqcup \Delta_n(Y),$$

so that

$$C_n^{\Delta}(X \sqcup Y) = \mathbb{Z}\{\Delta_n(X \sqcup Y)\} \cong \mathbb{Z}\{\Delta_n(X)\} \oplus \mathbb{Z}\{\Delta_n(Y)\} = C_n^{\Delta}(X) \oplus C_n^{\Delta}(Y).$$

Moreover, the differential is compatible with this splitting, in the sense that we have the commutative diagram

$$\begin{tikzcd}
C_n^{\Delta}(X \sqcup Y) \arrow[r, "\partial_n"] \arrow[d, "\cong"'] & C_{n-1}^{\Delta}(X \sqcup Y) \arrow[d, "\cong"] \\
C_n^{\Delta}(X) \oplus C_n^{\Delta}(Y) \arrow[r, "\partial_n \oplus \partial_n"'] & C_{n-1}^{\Delta}(X) \oplus C_{n-1}^{\Delta}(Y)
\end{tikzcd}$$

This shows that $\mathrm{H}_n^{\Delta}(X \sqcup Y) \cong \mathrm{H}_n^{\Delta}(X) \oplus \mathrm{H}_n^{\Delta}(Y)$ for all n. ■

Another way we might think of this result is that we have the two inclusions $\iota_X : X \hookrightarrow X \sqcup Y$ and $\iota_Y : Y \hookrightarrow X \sqcup Y$. We might expect each of these maps to induce a map on homology, such as $\mathrm{H}_*(\iota_X) : \mathrm{H}_*(X) \longrightarrow \mathrm{H}_*(X \sqcup Y)$, and that the isomorphism of Proposition 7.1 is simply the sum $\mathrm{H}_*(\iota_X) + \mathrm{H}_*(\iota_Y)$. This raises the question:

Question 7.2. *Is homology a functor?*

The answer depends on how you interpret the question. So far, we have only defined homology of Δ-complexes. So we can ask if each H_n^{Δ} defines a functor

$$\mathrm{H}_n^{\Delta} : \Delta\mathbf{Top} \longrightarrow \mathbf{AbGp}$$

for some suitable category $\Delta\mathbf{Top}$ of Δ-complexes. The morphisms in this category, which we will call the Δ-maps, are maps satisfying the following condition: for each simplex $\sigma : \Delta^n \longrightarrow X$ of X, the composition $\Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y$ is an n-simplex of Y. Note that when we say "is an n-simplex", we also mean with its given orientation. Now by the definition of a Δ-map, f will induce a function

$$\hat{f} : \Delta^n(X) \longrightarrow \Delta^n(Y)$$

for each n and therefore also a homomorphism

$$f_* : C_n^{\Delta}(X) \longrightarrow C_n^{\Delta}(Y)$$

for each n. We would like to say that this gives rise to homomorphisms on homology. In order to conclude this, we need to know how f_* interacts with the differential (boundary operator).

Note that if $d^i : \Delta^{n-1} \hookrightarrow \Delta^n$ is the ith face inclusion, the composition with d^i induces a function $d_i : \Delta^n(X) \longrightarrow \Delta^{n-1}(X)$. Since d_i and $\hat{f}$ are given by composition with d^i and f, respectively, we conclude that the diagram

$$\begin{tikzcd} \Delta^n(X) \arrow[r, "\hat{f}"] \arrow[d, "d_i"'] & \Delta^n(Y) \arrow[d, "d_i"] \\ \Delta^{n-1}(X) \arrow[r, "\hat{f}"'] & \Delta^{n-1}(Y) \end{tikzcd}$$

commutes for each n. This implies that the diagram

$$\begin{tikzcd} C_n^{\Delta}(X) \arrow[r, "f_*"] \arrow[d, "d_i"'] & C_n^{\Delta}(Y) \arrow[d, "d_i"] \\ C_{n-1}^{\Delta}(X) \arrow[r, "f_*"'] & C_{n-1}^{\Delta}(Y) \end{tikzcd}$$

commutes for each n. This is precisely the notion of a map of chain complexes.

Definition 7.3. Let (C_*, ∂_*^C) and (D_*, ∂_*^D) be chain complexes. Then a **chain map** $f_* : (C_*, \partial_*^C) \longrightarrow (D_*, \partial_*^D)$ is a sequence of homomorphisms $f_n : C_n \longrightarrow D_n$, for each n, such that each diagram

$$\begin{tikzcd} C_n \arrow[r, "f_n"] \arrow[d, "\partial_n^C"'] & D_n \arrow[d, "\partial_n^D"] \\ C_{n-1} \arrow[r, "f_{n-1}"'] & D_{n-1} \end{tikzcd}$$

commutes for each n.

There is an obvious way to compose chain maps, so that chain complexes and chain maps form a category $\mathbf{Ch}_{\geq \mathbf{0}}(\mathbb{Z})$.

Proposition 7.4. *The assignment $X \mapsto (C_*^{\Delta}(X), \partial_*)$ and $f \mapsto f_*$ defines a functor*

$$C_*^{\Delta} : \Delta\mathbf{Top} \longrightarrow \mathbf{Ch}_{\geq \mathbf{0}}(\mathbb{Z}).$$

Given the above discussion, it only remains to show that this construction takes identity morphisms to identity morphisms and that it preserves composition. We leave this as an exercise.

Why are chain maps a good notion of morphism between chain complexes?

Proposition 7.5. *A chain map $f_* : (C_*, \partial_*^C) \longrightarrow (D_*, \partial_*^D)$ induces homomorphisms $f_n : \mathrm{H}_n(C_*, \partial_*^C) \longrightarrow \mathrm{H}_n(D_*, \partial_*^D)$ for each n.*

Proof. Let $x \in C_n$ be a cycle, meaning that $\partial^C(x) = 0$. Then $\partial^D(f_n(x)) = f_{n-1}(\partial^C(x)) = f_{n-1}(0) = 0$, so that $f_n(x)$ is a cycle in D_n. In order to get a well-defined map on homology, we need to show that if x is in the image of ∂_{n+1}^C, then $f_n(x)$ is in the image of ∂_{n+1}^D. But if $x = \partial_{n+1}^C(y)$, then $f_n(x) = f_n(\partial_{n+1}^C(y)) = \partial_{n+1}^D f_{n+1}(y)$, which shows that $f_n(x)$ is a boundary. ∎

Note that the sequence of homology groups $\mathrm{H}_n(C_*, \partial_*^C)$ of a chain complex is not quite a chain complex, since there are no differentials between the homology groups. You can think of this as a degenerate case of a chain complex, in which all differentials are zero. But it is more common to simply call this a **graded abelian group**. If X_* and Y_* are graded abelian groups, then a graded map $f_* : X_* \longrightarrow Y_*$ is simply a collection of homomorphisms $f_n : X_n \longrightarrow Y_n$. Graded maps compose in the obvious way, so that we get a category **GrAb** of graded abelian groups. Then Proposition 7.5 is the main step in proving

Proposition 7.6. *Homology defines a functor*

$$\mathrm{H}_* : \mathbf{Ch}_{\geq 0}(\mathbb{Z}) \longrightarrow \mathbf{GrAb}.$$

The composition of two functors is always a functor. Thus Proposition 7.4 and Proposition 7.6 combine to yield

Proposition 7.7. *Simplicial homology defines a functor*

$$\mathrm{H}_*^{\Delta} : \Delta\mathbf{Top} \longrightarrow \mathbf{GrAb}.$$

This means that simplicial homology is a reasonably well-behaved construction. Still, the notion of Δ-map is quite restrictive. For instance, if X is a Δ-complex with at least one simplex that is not 0-dimensional, then there is no Δ-map $X \longrightarrow *$. It would be great to have functoriality with respect to a larger collection of maps between spaces.

8. Mon, Sept. 14

There is another variant that is more convenient when working with *based* spaces. Thus let X be a Δ-complex, with a particular 0-simplex x_0 identified as the basepoint. Then the inclusion $\{x_0\} \hookrightarrow X$ *is* Δ-map, so that we get a well-defined homomorphism $\mathrm{H}_*(\{x_0\}) \longrightarrow \mathrm{H}_*(X)$.

Definition 8.1. We define the **reduced homology** groups $\widetilde{\mathrm{H}}_*^{\Delta}(X)$ of (X, x_0) to be the cokernel of this map $\mathrm{H}_*(\{x_0\}) \longrightarrow \mathrm{H}_*(X)$.

Since $\mathrm{H}_n(\{x_0\}) = 0$ if $n > 0$, the reduced homology groups are the same as the ordinary homology groups, except in degree 0. We have simply reduced away the subgroup of $\mathrm{H}_0(X)$ generated by the basepoint. In fact, this subgroup is infinite. To see this, consider the chain maps

$$C_*^{\Delta}(\{x_0\}) \xrightarrow{\iota_*} C_*^{\Delta}(X) \xrightarrow{\varepsilon} C_*^{\Delta}(\{x_0\}),$$

where ε_0 is the homomorphism that sends every 0-simplex to the generator x_0. To see that this makes ε into a chain map, it suffices to see that

$$\begin{tikzcd} C_1^{\Delta}(X) \arrow[r, "\varepsilon_1"] \arrow[d, "\partial_1"'] & C_1^{\Delta}(\{x_0\}) = 0 \arrow[d] \\ C_0^{\Delta}(X) \arrow[r, "\varepsilon_0"'] & C_0^{\Delta}(\{x_0\}) = \mathbb{Z}\{x_0\} \end{tikzcd}$$

commutes. But if e is a 1-simplex from v_0 to v_1, then $\varepsilon\partial_1(e) = \varepsilon(v_1 - v_0) = x_0 - x_0 = 0$ as desired. Since $\varepsilon \circ \iota_* = \mathrm{id}_{C^{\Delta}(\{x_0\})}$, the same must be true after passage to homology (by Prop. 7.6), giving a splitting

$$\mathbb{Z} \cong \mathrm{H}_0^{\Delta}(\{x_0\}) \longrightarrow \mathrm{H}_0^{\Delta}(X) \longrightarrow \mathrm{H}_0^{\Delta}(\{x_0\}).$$

Thus we have

$$\mathrm{H}_0^{\Delta}(X) \cong \widetilde{\mathrm{H}}_0^{\Delta}(X) \oplus \mathbb{Z}.$$

Let us try to understand some of the homology group functors more closely.

Proposition 8.2. *For any Δ-complex X, the group $\mathrm{H}_0^{\Delta}(X)$ is (isomorphic to) the free abelian group on the set $\pi_0(X)$ of path components of X. In particular, for any path-connected space, this group is just $\mathbb{Z}$.*

Proof. Let $X' \subseteq X$ be the union of all 1-simplices in X, and let $\iota : X' \hookrightarrow X$ be the inclusion.

Lemma 8.3. *The inclusion induces a bijection $\iota_* : \pi_0(X') \cong \pi_0(X)$.*

Proof. We define $r : \pi_0(X) \longrightarrow \pi_0(X')$ as follows: for any $x \in X$, pick a simplex σ containing x. The define $r(x)$ to be the path-component in X' of any point y lying in a 1-dimensional face of σ. This does not depend on the choice of y since the union of the 1-dimensional faces of σ is path-connected. It also does not depend on the choice of σ, since if σ' is another such choice, then $\sigma \cap \sigma'$ is a simplex containing x, and we can pick our y from this intersection.

It is clear that $r \circ \iota_*$ is the identity on $\pi_0(X')$. On the other hand, if $x \in X$ then any representative y for $r(x)$ must lie in some simplex σ in X that also contains x. Since σ is path-connected, this imples that $\iota \circ r$ is the identity of $\pi_0(X)$. ■

Note that the inclusion $\iota : X' \hookrightarrow X$ also induces isomorphisms $C_i^\Delta(X') \cong C_i^\Delta(X)$ for $i = 0, 1$, which is all that is relevant for calculation of H_0. Thus, by the above lemma, we may without loss of generality replace X by X'.

Recall that $\mathrm{H}_0^\Delta(X) = C_0^\Delta(X)/\operatorname{im}(\partial_1)$. Let $p : \Delta^0(X) \longrightarrow \pi_0(X)$ be the function that sends each vertex of X to its path-component. This induces a homomorphism $p_* : C_0^\Delta(X) \longrightarrow \mathbb{Z}\{\pi_0(X)\}$, since the free abelian group construction is a functor. If $e \in \Delta^1(X)$ is a 1-simplex in X, then both endpoints of e lie in the same path component of X, since e is precisely a path from one endpoint to the other. It follows that $p_*(\partial_1(e)) = 0$ in $\mathbb{Z}\{\pi_0(X)\}$. This shows that p_* induces a homomorphism

$$p_* : \mathrm{H}_0^\Delta(X;\mathbb{Z}) \longrightarrow \mathbb{Z}\{\pi_0(X)\}.$$

Note that each path-component of X must contain a vertex, since if $x \in X$, then x must lie in some 1-simplex σ of X. But there is a straight-line path in the simplex σ from x to either endpoint of σ, showing that the vertex lies in the same path-component as x. This shows that p_* is surjective.

9. Wed, Sept. 16

Making a choice of 0-simplex in each path-component of X provides a function $s : \pi_0(X) \longrightarrow \Delta^0(X)$ and therefore a function

$$s_* : \mathbb{Z}\{\pi_0(X)\} \longrightarrow C_0^\Delta(X) \twoheadrightarrow \mathrm{H}_0(X;\mathbb{Z}).$$

It remains to show that the composition

$$\mathrm{H}_0^\Delta(X;\mathbb{Z}) \xrightarrow{p_*} \mathbb{Z}\{\pi_0(X)\} \xrightarrow{s_*} \mathrm{H}_0^\Delta(X;\mathbb{Z})$$

is the identity. For any 0-chain $\sum_i n_i x_i$ in X, the composition produces the 0-chain $\sum_i n_i s(x_i)$, so it suffices to show these two 0-chains agree modulo the image of ∂_1. It suffices to show that $x_i - s(x_i)$ is in the image of ∂_1. But x_i and $s(x_i)$ are both 0-simplices lying in the same component of X, so that there must be a path between them which is a finite union of 1-simplices (since paths are compact). Applying ∂_1 to the corresponding finite sum of 1-simplices produces the difference $x_i - s(x_i)$. ■

Proposition 8.2 is not stated optimally, in the sense that it does not say to what extent this depends on X. That is, both $\mathrm{H}_0(-;\mathbb{Z})$ and $\mathbb{Z}\{\pi_0(-)\}$ can be viewed as functors $\Delta\mathbf{Top} \longrightarrow \mathbf{AbGp}$. A stronger version of the proposition would say that these are isomorphic *as functors*. This brings up the question of what should be the notion of a "morphism between functors".

Natural Transformations

Definition 9.1. Let $F, G : \mathscr{C} \to \mathscr{D}$ be functors. A **natural transformation** $\eta : F \to G$ is a collection of maps $\eta_C : F(C) \to G(C)$, one for each $C \in \mathscr{C}$, such that for any $C, C' \in \mathscr{C}$ and any $f \in \operatorname{Hom}_{\mathscr{C}}(C, C')$, the following diagram commutes:

$$\begin{tikzcd} F(C) \arrow[r, "F(f)"] \arrow[d, "\eta_C"'] & F(C') \arrow[d, "\eta_{C'}"] \\ G(C) \arrow[r, "G(f)"] & G(C') \end{tikzcd}$$

The morphism η_C is sometimes called the **component** of η at the object C.

Example 9.2.

(1) We previously described abelianization as a functor $(-)_{ab} : \mathbf{Gp} \longrightarrow \mathbf{AbGp}$. Now $\mathbf{AbGp}$ includes in $\mathbf{Gp}$ as a subcategory, so we can think of abelianization as giving a functor $(-)_{ab} : \mathbf{Gp} \longrightarrow \mathbf{Gp}$. The identity functor $\mathrm{Id}_{\mathbf{Gp}} : \mathbf{Gp} \longrightarrow \mathbf{Gp}$ is another functor with the same domain and codomain. For any group G, the abelianization G_{ab} is defined as a quotient of G, so that there is a quotient homomorphism $\eta : G \longrightarrow G_{ab}$. This homomorphism is "natural in G", in the sense that there is a natural transformation $\eta : \mathrm{Id}_{\mathbf{Gp}} \longrightarrow (-)_{ab}$ whose components are η_G. In other words, for each group homomorphism $\varphi : H \longrightarrow G$, the diagram

$$\begin{tikzcd} H \arrow[r, "\varphi"] \arrow[d, "\eta_H"'] & G \arrow[d, "\eta_G"] \\ H_{ab} \arrow[r, "\varphi_{ab}"'] & G_{ab} \end{tikzcd}$$

commutes. If you look back at Example 2.11, this was precisely the diagram used to define the morphism φ_{ab}.

(2) Recall that for any based Δ-complex (X, x_0), we have a quotient homomorphism

$$\mathrm{H}_n^{\Delta}(X) \longrightarrow \widetilde{\mathrm{H}}_n^{\Delta}(X, x_0).$$

This is a natural transformation of functors $\Delta\mathbf{Top}_* \longrightarrow \mathbf{AbGp}$. In order to make sense of this claim, we first need to discuss the functoriality of reduced homology. Let $f : X \longrightarrow Y$ be a based Δ-map. Then the induced map on reduced homology is defined to be the dashed arrow coming from the universal property of the quotient:

$$\begin{tikzcd} \mathrm{H}_n^{\Delta}(x_0) \arrow[r] \arrow[d] & \mathrm{H}_n^{\Delta}(y_0) \arrow[d] \\ \mathrm{H}_n^{\Delta}(X) \arrow[r, "f_*"] \arrow[d] & \mathrm{H}_n^{\Delta}(Y) \arrow[d] \\ \widetilde{\mathrm{H}}_n^{\Delta}(X, x_0) \arrow[r, dashed, "f_*"'] & \widetilde{\mathrm{H}}_n^{\Delta}(Y, y_0). \end{tikzcd}$$

Note the the commutativity of the bottom square is precisely the statement that the quotient $\mathrm{H}_n^{\Delta} \longrightarrow \widetilde{\mathrm{H}}_n^{\Delta}$ is a natural transformation.

10. Fri, Sept. 18

(3) Let k be a field. For any vector space V over k, we define the dual vector space

$$V^* := \mathrm{Hom}_k(V, k).$$

This is the vector space of linear functionals on V. In fact the assignment $V \mapsto V^*$ determines a contravariant functor $(-)^* : \mathbf{Vect}_k \to \mathbf{Vect}_k$. Composing this functor with itself gives a covariant functor $(-)^{**} : \mathbf{Vect}_k \to \mathbf{Vect}_k$ which sends a vector space to its double dual. Because we will need this below, we note that if $\phi : V \longrightarrow W$ is a linear map, then the induced linear map $\phi^{**} : V^{**} \longrightarrow W^{**}$ is given by $\phi^{**}(X)(\lambda) = X(\lambda \circ \phi)$.

Now fix $v \in V$. We define a function $eval_v : V^* \to k$ by $eval_v(\lambda) = \lambda(v)$. This is in fact k-linear and so determines an element of $(V^*)^*$. But now the assignment $v \mapsto eval_v$ can also be seen to be k-linear, so we have a homomorphism $eval_V : V \to V^{**}$. This map is an isomorphism if V is finite dimensional. Moreover, the homomorphisms $V \to V^{**}$ fit together to determine a natural transformation of functors $\mathrm{Id} \to (-)^{**}$. Again, this means that for every linear map $\phi : V \longrightarrow W$, the diagram

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "eval_V"'] & W \arrow[d, "eval_W"] \\ V^{**} \arrow[r, "\phi^{**}"'] & W^{**} \end{tikzcd}$$

commutes. To see this, let $\lambda : W \longrightarrow k$ be an element of W^*. Then

$$[eval_W \circ \phi](v)(\lambda) = \lambda(\phi(v)) = eval_V(v)(\lambda \circ \phi) = \phi^{**}(eval_V(v))(\lambda) = [\phi^{**} \circ eval_V](v)(\lambda)$$

This is a precise version of the statement that a finite-dimensional vector space is *canonically* isomorphic to its double dual.

Remark 10.1. For finite-dimensional vector spaces, it is also true that V is isomorphic to V^*, but to construct such an isomorphism one must first choose a basis for V. Thus the isomorphism $V \cong V^*$ cannot be natural.

We saw that if we restrict ourselves to $(\mathbf{Vect}_k)_{\text{f.d.}}$, then *eval* determines a natural transformation $\mathrm{Id} \to (-)^{**}$ in which each component $V \to V^{**}$ is an isomorphism. More generally, a natural transformation $\eta : F \to G$ between functors $F, G : \mathscr{C} \to \mathscr{D}$ is called a **natural isomorphism** if $\eta_C : F(C) \to G(C)$ is an isomorphism for each $C \in \mathscr{C}$. This is equivalent to asking that there be a natural transformation $\delta : G \longrightarrow F$ such that $\delta \circ \eta = \mathrm{id}_F$ and $\eta \circ \delta = \mathrm{id}_G$.

Proposition 10.2. *The isomorphisms of Propsition 8.2 assemble together to yield a natural isomorphism of functors* $\mathrm{H}_0^\Delta(-;\mathbb{Z}) \cong \mathbb{Z}\{\pi_0(-)\}$.

Proof. We must show that for each Δ-map of Δ-complexes $f : X \longrightarrow Y$, the square

$$\begin{tikzcd} \mathrm{H}_0^\Delta(X;\mathbb{Z}) \arrow[r, "f_*"] \arrow[d] & \mathrm{H}_0^\Delta(Y;\mathbb{Z}) \arrow[d] \\ \mathbb{Z}\{\pi_0(X)\} \arrow[r, "\mathbb{Z}\{\pi_0(f)\}"'] & \mathbb{Z}\{\pi_0(Y)\} \end{tikzcd}$$

commutes. The vertical maps are induced by maps out of C_0^Δ, so that it suffices to check that

$$\begin{tikzcd} C_0^\Delta(X;\mathbb{Z}) \arrow[r, "f_*"] \arrow[d] & C_0^\Delta(Y;\mathbb{Z}) \arrow[d] \\ \mathbb{Z}\{\pi_0(X)\} \arrow[r, "\mathbb{Z}\{\pi_0(f)\}"'] & \mathbb{Z}\{\pi_0(Y)\} \end{tikzcd}$$

commutes. Starting with a 0-chain $\sum_i n_i x_i$, either composition gives the element $\sum_i n_i f(x_i)$. ∎

We have now given a description of the *functor* $H_0^\Delta(-;\mathbb{Z})$. What about H_1^Δ (or higher homology)? There is a nice answer for H_1, but it is more convenient to address using a different model for homology.

Singular Homology

Simplicial homology is great because, as we have seen, it is very computable. On the other hand, it has the serious defect that it is only defined on Δ-complexes (and Δ-maps). We introduce here a variant that is defined on all spaces.

The basic idea is this: in defining simplicial homology, we took the chains to be free abelian on the set $\Delta^n(X)$ of simplices of X, which we noted could be thought of as maps $\Delta^n \longrightarrow X$. If you look at the formula for the differential, it only uses the formulation as maps from simplices to X.

Definition 10.3. Given a space X, define a **singular n-simplex** of X to be any continuous map $\Delta^n \longrightarrow X$. We define the group of **singular n-chains** on X to be

$$C_n(X) := \mathbb{Z}\{\mathbf{Top}(\Delta^n, X)\}.$$

We sometimes write $\mathrm{Sing}_n(X) := \mathbf{Top}(\Delta^n, X)$. Again, the formula for the differential in Definition 4.4 makes just as much sense in the singular context.

Definition 10.4. Given a space X, we define the **singular homology groups** of X to be the homology groups of the chain complex $(C_*(X), \partial)$.

If X is a Δ-complex, then any simplex of X may be thought of as a singular simplex. This gives natural maps $C_*^\Delta(X) \longrightarrow C_*(X)$ of chain complexes and therefore natural maps of graded groups $H_*^\Delta(X) \longrightarrow H_*(X)$. We will see later that these are isomorphisms.

Notice that the groups $C_*(X)$ are *much* bigger than the groups $C_*^\Delta(X)$. For a Δ-complex with finitely many simplices, the latter groups all have finite rank, whereas this is almost never the case for the groups $C_*(X)$.

Example 10.5. Consider $X = *$. Then $C_n(\{*\}) = \mathbb{Z}\{\mathbf{Top}(\Delta^n, \{*\})\} \cong \mathbb{Z}$ for all n. The differential $\partial_n : C_n(\{*\}) \longrightarrow C_{n-1}(\{*\})$ takes the (constant) singular n-simplex c_n to the alternating sum

$$\sum_i (-1)^i c_{n-1} = \begin{cases} c_{n-1} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}.$$

In other words, the chain complex is

$$\ldots \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z},$$

so that the only nonzero homology group is $H_0(*) \cong \mathbb{Z}$.

But already for $X = \Delta^1$, the chain groups are infinite rank, and computing becomes impractical. On the other hand, the singular homology groups have much better properties.

Proposition 10.6. *Singular homology defines a functor*

$$H_* : \mathbf{Top} \longrightarrow \mathbf{GrAb}.$$

Proof. The proof strategy is the same as for Proposition 7.7. The main point is that, for *any* continuous map $f : X \longrightarrow Y$, composition with f defines a function $\hat{f} : \mathrm{Sing}_n(X) \longrightarrow \mathrm{Sing}_n(Y)$. The rest of the argument is the same. ■

This implies, for instance, that homeomorphic spaces have isomorphic singular homology groups. But now that we've been given an inch, we want a whole yard. We will show that homology factors through the homotopy category.

It is *not* true that the singular chains functors $C_n(-) : \mathbf{Top} \longrightarrow \mathbf{Ch}_{\geq 0}(\mathbb{Z})$ factor through the homotopy category, so a new idea is needed, that of a chain homotopy between chain maps of chain complexes.

Definition 11.1. Let $f, g : C_* \rightrightarrows D_*$ be chain maps. Then a **chain homotopy** h is a sequence of homomorphisms $h_n : C_n \longrightarrow D^{n+1}$ satisfying

$$\partial^D_{n+1}(h_n(c)) + h_{n-1}(\partial^C_n c) = g(c) - f(c).$$

$$\begin{tikzcd}
C_{n+1} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "\partial^C_{n+1}"'] & D_{n+1} \arrow[d, "\partial^D_{n+1}"] \\
C_n \arrow[ur, "h_n"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "\partial^C_n"'] & D_n \arrow[d, "\partial^D_n"] \\
C_{n-1} \arrow[ur, "h_{n-1}"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & D_{n-1}
\end{tikzcd}$$

Remark 11.2. It is probably not apparent why this notion deserves the name "chain homotopy". A homotopy in topology means a map $I \times X \longrightarrow Y$, and it turns out that there is a chain complex I_* such that a chain homotopy in the sense given above is the same as a chain map $I_* \otimes X_* \longrightarrow Y_*$, where here $\otimes$ means the tensor product of chain complexes.

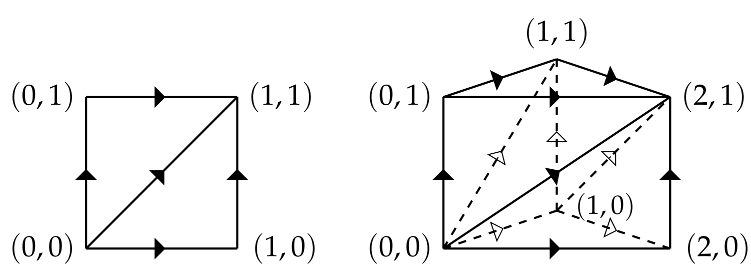
Proposition 11.3. *Let $h : X \times I \longrightarrow Y$ be a homotopy between $f = h_0$ and $g = h_1$. Then there exists a chain homotopy h^C_* between $C_*(f)$ and $C_*(g)$.*

Proof. If σ is a singular n-simplex of X, then h gives the composite

$$\Delta^n \times I \xrightarrow{\sigma \times \mathrm{id}} X \times I \xrightarrow{h} Y.$$

Lemma 11.4. *The product $\Delta^n \times I$ has a canonical Δ-complex structure with $n+1$ (simplicial) $(n+1)$-simplices.*

Proof. We sketch this structure for $n = 1$ and 2.



Vertices of the simplices of $\Delta^n \times I$ are labelled by pairs (j, k), where $0 \leq j \leq n$ and $0 \leq k \leq 1$. The $(n+1)$-simplices each include a single "vertical" 1-simplex with endpoints $(i, 0)$ and $(i, 1)$. We denote by $p_i : \Delta^{n+1} \hookrightarrow \Delta^n \times I$ the inclusion of the simplex which includes the vertical edge at $(i, 0)$. ■

We abuse notation and write $p_i(\sigma)$ for the composition

$$\Delta^{n+1} \xrightarrow{p_i} \Delta^n \times I \xrightarrow{\sigma \times \mathrm{id}} X \times I.$$

We then define

$$h^C_n(\sigma) = \sum_{i=0}^{n} (-1)^i h p_i(\sigma).$$

To verify that this is a chain homotopy as claimed, we make several observations:

(1) The Δ-complex $\Delta^n \times I$ has n "internal" n-simplices, with vertices

$$(0,0),(1,0),\dots,(i,0),(i+1,1),\dots,(n,1).$$

When calculating $\partial_{n+1}(h_n^C(\sigma))$, this n-simplex shows up as both $p_i(\sigma)\circ d^i$ and $p_{i+1}(\sigma)\circ d^i$. Since $p_i(\sigma)$ and $p_{i+1}(\sigma)$ appear with opposite signs in $h_n^C(\sigma)$, these two will cancel out in $\partial_{n+1}(h_n^C(\sigma))$.

Thus the only terms that remain in $\partial_{n+1}(h_n^C(\sigma))$ are the "external" n-simplices, which contain a vertical edge, as well as the "horizontal" n-simplices $g(\sigma)$ and $f(\sigma)$.

(2) Each of the external n-simplices occurs as the face of a single $n+1$-simplex and thus appears only once in $\partial_{n+1}(h_n^C(\sigma))$. Moreover, each of these can be written in the form $p_i(\sigma\circ d^j)$ and therefore appears in $h_{n-1}^C(\partial_n(\sigma))$. In fact, every term of $h_{n-1}^C(\partial_n(\sigma))$ arises in this way.

■

Proposition 11.5. *If $f, g : C_* \rightrightarrows D_*$ are chain-homotopic, then $\mathrm{H}_*(f) = \mathrm{H}_*(g)$.*

Proof. It suffices to show that for any n-cycle c, the difference $g(c) - f(c)$ is in the image of the boundary map. But this comes directly from the definition of chain-homotopy, since $h_{n-1}(\partial_n^C(c)) = h_{n-1}(0) = 0$. ■

Combining propositions 11.3 and 11.5 gives

Proposition 11.6 (Homotopy invariance)**.** *If $f, g : X \rightrightarrows Y$ are homotopic, then $\mathrm{H}_*(f) = \mathrm{H}_*(g)$.*

Corollary 11.7. *If $X \simeq Y$, then $\mathrm{H}_*(X) \cong \mathrm{H}_*(Y)$.*

So the homology of any contractible space agrees with the homology of a point. Said differently, the reduced homology of any contractible space is zero.

12. Wed, Sept. 23

Coefficients

Recall that when we originally introduced homology, we wrote $\mathrm{H}_*(X;\mathbb{Z})$. We know how to let X vary, but the notation suggests that we should also be able to substitute for the $\mathbb{Z}$ as well.

Definition 12.1. Given an abelian group A, we define the group of singular chains **with coefficients in** A to be

$$C_n(X;A) := C_n(X)\otimes_{\mathbb{Z}} A.$$

The **singular homology groups with coefficients in** A are then defined by

$$\mathrm{H}_n(X;A) := \mathrm{H}_n(C_*(X;A)).$$

Similarly, the **simplicial homology groups with coefficients in** A are defined by

$$\mathrm{H}_n^\Delta(X;A) := \mathrm{H}_n(C_*^\Delta(X;\mathbb{Z})\otimes A).$$

This simply means that when we write an n-chain as a linear combination $\sum_i n_i\sigma_i$, each n_i should be in A rather than $\mathbb{Z}$. The

The most common choices for A, other than $\mathbb{Z}$, are the fields $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{F}_p$.

Example 12.2. $X = S^1$. If we take the Δ-complex having a single 0-simplex and single 1-simplex, then the chain complex with coefficients in A is just

$$\begin{tikzcd} C_1^\Delta(S^1) \otimes A \arrow[r, "\partial_1"] \arrow[d, equal] & C_0^\Delta(S^1) \otimes A \arrow[d, equal] \\ A\{e\} & A\{v\}, \end{tikzcd}$$

where $\partial_1 = 0$. It follows that $\mathrm{H}_1^\Delta(S^1; A) = A$ and $\mathrm{H}_0^\Delta(S^1; A)$.

A more interesting example is

Example 12.3. $X = \mathbb{RP}^2$, $A = k$ is a field. The chain complex with coefficients in k is

$$\begin{tikzcd} C_2^\Delta(\mathbb{RP}^2) \arrow[r, "\partial_2"] \arrow[d, equal] & C_1^\Delta(\mathbb{RP}^2) \arrow[r, "\partial_1"] \arrow[d, equal] & C_0^\Delta(\mathbb{RP}^2) \arrow[d, equal] \\ k\{z_1, z_2\} \arrow[r, "{\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}}"'] & k\{y_1, y_2, y_3\} \arrow[r, "{\begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix}}"'] & k\{x_1, x_2\}. \end{tikzcd}$$

The Smith Normal Form that we previously found over $\mathbb{Z}$ gives a reduced echelon form over k. The echelon form for ∂_1 is $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, but the Smith Normal Form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{smallmatrix}\right)$ for ∂_2 gives a reduced echelon form of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ if $\mathrm{char}(k) \neq 2$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ if $\mathrm{char}(k) = 2$. Thus we read off the homology groups

$$\mathrm{H}_0^\Delta(\mathbb{RP}^2; \mathbb{F}_2) \cong \mathbb{F}_2, \qquad \mathrm{H}_1^\Delta(\mathbb{RP}^2; \mathbb{F}_2) \cong \mathbb{F}_2, \qquad \mathrm{H}_2^\Delta(\mathbb{RP}^2; \mathbb{F}_2) \cong \mathbb{F}_2$$

and

$$\mathrm{H}_0^\Delta(\mathbb{RP}^2; k) \cong k, \qquad \mathrm{H}_1^\Delta(\mathbb{RP}^2; k) = 0, \qquad \mathrm{H}_2^\Delta(\mathbb{RP}^2; k) = 0$$

if $\mathrm{char}(k) \neq 2$.

[We had a discussion of tensor products today.]

13. Fri, Sept. 25

We spent some time discussing tensor products last time. This construction also has a universal property:

Proposition 13.1. *The homomorphism $u : A \oplus B \longrightarrow A \otimes B$ defined by $u(a, b) = a \otimes b$ is the universal example of a bilinear map out of $A \oplus B$. That is, if $f : A \oplus B \longrightarrow C$ is also bilinear, then there is a unique homomorphism $\overline{f} : A \otimes B \longrightarrow C$ making the diagram commute.*

$$\begin{tikzcd} A \oplus B \arrow[rr, "f"] \arrow[dr, "u"'] & & C \\ & A \otimes B \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

Beware that $u : A \oplus B \longrightarrow A \otimes B$ is *not* surjective in general. For instance $\mathbb{Z}^2 \otimes \mathbb{Z}^3 \cong \mathbb{Z}^6$.

There is another important property satisfied by the tensor product; namely, its interaction with Hom.

Proposition 13.2. *Given abelian groups A, B, and C, there is an isomorphism*

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B, C))$$

that is natural in A, B, and C.

This is precisely analogous to the relationship between cartesian product and mapping spaces in topology.

For a given space X, the assignment $A \mapsto \mathrm{H}_n(X; A)$ is functorial in A, meaning that any homomorphism $\varphi : A \longrightarrow B$ induces a homomorphism $\varphi_* : \mathrm{H}_n(X; A) \longrightarrow \mathrm{H}_n(X; B)$ by simply applying φ to the coefficients in any n-chain in X. Even better, the homomorphisms φ_* are *natural* in X. But there is an even stronger connection between the $\mathrm{H}_n(X; A)$ as A varies.

Recall that a **short exact sequence** is a chain complex

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{q} C \longrightarrow 0$$

that is *exact* (has no homology). Exactness at the three spots means

(1) $\ker(i) = 0$, so that i is injective
(2) $\ker(q) = \mathrm{im}(i)$, and
(3) $\mathrm{im}(q) = C$, so that q is surjective.

A standard example is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0.$$

The question is what does this short exact sequence of coefficients buy for us at the level of homology? Let's first consider what happens at the level of chain complexes. The first observation is that we get a short exact sequence of chain complexes

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C_{n+1}(X) \arrow[r, "p"] \arrow[d, "\partial_{n+1}"'] & C_{n+1}(X) \arrow[r] \arrow[d, "\partial_{n+1}"'] & C_{n+1}(X) \otimes \mathbb{Z}/p\mathbb{Z} \arrow[r] \arrow[d, "\partial_{n+1}"'] & 0 \\
0 \arrow[r] & C_n(X) \arrow[r, "p"] \arrow[d, "\partial_n"'] & C_n(X) \arrow[r] \arrow[d, "\partial_n"'] & C_n(X) \otimes \mathbb{Z}/p\mathbb{Z} \arrow[r] \arrow[d, "\partial_n"'] & 0 \\
0 \arrow[r] & C_{n-1}(X) \arrow[r, "p"] \arrow[d, "\partial_{n-1}"'] & C_{n-1}(X) \arrow[r] \arrow[d, "\partial_{n-1}"'] & C_{n-1}(X) \otimes \mathbb{Z}/p\mathbb{Z} \arrow[r] \arrow[d, "\partial_{n-1}"'] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

This means that each row is a short exact sequence and that moreover all squares in the above diagram commute. (Note that the fact that each row is exact relies on the fact that each group $C_n(X)$ is free abelian.)

Proposition 13.3. *A short exact sequence* $0 \longrightarrow A_* \xrightarrow{i} B_* \xrightarrow{q} C_* \longrightarrow 0$ *of chain complexes induces a long exact sequence in homology*

$$\ldots \longrightarrow \mathrm{H}_{n+1}(C) \xrightarrow{\delta} \mathrm{H}_n(A) \xrightarrow{i_*} \mathrm{H}_n(B) \xrightarrow{q_*} \mathrm{H}_n(C) \xrightarrow{\delta} \mathrm{H}_{n-1}(A) \longrightarrow \ldots$$

Proof. We start with the construction of the "connecting homomorphism δ". Thus let $c \in C_n$ be a cycle. Choose a lift $b \in B_n$, meaning that $q(b) = c$. We then have $q(\partial_n(b)) = \partial_n(q(b)) = \partial_n(c) = 0$. Since the rows are exact, we have $\partial_n(b) = i(a)$ for some unique $a \in A_{n-1}$, and we define

$$\delta(c) := a.$$

$$\begin{tikzcd} & b \arrow[r, mapsto] \arrow[d, mapsto] & c \arrow[d, mapsto] \\ a \arrow[r, mapsto] & \partial(b) \arrow[r, mapsto] & 0 \end{tikzcd}$$

It remains to see how a depends on the choice of b. Thus let $d \in \ker(q)$, so that $q(b+d) = c$. By exactness, we have $d = i(e)$ for some $e \in A_n$. Then

$$i(a + \partial_n(e)) = \partial_n(b) + i(\partial_n(e)) = \partial_n(b) + \partial_n(i(e)) = \partial_n(b) + \partial_n(d) = \partial_n(b+d),$$

so that $\delta(c) = a + \partial_n(e) \sim a$. In other words, a specifies a well-defined homology class.

Since we want δ to be well-defined not only on cycles but also on homology, we need to show that if c is a boundary, then $\delta(c) \sim 0$. Thus suppose $c = \partial(c')$. We can then choose b' such that $q(b') = c'$. It follows that $\partial(b')$ would be a suitable choice for b. But then $\partial(b) = \partial(\partial(b')) = 0$, so that $\delta(c) = 0$.

14. Mon, Sept. 28

Exactness at B: First, we see that $q_* \circ i_* = 0$ since this is already true at the chain level. Now suppose that $b \in \ker(q_*)$. This means that $q(b) = \partial(c)$ for some $c \in C_{n+1}$. Now choose a lift $d \in B_{n+1}$ of c. Then we know

$$q(\partial(d)) = \partial(q(d)) = \partial(c) = q(b).$$

In other words, $q(b - \partial(d)) = 0$, so that we must have $b - \partial(d) = i(a)$ for some a. Since $b \sim b - \partial(d)$, we are done.

Exactness at C: We first show that $\delta \circ q_* = 0$. Thus let $b \in B_n$ be a cycle. We wish to show that $\delta(q_*(b)) = 0$. But the first step in constructing $\delta(q(b))$ is to choose a lift for $q(b)$, which we can of course take to be b. Then $\partial(b) = 0$, so that $a = 0$ as well.

Now suppose that $c \in C_n$ is a cycle that lives in the kernel of δ. This means that $a = \partial(e)$ for some e. But then $b - i(e)$ is a cycle, and $q(b - i(e)) = c$, so c is in the image of q_*.

Exactness at A: First, we show that $i_* \circ \delta = 0$. Let $c \in C_n$ be a cycle. Then if $\delta(c) = a$, then by construction, we have $i(a) = \partial(b) \sim 0$, so that $i_* \circ \delta = 0$.

Finally, suppose that $a \in A_n$ is a cycle that lives in $\ker i_*$. Then $i(a) = \partial(b)$ for some b, but then $a = \delta(q(b))$. ■

Example 14.1. The short exact sequence $0 \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$ gives rise to a short exact sequence of chain complexes

$$0 \longrightarrow C_*^{\Delta}(X) \xrightarrow{p} C_*^{\Delta}(X) \xrightarrow{q_*} C_*^{\Delta}(X) \otimes \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

and therefore to a long exact sequence

$$\dots \longrightarrow H_{n+1}^{\Delta}(X;\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\delta} H_n^{\Delta}(X;\mathbb{Z}) \xrightarrow{p} H_n^{\Delta}(X;\mathbb{Z}) \xrightarrow{q_*} H_n^{\Delta}(X;\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\delta} H_{n-1}^{\Delta}(X;\mathbb{Z}) \longrightarrow \dots$$

Taking $X = \mathbb{RP}^2$, this long exact sequence takes the form

$$0 \longrightarrow H_2^{\Delta}(\mathbb{RP}^2;\mathbb{Z}) \xrightarrow{p} H_2^{\Delta}(\mathbb{RP}^2;\mathbb{Z}) \xrightarrow{q_*} H_2^{\Delta}(\mathbb{RP}^2;\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\delta} H_1^{\Delta}(\mathbb{RP}^2;\mathbb{Z}) \xrightarrow{p} H_1^{\Delta}(\mathbb{RP}^2;\mathbb{Z})$$

$$\xrightarrow{q_*} H_1^{\Delta}(\mathbb{RP}^2;\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\delta} H_0^{\Delta}(\mathbb{RP}^2;\mathbb{Z}) \xrightarrow{p} H_0^{\Delta}(\mathbb{RP}^2;\mathbb{Z}) \xrightarrow{q_*} H_0^{\Delta}(\mathbb{RP}^2;\mathbb{Z}/p\mathbb{Z}) \longrightarrow 0.$$

If p is odd, this sequence becomes

$$0 \longrightarrow 0 \xrightarrow{p} 0 \xrightarrow{q_*} H_2^{\Delta}(\mathbb{RP}^2;\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\delta} \mathbb{Z}/2\mathbb{Z} \xrightarrow{p} \mathbb{Z}/2\mathbb{Z}$$

$$\xrightarrow{q_*} H_1^{\Delta}(\mathbb{RP}^2;\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\delta} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{q_*} H_0^{\Delta}(\mathbb{RP}^2;\mathbb{Z}/p\mathbb{Z}) \longrightarrow 0.$$

Since $\mathbb{Z}/2\mathbb{Z} \xrightarrow{p} \mathbb{Z}/2\mathbb{Z}$ is an isomorphism, we conclude that $\mathrm{H}_2^{\Delta}(\mathbb{RP}^2;\mathbb{Z}/p\mathbb{Z}) = 0$ and $\mathrm{H}_1^{\Delta}(\mathbb{RP}^2;\mathbb{Z}/p\mathbb{Z}) = 0$. We also get that $\mathrm{H}_0^{\Delta}(\mathbb{RP}^2;\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$.

On the other hand, for $p = 2$, we get the sequence

$$0 \longrightarrow 0 \xrightarrow{p} 0 \xrightarrow{q_*} \mathrm{H}_2^{\Delta}(\mathbb{RP}^2;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\delta} \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/2\mathbb{Z}$$

$$\xrightarrow{q_*} \mathrm{H}_1^{\Delta}(\mathbb{RP}^2;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\delta} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{q_*} \mathrm{H}_0^{\Delta}(\mathbb{RP}^2;\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0.$$

Since $\mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/2\mathbb{Z}$ is zero, we get $\mathrm{H}_2^{\Delta}(\mathbb{RP}^2;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \cong \mathrm{H}_1^{\Delta}(\mathbb{RP}^2;\mathbb{Z}/2\mathbb{Z})$. We also get $\mathrm{H}_0^{\Delta}(\mathbb{RP}^2;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ as before.

The Long Exact Sequence

Let $A \subseteq X$ be a subspace. Define the group of relative n-chains by

$$C_n(X, A) := C_n(X)/C_n(A).$$

Definition 14.2. Given $A \subseteq X$ and an abelian group M, we define the **relative homology groups** to be

$$\mathrm{H}_n(X, A; M) := \mathrm{H}_n(C_*(X, A) \otimes M).$$

Given our discussion from above, we easily derive

Proposition 14.3. *For any subspace $A \subseteq X$ and abelian group M, there is a long exact sequence*

$$\dots \mathrm{H}_n(A; M) \xrightarrow{i_*} \mathrm{H}_n(X; M) \longrightarrow \mathrm{H}_n(X, A; M) \xrightarrow{\delta} \mathrm{H}_{n-1}(A; M) \longrightarrow \dots$$

Proof. We have a short exact sequence of chain complexes

$$0 \longrightarrow C_*(A) \longrightarrow C_*(X) \longrightarrow C_*(X)/C_*(A) \longrightarrow 0.$$

Moreover, we have a levelwise splitting $C_n(X) \longrightarrow C_n(A)$ which sends a simplex σ to either σ or 0, depending on whether or not the image of σ lies entirely in A. It follows that we have a short exact sequence of chain complexes

$$0 \longrightarrow C_*(A) \otimes M \longrightarrow C_*(X) \otimes M \longrightarrow C_*(X, A) \otimes M \longrightarrow 0.$$

The result is now a direct application of Prop 13.3. ■

15. WED, SEPT. 30

Example 15.1. If (X, x_0) is a based space, then we get a long exact sequence

$$\dots \mathrm{H}_n(x_0) \xrightarrow{i_*} \mathrm{H}_n(X) \longrightarrow \mathrm{H}_n(X, x_0) \xrightarrow{\delta} \mathrm{H}_{n-1}(x_0) \longrightarrow .$$

Moreover, the map $p : X \longrightarrow x_0$ induces a splitting $p_* : \mathrm{H}_n(X) \longrightarrow \mathrm{H}_n(x_0)$ to i_*. It follows that each connecting homomorphism δ is zero, so that the long exact sequence breaks up into a bunch of short exact sequences

$$0 \longrightarrow \mathrm{H}_n(x_0) \longrightarrow \mathrm{H}_n(X) \longrightarrow \mathrm{H}_n(X, x_0) \longrightarrow 0.$$

Since reduced homology was defined to be the cokernel of i_*, we conclude that

$$\widetilde{\mathrm{H}}_n(X) \cong \mathrm{H}_n(X, x_0).$$

However, in general the long exact sequence is of limited use unless we can compute the relative groups. One of the main tools for computing relative homology is the Excision Theorem.

Definition 15.2. An **excisive triad** is a triple $(X; A, B)$, where $A, B \subseteq X$ and $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$.

Theorem 15.3 (Excision)**.** *Let $(X; A, B)$ be an excisive triad. Then the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism*

$$\mathrm{H}_n(A, A \cap B; M) \cong \mathrm{H}_n(X, B; M)$$

for any coefficient group M.

Example 15.4. We use the Excision Theorem to compute $\mathrm{H}_k(S^n)$. We write S^n as a union

$$S^n = S^n_+ \cup S^n_-,$$

where S^n_+ and S^n_- are the upper and lower hemispheres (extended by a collar around the equator, so that the equator lies in the interior of each). The intersection $S^n_+ \cap S^n_-$ is a thickened version of the equator, but we simply write S^{n-1}, since these are homotopy equivalent. Now the long exact sequence for the pair (S^n, S^n_-) takes the form

$$\longrightarrow \mathrm{H}_k(S^n_-) \longrightarrow \mathrm{H}_k(S^n) \longrightarrow \mathrm{H}_k(S^n, S^n_-) \xrightarrow{\delta} \mathrm{H}_{k-1}(S^n_-) \longrightarrow .$$

Since the hemisphere S^n_- is contractible, the outer two groups are zero if $k \geq 2$. Thus

$$\mathrm{H}_k(S^n) \cong \mathrm{H}_k(S^n, S^n_-) \quad \text{if } k \geq 2.$$

In the case $k = 1$, this part of the sequence is

$$0 = \mathrm{H}_1(S^n_-) \longrightarrow \mathrm{H}_1(S^n) \longrightarrow \mathrm{H}_1(S^n, S^n_-) \xrightarrow{\delta} \mathrm{H}_0(S^n_-) \longrightarrow \mathrm{H}_0(S^n).$$

The rightmost map is an isomorphism $\mathbb{Z} \cong \mathbb{Z}$, so that $\delta = 0$. We conclude that $\mathrm{H}_1(S^n) \cong \mathrm{H}_1(S^n, S^n_-)$.

Now excision gives $\mathrm{H}_k(S^n, S^n_-) \cong \mathrm{H}_k(S^n_+, S^{n-1})$, and the long exact sequence for the pair (S^n_+, S^{n-1}) is

$$\longrightarrow \mathrm{H}_k(S^n_+) \longrightarrow \mathrm{H}_k(S^n_+, S^{n-1}) \xrightarrow{\delta} \mathrm{H}_{k-1}(S^{n-1}) \longrightarrow \mathrm{H}_{k-1}(S^n_+) \longrightarrow .$$

Again, the hemisphere S^n_+ is contractible, so the outer two groups are zero if $k \geq 2$. We have shown that

$$\mathrm{H}_k(S^n) \cong \mathrm{H}_k(S^n, S^n_-) \cong \mathrm{H}_k(S^n_+, S^{n-1}) \cong \mathrm{H}_{k-1}(S^{n-1}) \quad \text{if } k \geq 2.$$

16. Fri, Oct. 2

If $k = 1$, this becomes

$$0 = \mathrm{H}_1(S^n_+) \longrightarrow \mathrm{H}_1(S^n_+, S^{n-1}) \xrightarrow{\delta} \mathrm{H}_0(S^{n-1}) \longrightarrow \mathrm{H}_0(S^n_+).$$

If $n \geq 2$, then the right map is an isomorphism $\mathbb{Z} \cong \mathbb{Z}$, so that $\mathrm{H}_1(S^n) \cong \mathrm{H}_1(S^n_+, S^{n-1}) = 0$. The other possible case is $n = 1$, in which case the right map is the fold map $\mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}$, so that $\mathrm{H}_1(S^1) \cong \mathrm{H}_1(S^1_+, S^0)$ is identified with the kernel of the fold map, which is isomorphic to $\mathbb{Z}$.

Combining the above results, if $k > n$, then

$$\mathrm{H}_k(S^n) \cong \mathrm{H}_{k-1}(S^{n-1}) \cong \ldots \cong \mathrm{H}_{k-n+1}(S^1) = 0.$$

If $k = n$, we have

$$\mathrm{H}_n(S^n) \cong \mathrm{H}_{n-1}(S^{n-1}) \cong \ldots \cong \mathrm{H}_1(S^1) \cong \mathbb{Z}.$$

If $k < n$, we have

$$\mathrm{H}_k(S^n) \cong \mathrm{H}_{k-1}(^{n-1}) \cong \ldots \cong \mathrm{H}_1(S^{n-k+1}) \cong 0.$$

In summary, if $k, n \geq 1$, then

$$\mathrm{H}_k(S^n) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n. \end{cases}$$

If we switch to reduced homology, the statement holds and extends to include the $n = 0$ case.

Example 16.1. $X = \mathbb{RP}^2$. Recall that we can build $\mathbb{RP}^2$ as a CW complex in which we start with a single 1-cell and attach a 2-cell via the attaching map $S^1 \xrightarrow{2} S^1$.

Let x be a point in the interior of the attached 2-cell. Then $\mathbb{RP}^2 - \{x\}$ deformation retracts onto the 1-skeleton S^1. Write $U = \mathbb{RP}^2 - \{x\}$, and let V be the interior of the 2-cell. Then $U \cap V = V - \{x\} \simeq S^1$. The long exact sequence takes the form

$$\longrightarrow \mathrm{H}_2(U) \longrightarrow \mathrm{H}_2(\mathbb{RP}^2) \longrightarrow \mathrm{H}_2(\mathbb{RP}^2, U) \xrightarrow{\delta} \mathrm{H}_1(U) \longrightarrow \mathrm{H}_1(\mathbb{RP}^2) \longrightarrow \mathrm{H}_1(\mathbb{RP}^2, U) \xrightarrow{\delta} \mathrm{H}_0(U) \longrightarrow \mathrm{H}_0(\mathbb{RP}^2).$$

Since $U \simeq S^1$, we know that $\mathrm{H}_k(U) = 0$ for $k \geq 2$, so that $\mathrm{H}_k(\mathbb{RP}^2) \cong \mathrm{H}_k(\mathbb{RP}^2, U)$ for all $k \geq 2$. We have previously identified $\mathrm{H}_0(X)$ with $\mathbb{Z}\{\pi_0(X)\}$, so the last map is an isomorphism $\mathbb{Z} \cong \mathbb{Z}$. It follows that the last δ must be zero, so we can replace our sequence with

$$0 \longrightarrow \mathrm{H}_2(\mathbb{RP}^2) \longrightarrow \mathrm{H}_2(\mathbb{RP}^2, U) \xrightarrow{\delta} \mathbb{Z} \longrightarrow \mathrm{H}_1(\mathbb{RP}^2) \longrightarrow \mathrm{H}_1(\mathbb{RP}^2, U) \longrightarrow 0.$$

We use excision to calculate these relative groups. Excision identifies the above relative groups with the relative groups for $(V, V \cap U) \simeq (D^2, S^1)$. These groups sit in a long exact sequence

$$\mathrm{H}_2(D^2) \longrightarrow \mathrm{H}_2(D^2, S^1) \xrightarrow{\delta} \mathrm{H}_1(S^1) \longrightarrow \mathrm{H}_1(D^2) \longrightarrow \mathrm{H}_1(D^2, S^1) \xrightarrow{\delta} \mathrm{H}_0(S^1) \longrightarrow \mathrm{H}_0(D^2).$$

Since $\mathrm{H}_k(D^2)$ and $\mathrm{H}_k(S^1)$ both vanish for $k \geq 2$, it follows that the relative groups vanish for $k \geq 3$. By the above, this shows that $\mathrm{H}_k(\mathbb{RP}^2) = 0$ for $k \geq 3$. Next, we we identify the above sequence with

$$0 \longrightarrow \mathrm{H}_2(D^2, S^1) \xrightarrow{\delta} \mathbb{Z} \longrightarrow 0 \longrightarrow \mathrm{H}_1(D^2, S^1) \xrightarrow{\delta} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z}.$$

It follows that $\mathrm{H}_2(D^2, S^1) \cong \mathbb{Z}$ and $\mathrm{H}_1(D^2, S^1) \cong 0$. Plugging this back in above gives the exact sequence

$$0 \longrightarrow \mathrm{H}_2(\mathbb{RP}^2) \longrightarrow \mathbb{Z} \xrightarrow{\delta} \mathbb{Z} \longrightarrow \mathrm{H}_1(\mathbb{RP}^2) \longrightarrow 0 \longrightarrow 0.$$

Now we cheat, and **assume** $\mathrm{H}_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}$. We will see later that this follows from the Hurewicz theorem. This implies that δ must be multiplication by 2 and so $\mathrm{H}_2(\mathbb{RP}^2) = 0$.

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In order to prove the excision theorem, we introduce a new chain complex: let $C_n^{A,B}(X)$ be the free abelian group on (singular) n-simplices of X whose image lies entirely in either A or B. This condition is preserved by the differential of $C_*(X)$, so that $C_*^{A,B}(X) \subseteq C_*(X)$ is a sub-chain complex.

Proposition 17.1. *The inclusion $C_*^{A,B}(X) \hookrightarrow C_*(X)$ is a chain homotopy equivalence.*

Proof. We only give a brief indication. For a full (and lengthy) proof, see Prop 2.21 of Hatcher.

We need to define a homotopy inverse $f : C_*(X) \longrightarrow C_*^{A,B}(X)$. The idea is to use barycentric subdivision. The subdivision of an n-simplex expresses it as the union of smaller n-simplices. By the Lebesgue Number Lemma, repeated barycentric subdivision will eventually decompose any singular n-simplex of X into a collection of n-simplices, each of which is either contained in A or in B. This subdivision allows you to define a chain map f. You then show that subdivision of simplices is chain-homotopic to the identity. ■

Proof of Theorem 15.3. The chain homotopy equivalence $C_*^{A,B}(X) \simeq C_*(X)$ carries $C_*(B)$ into itself, so that we get a chain homotopy equivalence

$$C_*^{A,B}(X)/C_*(B) \simeq C_*(X)/C_*(B).$$

But the inclusion $C_*(A) \hookrightarrow C_*^{A,B}(X)$ induces an isomorphism

$$C_*(A)/C_*(A \cap B) \cong C_*^{A,B}(X)/C_*(B),$$

since both quotients can be identified with the free abelian group on n-simplices in A that are not entirely contained in B. These chain homotopy equivalences are carried over after tensoring with M, which gives the theorem. ■

Recall that, given a map $f : A \longrightarrow X$, the **mapping cone** $C(f)$ on f is defined to be

$$C(f) := X \cup_A C(A).$$

Proposition 17.2. *In general, we have* $\mathrm{H}_n(X, A) \cong \widetilde{\mathrm{H}}_n(C(f))$*, so that the long exact sequence may be written*

$$\ldots \mathrm{H}_n(A; M) \xrightarrow{i_*} \mathrm{H}_n(X; M) \longrightarrow \widetilde{\mathrm{H}}_n(C(f); M) \xrightarrow{\delta} \mathrm{H}_{n-1}(A; M) \longrightarrow \ldots$$

Proof. We write c for the cone point in $C(A) \subseteq C(f)$. Since $C(A) \simeq *$, we have $\widetilde{\mathrm{H}}_n(C(f)) \cong \mathrm{H}_n(C(f), C(A))$. Excision then gives

$$\mathrm{H}_n(C(f), C(A)) \cong \mathrm{H}_n(C(f) - \{c\}, C(A) - \{c\}).$$

But we can deformation retract $C(f) - \{c\}$ onto X and similarly $C(A) - \{c\}$ onto A, so that the latter relative homology group can be identified with $\mathrm{H}_n(X, A)$. ■

In many "nice" situations, the cofiber $C(f)$ is homotopy equivalent to the quotient X/A. For example, if $A \subseteq X$ is a subcomplex of a CW complex, then this follows from [Hatcher, Prop. 0.17] applied to the pair $(C(f), C(A))$.

Hatcher introduces a weaker notion, called "good pairs". The precise definition of a good pair (X, A) is that A is closed (and nonempty) and that there is a neighborhood $A \subseteq U$ of A in X, such that U deformation retracts onto A. The point is that this is enough [Hatcher, Prop 2.22] to conclude that $\widetilde{\mathrm{H}}_n(X/A) \cong \widetilde{\mathrm{H}}_n(C(f)) \cong \mathrm{H}_n(X, A)$. In the case that $A = x_0$ is a basepoint, we say that X is "well-based".

Proposition 17.3 (Suspension isomorphism)**.** *If X is a based space, then*

$$\widetilde{\mathrm{H}}_n(X) \cong \widetilde{\mathrm{H}}_{n+1}(SX),$$

where $SX = CX \cup_X CX$ is the (unreduced) suspension and we take one of the cone points as the basepoint.

Proof. Consider the pair (CX, X). The quotient $C(X)/X$ is the (unreduced) suspension $S(X)$, and (CX, X) is a "good pair". The long exact therefore takes the form

$$\ldots \longrightarrow \mathrm{H}_{n+1}(CX) \longrightarrow \mathrm{H}_{n+1}(CX, X) \cong \widetilde{\mathrm{H}}_{n+1}(SX) \xrightarrow{\delta} \mathrm{H}_n(X) \longrightarrow \mathrm{H}_n(CX) \longrightarrow \ldots.$$

Since the outer two groups are zero for $n \geq 1$, we conclude that the connecting homomorphism is an isomorphism. This gives what we wanted if $n \geq 1$ since $\mathrm{H}_n(X) \cong \widetilde{\mathrm{H}}_n(X)$ for $n \geq 1$.

In the case $n = 0$, $\mathrm{H}_0(CX) \cong \mathbb{Z}$, and the connecting homomorphism identifies $\widetilde{\mathrm{H}}_1(SX)$ with the kernel of $\mathrm{H}_0(X) \longrightarrow \mathrm{H}_0(CX)$, which is precisely the group $\widetilde{\mathrm{H}}_0(X)$. ■

The unreduced suspension has no canonical basepoint, so the above result is usually stated instead in terms of the reduced suspension.

Proposition 17.4 (Suspension isomorphism)**.** *If X is a well-based space, then*

$$\widetilde{\mathrm{H}}_n(X) \cong \widetilde{\mathrm{H}}_{n+1}(\Sigma X),$$

where $\Sigma X = S^1 \wedge X$ is the (reduced) suspension.

The reduced suspension is $\Sigma X = SX/(I \times \{x_0\})$. If X is well-based, then $(SX, I \times \{x_0\})$ is a good pair, so that the reduced homology of the two versions of suspension are the same.

Proposition 18.1 (Wedge isomorphism). *If $\{X_\alpha\}_{\alpha\in A}$ are based spaces, with "good" basepoints, then the inclusions $X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ induce an isomorphism*

$$\bigoplus_\alpha \widetilde{\mathrm{H}}_n(X_\alpha) \cong \widetilde{\mathrm{H}}_n(\bigvee_\alpha X_\alpha).$$

Proof. We apply proposition 17.2 with $X = \coprod_\alpha X_\alpha$ and $A = \coprod_\alpha *$. We have a long exact sequence

$$\longrightarrow \mathrm{H}_n(A) \longrightarrow \mathrm{H}_n(X) \cong \bigoplus_\alpha \mathrm{H}_n(X_\alpha) \longrightarrow \widetilde{\mathrm{H}}_n(\bigvee_\alpha X_\alpha) \xrightarrow{\delta} \mathrm{H}_{n-1}(A) \longrightarrow .$$

The outer two groups are zero if $n \geq 2$, so that the middle map becomes an isomorphism. The same conclusion holds when $n = 1$ since $\mathrm{H}_0(A) \longrightarrow \mathrm{H}_0(X)$ is injective, so that the connecting homomorphism must be zero. For $n = 0$, we get a short exact sequence

$$0 \longrightarrow \mathrm{H}_0(A) \cong \bigoplus_\alpha \mathbb{Z} \longrightarrow \mathrm{H}_0(X) \cong \bigoplus_\alpha \mathrm{H}_0(X_\alpha) \longrightarrow \widetilde{\mathrm{H}}_0(\bigvee_\alpha X_\alpha) \longrightarrow 0,$$

which gives the desired conclusion. ∎

The Eilenberg-Steenrod Axioms

By the category of pairs of CW complexes, we mean the category in which the objects are a pair (X, A), where X is CW and A is a subcomplex, and a morphism $f : (X, A) \longrightarrow (Y, B)$ is a map $f : X \longrightarrow Y$ such that $f(A) \subseteq B$.

Definition 18.2. A **homology theory** on CW complexes is a sequence of functors $h_n(X, A)$ on pairs of CW complexes and natural transformations $\delta : h_n(X, A) \longrightarrow h_{n-1}(A, \varnothing)$ satisfying the following axioms:

(1) (Homotopy) If $f \simeq g$, then $f_* = g_*$
(2) (Long exact sequence) Writing $h_n(X) := h_n(X, \varnothing)$, the sequence

$$\dots h_n(A) \longrightarrow h_n(X) \longrightarrow h_n(X, A) \xrightarrow{\delta} h_{n-1}(A) \longrightarrow \dots$$

is exact
(3) (Excision) If X is the union of subcomplexes A and B, then the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism

$$h_n(A, A \cap B) \cong h_n(X, B)$$

(4) (Additivity) If (X, A) is the disjoint union of pairs (X_i, A_i), then the inclusions $(X_i, A_i) \longrightarrow (X, A)$ induce an isomorphism

$$\bigoplus_i h_n(X_i, A_i) \cong h_n(X, A).$$

An **ordinary homology theory** is one that also satisfies the additional axiom

(5) (Dimension) $h_n(pt) = 0$ if $n \neq 0$.

It turns out that if h is an ordinary homology theory and $G := h_0(pt, \varnothing)$, then $h_n(X, A) \cong \mathrm{H}_n(X, A; G)$. In other words, singular homology is essentially the only ordinary homology theory. There are many "extraordinary" homology theores (**K**-theory, bordism, stable homotopy ...) but we will not study these in this course.

The Mayer-Vietoris sequence

It is sometimes convenient to combine the long exact sequence and excision into a different form. On your homework, you are asked to deduce the following Mayer-Vietoris exact sequence simply from the axioms. We give a chain-level argument here.

Let $(X; A, B)$ be an excisive triad and recall that the group $C_n^{A,B}(X)$ defined in Prop. 17.1 is chain-homotopy equivalent to X.

We have a surjection $\varphi : C_n(A) \oplus C_n(B) \longrightarrow C_n^{A,B}(X)$ given by $\varphi(x,y) = x + y$. The kernel consists of pairs of the form $(x, -x)$. But then x is a chain in both A and B, so it is a chain in $A \cap B$. We conclude that we have a short exact sequence of chain complexes

$$0 \longrightarrow C_*(A \cap B) \xrightarrow{\kappa} C_*(A) \oplus C_*(B) \xrightarrow{\varphi} C_*^{A,B}(X) \longrightarrow 0,$$

where $\kappa(x) = (x, -x)$. Again, use of Prop 13.3 gives rise to the **Mayer-Vietoris** long exact sequence

$$\dots \xrightarrow{\delta} \mathrm{H}_n(A \cap B) \xrightarrow{(j_A, -j_B)} \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) \xrightarrow{i_A + i_B} \mathrm{H}_n(X) \xrightarrow{\delta} \mathrm{H}_{n-1}(A \cap B) \longrightarrow \dots,$$

where $j_A : A \cap B \longrightarrow A$, $j_B : A \cap B \longrightarrow B$, $i_A : A \longrightarrow X$, and $i_B : B \longrightarrow X$ are the various inclusions.

The identification of Simplicial and Singular homology

If X is a Δ-complex, we can conisder the chain complexes $C_*^\Delta(X)$ and $C_*(X)$. In fact, there is a natural map $\eta : C_*^\Delta(X) \hookrightarrow C_*(X)$, which considers a simplex of X as a singular simplex. This works just as well in the relative case, and we will prove

Theorem 18.3. *Let X be a Δ-complex and $A \subseteq X$ a sub-Δ-complex. Then the chain map η induces an isomorphism*

$$\mathrm{H}_n^\Delta(X, A) \cong \mathrm{H}_n(X, A).$$

Proof. We only give the proof in the case that X is finite-dimensional and $A = \emptyset$. See [Hatcher, Theorem 2.27] for the general case.

For each $k \geq 0$, denote by X^k the k-skeleton of X, which is the union of all simplices of dimension k or less. We will argue by induction on k that $\eta : \mathrm{H}_*^\Delta(X^k) \longrightarrow \mathrm{H}_*(X^k)$ is an isomorphism. In the base case $k = 0$, this is clear since X^0 is discrete and we know that both versions of homology agree on discrete spaces.

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For the induction step, the inclusion $X^{k-1} \hookrightarrow X^k$ is a Δ-map, and we have a map of long exact sequences

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{H}_{n+1}^\Delta(X^k, X^{k-1}) \arrow[r, "\delta"] \arrow[d] & \mathrm{H}_n^\Delta(X^{k-1}) \arrow[r] \arrow[d] & \mathrm{H}_n^\Delta(X^k) \arrow[r] \arrow[d] & \mathrm{H}_n^\Delta(X^k, X^{k-1}) \arrow[r] \arrow[d] & \mathrm{H}_{n-1}^\Delta(X^{k-1}) \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & \mathrm{H}_{n+1}(X^k, X^{k-1}) \arrow[r, "\delta"] & \mathrm{H}_n(X^{k-1}) \arrow[r] & \mathrm{H}_n(X^k) \arrow[r] & \mathrm{H}_n(X^k, X^{k-1}) \arrow[r] & \mathrm{H}_{n-1}(X^{k-1}) \arrow[r] & \dots \end{tikzcd}$$

We first argue that the vertical maps at the relative groups are isomorphisms. By definition, the simplicial relative homology groups are the homology groups of the chain complex $C_*^\Delta(X^k)/C_*^\Delta(X^{k-1})$. But this quotient group is trivial in every degree except for k, in which case we have a free abelian group on the set of k-simplices of X^k. So this chain complex has zero differential, and the relative homology groups are again just $\mathbb{Z}(\Delta^k(X^k))$ concentrated in degree k.

For the relative singular groups, we have

$$\mathrm{H}_n(X^k, X^{k-1}) \cong \widetilde{\mathrm{H}}_n(X^k/X^{k-1}) \cong \widetilde{\mathrm{H}}_n(\bigvee_{\Delta_k(X)} S^k) \cong \bigoplus_{\Delta_k(X)} \widetilde{\mathrm{H}}_n(S^k) \cong \begin{cases} \mathbb{Z}\{\Delta_k(X)\} & k = n \\ 0 & k \neq n. \end{cases}$$

So the relative groups agree, and the map η sends generators to generators, so the vertical maps at the relative groups are isomorphisms.

Now for the induction step assume the vertical maps at X^{k-1} are isomorphisms. The theorem follows from the following important result from homological algebra:

Lemma 19.1 (5-lemma)**.** *If both rows in*

$$\begin{tikzcd} A_1 \arrow[r, "g_1"] \arrow[d, "f_1"', "\cong"] & A_2 \arrow[r, "g_2"] \arrow[d, "f_2"', "\cong"] & A_3 \arrow[r, "g_3"] \arrow[d, "f_3"'] & A_4 \arrow[r, "g_4"] \arrow[d, "f_4"', "\cong"] & A_5 \arrow[d, "f_5"', "\cong"] \\ B_1 \arrow[r, "h_1"'] & B_2 \arrow[r, "h_2"'] & B_3 \arrow[r, "h_3"'] & B_4 \arrow[r, "h_4"'] & B_5 \end{tikzcd}$$

are exact and all f_i except f_3 are isomorphisms, then f_3 is also an isomorphism.

Proof. We give the proof of injectivity. The proof of surjectivity is left as an exercise.

Suppose $x \in A_3$ and $f_3(x) = 0$. We wish to show that $x = 0$. Now $f_4(g_3(x)) = h_3(f_3(x)) = 0$. Since f_4 is injective, we know that $g_3(x) = 0$. Thus $x = g_2(w)$, some $w \in A_2$. Now $h_2(f_2(w)) = f_3(g_2(w)) = f_3(x) = 0$. It follows that $f_2(w) = h_1(y)$, some $y \in B_1$. Since f_1 is surjective, there is some $z \in A_1$ with $f_1(z) = y$.

$$\begin{tikzcd} z \arrow[r, "g_1"] \arrow[d, mapsto, "f_1"', "\cong"] & w \arrow[r, mapsto, "g_2"] \arrow[d, mapsto, "f_2"', "\cong"] & x \arrow[r, mapsto, "g_3"] \arrow[d, mapsto, "f_3"'] & g_3(x) = 0 \arrow[r, "g_4"] \arrow[d, mapsto, "f_4"', "\cong"] & A_5 \arrow[d, "f_5"', "\cong"] \\ y \arrow[r, mapsto, "h_1"'] & f_2(w) \arrow[r, mapsto, "h_2"'] & 0 \arrow[r, mapsto, "h_3"'] & 0 \arrow[r, "h_4"'] & B_5 \end{tikzcd}$$

Now $f_2(g_1(z)) = h_1(f_1(z)) = h_1(y) = f_2(w)$. Since f_2 is injective, it follows that $g_1(z) = w$. But then $x = g_2(w) = g_2(g_1(z)) = 0$. ∎

∎

20. MON, OCT. 12

Euler characteristic

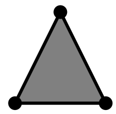
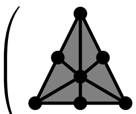
The Euler characteristic χ started from the simple formula

$$\chi(X) = V - E + F,$$

in the case of a 2-dimensional simplicial complex, where V, E, and F stand for the number of vertices, edges, and faces, respectively. An arbitrary simplicial (or Δ-) complex can have simplices of arbitrary dimension, and we can more generally define

$$\chi(X) := \sum_{i=0}^{\infty} (-1)^i (\text{number of } i\text{-simplices}).$$

If we want to define the Euler characteristic to be a **topological invariant**, meaning that any two homeomorphic simplicial complexes should have the same Euler characteristic, then you can already see why the alternating sum is a good idea: subdividing a simplex does not change the above formula.

χ () $= \chi$ () $= 1$ and χ () $= \chi$ () $= 1$

We can also define an algebraic version. Recall that the **rank** of a finitely generated abelian group is the rank of the free part. In other words, if $A \cong \mathbb{Z}^r \oplus$ torsion, then $\operatorname{rank}(A) := r$. This is also the same as the dimension of the $\mathbb{Q}$-vector space $A \otimes_{\mathbb{Z}} \mathbb{Q}$.

We also say that a chain complex C_* of abelian groups is **finite** if each group C_n is finitely generated and furthermore if only finitely many groups C_n are nonzero.

Definition 20.1. If C_* is a finite chain complex, we define

$$\chi(C_*) := \sum_{i \geq 0} (-1)^i \operatorname{rank}(C_i).$$

Proposition 20.2. *Let C_* be a finite chain complex. Then*

$$\chi(C_*) = \chi(\mathrm{H}_*(C_*)).$$

This will follow from

Lemma 20.3. *Tensoring with $\mathbb{Q}$ preserves (short) exact sequences. In other words, if*

$$0 \longrightarrow A \xrightarrow{i} B \longrightarrow C \xrightarrow{q} 0$$

is exact, then so is

$$0 \longrightarrow \mathbb{Q} \otimes A \longrightarrow \mathbb{Q} \otimes B \longrightarrow \mathbb{Q} \otimes C \longrightarrow 0.$$

Abelian groups with this property are called **flat**.

Proof. You are asked to show on your homework that for any abelian D, the sequence

$$D \otimes A \longrightarrow D \otimes B \longrightarrow D \otimes C \longrightarrow 0$$

is always exact. So it suffices to show that $\mathbb{Q} \otimes A \longrightarrow \mathbb{Q} \otimes B$ is injective. We will write φ for this map of $\mathbb{Q}$-vector spaces.

Let $x = \sum_i r_i \otimes a_i \in \mathbb{Q} \otimes A$ such that $\varphi(x) = 0$ in $\mathbb{Q} \otimes B$. We can clear denominators of the r_i by multiplying by some sufficiently large integer n. Thus nx is in the image of $A \longrightarrow \mathbb{Q} \otimes A$, $a \mapsto 1 \otimes a$. So we can write $nx = 1 \otimes a$ for some $a \in A$. Now

$$1 \otimes i(a) = \varphi(nx) = n\varphi(x) = 0$$

in $\mathbb{Q} \otimes B$, so $i(a)$ must be a torsion class in B. Since $i : A \hookrightarrow B$ was injective, it follows that a was torsion in A. But then $nx = 1 \otimes a = 0$ in $\mathbb{Q} \otimes A$. It follows that $x = \frac{1}{n} \cdot nx = 0$ as well. ∎

Corollary 20.4. *If*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is short exact, then $\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(C)$.

Proof. This follows from the lemma, given that $\operatorname{rank}(A) = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes A)$. ∎

Proof of Proposition 20.2. Let $Z_i := \ker(\partial_i) \subseteq C_i$ be the subgroup of cycles and $B_i = \operatorname{im}(\partial_{i+1}) \subseteq Z_i \subseteq C_i$ be the boundaries. The key is to note that we have short exact sequences

$$0 \longrightarrow Z_i \longrightarrow C_i \longrightarrow B_{i-1} \longrightarrow 0.$$

and

$$0 \longrightarrow B_i \longrightarrow Z_i \longrightarrow H_i \longrightarrow 0.$$

By the corollary, these tell us that

$$\operatorname{rank}(C_i) = \operatorname{rank}(Z_i) + \operatorname{rank}(B_{i-1})$$

and

$$\operatorname{rank}(Z_i) = \operatorname{rank}(B_i) + \operatorname{rank}(H_i).$$

So

$$\sum_i (-1)^i \operatorname{rank}(C_i) = \sum_i (-1)^i (\operatorname{rank}(B_i) + \operatorname{rank}(H_i) + \operatorname{rank}(B_{i-1})).$$

This is a telescoping sum, and we end up with $\chi(H_*)$. ■

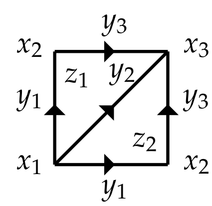
So this tells us that the Euler characteristic only depends on the homology of the space, not on the particular cellular model. This also allows us to define the Euler characteristic for any space (with "finite" homology), not only for simplicial complexes.

Definition 20.5. Let X be a space such that $\mathrm{H}_*(X)$ is a finite chain complex. We then define

$$\chi(X) := \chi(\mathrm{H}_*(X)).$$

By Proposition 20.2, this agrees with the previous notion for simplicial complexes.

Example 20.6.

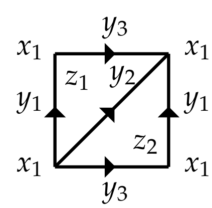


(1) $X = S^2$. We built the sphere as a Δ-complex by gluing together two 2-simplices. The leads to the Euler characteristic computation

$$\chi(S^2) = 3 - 3 + 2 = 2.$$

On the other hand, the computation via homology is

$$\chi(S^2) = \chi(\mathrm{H}_*(S^2)) = 1 - 0 + 1 = 2.$$

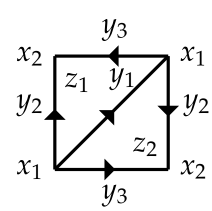


(2) $X = T^2$. The torus was similarly built by gluing two 2-simplices. We have, on the one hand

$$\chi(T^2) = 1 - 3 + 2 = 0$$

and on the other

$$\chi(\mathrm{H}_*(T^2)) = 1 - 2 + 1 = 0.$$



(3) $X = \mathbb{RP}^2$. The projective plane was built from two simplices as in the picture to the right. So

$$\chi(\mathbb{RP}^2) = 2 - 3 + 2 = 1$$

and

$$\chi(\mathbb{RP}^2) = \operatorname{rank}(\mathbb{Z}) - \operatorname{rank}(\mathbb{Z}/2\mathbb{Z}) = 1$$

Degree

The next topic is yet another variant of homology, this one defined for CW complexes. It will be convenient to first discuss the notion of "degree" of a map of spheres.

Definition 20.7. For $n > 0$, let $f : S^n \longrightarrow S^n$ be any map. This induces a map

$$\mathbb{Z} \cong \widetilde{\mathrm{H}}_n(S^n) \xrightarrow{f_*} \widetilde{\mathrm{H}}_n(S^n) \cong \mathbb{Z}$$

which is necessarily of the form $i \mapsto k \cdot i$ for some $k \in Z$. This integer k is called the **degree** of the map f.

Note that there are two possible choices of isomorphism $\widetilde{\mathrm{H}}_n(S^n) \cong \mathbb{Z}$, corresponding to the two generators for the infinite cyclic group. But as long as we use the same choice in both the domain and codomain of f_*, this makes the notion of degree well-defined. Here are some properties of the degree of a map of spheres.

Proposition 20.8.
(1) $\deg(f)$ *only depends on the homotopy class of* f
(2) *The degree defines a homomorphism* $\deg : \pi_n(S^n) \longrightarrow \widetilde{\mathrm{H}}_n(S^n) \cong \mathbb{Z}$.
(3) $\deg(\mathrm{id}) = 1$.
(4) $\deg(g \circ f) = \deg(g) \cdot \deg(f)$
(5) *Any reflection has degree* -1.
(6) *The antipodal map has degree* $(-1)^{n+1}$.
(7) *If* f *is fixed-point free, then* $\deg(f) = (-1)^{n+1}$.

Proof.
(1) This follows from homotopy-invariance of homology
(2) Recall that the sum $f+g$ of two elements of the homotopy group is defined to be the composite

$$S^n \xrightarrow{p} S^n \vee S^n \xrightarrow{f \vee g} S^n,$$

where p is a pinch map. Applying homology gives

$$\widetilde{\mathrm{H}}_n(S^n) \xrightarrow{p_*} \widetilde{\mathrm{H}}_n(S^n \vee S^n) \cong \widetilde{\mathrm{H}}_n(S^n) \oplus \widetilde{\mathrm{H}}_n(S^n) \xrightarrow{f_* \oplus g_*} \widetilde{\mathrm{H}}_n(S^n).$$

The isomorphism $\widetilde{\mathrm{H}}_n(S^n \vee S^n) \cong \widetilde{\mathrm{H}}_n(S^n) \oplus \widetilde{\mathrm{H}}_n(S^n)$ is induced by the two collapse maps $c_i : S^n \vee S^n \longrightarrow S^n$. These compose with the pinch map p to give maps (based-)homotopic to the identity, so that the above sequence is isomorphic to

$$\widetilde{\mathrm{H}}_n(S^n) \xrightarrow{\Delta} \widetilde{\mathrm{H}}_n(S^n) \oplus \widetilde{\mathrm{H}}_n(S^n) \xrightarrow{f_* \oplus g_*} \widetilde{\mathrm{H}}_n(S^n),$$

which simplifies to the sum $f_* + g_*$.
(3) Since $\widetilde{\mathrm{H}}_n$ is a functor, we know that $\widetilde{\mathrm{H}}_n(\mathrm{id}_{S^n}) = \mathrm{id}_{\widetilde{\mathrm{H}}_n(S^n)}$, so that the multiplier is just 1.
(4) This again comes from the fact that $\widetilde{\mathrm{H}}_n$ is a functor! We know that $(g \circ f)_* = g_* \circ f_*$, so that

$$\deg(g \circ f) \cdot 1 = (g \circ f)_*(1) = g_*(f_*(1)) = g_*(\deg(f) \cdot 1) = \deg(f) \cdot g_*(1) = \deg(f) \cdot \deg(g) \cdot 1.$$

(5) If r is a reflection in a hyperplane H, then we can think of $H \cap S^n$ as an equator and describe S^n as a Δ-complex obtained by gluing together two Δ^n's s_1 and s_2 along this equator. The difference of the chains $s_1 - s_2$ is a cycle that represents the generator of $\widetilde{\mathrm{H}}_n(S^n)$. But $r_*(s_1 - s_2) = s_2 - s_1 = -(s_1 - s_2)$, so $\deg(r) = -1$.
(6) The antipodal map is the composite of reflection in each of the $n+1$ coordinates.
(7) If f is fixed-point free, then the line from $f(x)$ to $-x$ never passes through $\mathbf{0}$. It follows that, by suitably normalizing the (nonzero) vectors on this line, that we can define a straight-line homotopy from f to the antipodal map.

■

Proposition 20.9. $\pi_n(S^n) \cong \mathbb{Z} \oplus ?$.

Proof. We have a homomorphism $\deg : \pi_n(S^n) \longrightarrow \mathbb{Z}$. There are two possibilities: either it is the zero homomorphism, or it is surjective. Since $\deg(\mathrm{id}) = 1$, it must be surjective. But then we have a splitting $s : \mathbb{Z} \longrightarrow \pi_n(S^n)$ defined by $s(n) = n \cdot \mathrm{id}_{S^n}$. As we have discussed, the splitting induces a direct sum decomposition. ■

In fact, the ? is trivial, so that $\pi_n(S^n) \cong \mathbb{Z}$ for all $n \geq 1$.

Proposition 21.1. *$\mathbb{Z}/2\mathbb{Z}$ is the only (nontrivial) group that can act freely on S^n if n is even.*

Recall that an action of G on X is **free** if the mapping $x \mapsto g \cdot x$ has no fixed points when $g \neq e$.

Proof. Let n be even and suppose that G acts freely on S^n. For each $g \in G$, denote by $m_g : S^n \longrightarrow S^n$ the mapping $x \mapsto g \cdot x$. Then m_g is a homeomorphism with inverse $m_{g^{-1}}$. By proposition 20.8, we have

$$\deg(m_g) \cdot \deg(m_{g^{-1}}) = \deg(m_g \circ m_{g^{-1}}) = \deg(\mathrm{id}) = 1,$$

so it follows that $\deg(m_g) = \pm 1$. Moreover, we have $m_{gh} = m_g \circ m_h$, so that

$$\deg(m_{gh}) = \deg(m_g \circ m_h) = \deg(m_g) \cdot \deg(m_h).$$

Summing up, we see that deg defines a homomorphism $\deg : G \longrightarrow \mathbb{Z}^\times \cong \mathbb{Z}/2\mathbb{Z}$.

But now if the action is free, again by the proposition we have that for any $g \neq e$, then $\deg(g) = (-1)^{n+1}$. If n is even, this means that $\deg(g) = -1$. In other words, the kernel of $\deg : G \longrightarrow \mathbb{Z}/2\mathbb{Z}$ is trivial, so $G = \mathbb{Z}/2\mathbb{Z}$ or $G = 0$. ∎

Of course, we do know of a free action of $\mathbb{Z}/2\mathbb{Z}$ on a sphere of arbitrary dimension: the antipodal action in which $-1 \in \mathbb{Z}^\times$ acts on S^n as the antipodal map. The antipodal map has no fixed points, so the action is free.

In the case of a sphere of odd dimension S^{2n-1}, we can think of it as the unit sphere inside $\mathbb{R}^{2n} \cong \mathbb{C}^n$. If ζ is a primitive k-th root of unity, then we can define a free action of $\mathbb{Z}/k\mathbb{Z}$ on S^{2n-1} by letting the generator act as multiplication by ζ in each complex coordinate. This is a free action, since if $(z_1, \dots, z_n)$ is a point on the sphere written in terms of complex coordinates, then some z_i must be nonzero. Then $\zeta \cdot z_i \neq z_i$, so this point is not fixed by the action.

When $k = 2$, this is again the antipodal action, and the quotient is $\mathbb{RP}^n$, by definition. When k is odd, the resulting quotient is known as a "lens" space. You computed the homology of this space when $n = 2$ and $k = 3$ on homework 2.

Cellular homology

We now introduce our third version of homology, this one defined for CW complexes. The idea is to define the cellular chain complex by

$$C_n^{\mathrm{cell}}(X) := \mathbb{Z}\{n\text{-cells of } X\}.$$

For the differential $\partial_n^{\mathrm{cell}} : C_n^{\mathrm{cell}}(X) \longrightarrow C_{n=1}^{\mathrm{cell}}(X)$, let e_α^n be an n-cell of X. Then e_α^n is determined by its attaching map $\varphi_\alpha : S^{n-1} \longrightarrow X^{n-1}$. The idea is that $\partial_n^{\mathrm{cel}}(e_\alpha^n)$ should capture how the attaching map interacts with the various $(n-1)$-cells. If we write

$$\partial_n^{\mathrm{cell}}(e_\alpha^n) = \sum_\beta d_{\alpha\beta}[\beta],$$

where β are the $(n-1)$-cells of X, then we take $d_{\alpha\beta}$ to be the degree of the map

$$S^{n-1} \xrightarrow{\varphi_\alpha} X^{n-1} \longrightarrow X^{n-1}/X^{n-2} \cong \bigvee_\beta S^{n-1} \xrightarrow{p_\beta} S^{n-1}.$$

It remains to show that $\partial_{n-1}^{\mathrm{cell}} \circ \partial_n^{\mathrm{cell}} = 0$ and to then define cellular homology as the homology of this cellular chain complex. This can be done, but there is another, slick, approach, using the machinery we have already built up.

We wanted to define

$$C_n^{\mathrm{cell}}(X) := \mathbb{Z}\{n\text{-cells of } X\}.$$

Now in a truly perverse act, we can rewrite this as

$$\mathbb{Z}\{n\text{-cells of } X\} \cong \widetilde{\mathrm{H}}_n(\bigvee_\beta S^n) \cong \widetilde{\mathrm{H}}_n(X^n/X^{n-1}) \cong \mathrm{H}_n(X^n, X^{n-1}),$$

and we now instead choose to define

$$C_n^{\mathrm{cell}}(X) := \mathrm{H}_n(X^n, X^{n-1}).$$

The differential is defined as the composite

$$C_n^{\mathrm{cell}}(X) = \mathrm{H}_n(X^n, X^{n-1}) \xrightarrow{\delta} \mathrm{H}_{n-1}(X^{n-1}) \longrightarrow \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}) = C_{n-1}^{\mathrm{cell}}(X).$$

But now with this definition, it is simple to check that $\partial_n^{\mathrm{cell}} \circ \partial_{n+1}^{\mathrm{cell}} = 0$: this composition is displayed in the diagram

$$\begin{tikzcd}
C_{n+1}^{\mathrm{cell}}(X) \arrow[r, equal] & \mathrm{H}_{n+1}(X^{n+1}, X^n) \arrow[r, "\delta"] & \mathrm{H}_n(X^n) \arrow[dr] & & \\
 & & & \mathrm{H}_n(X^n, X^{n-1}) \arrow[r, equal] \arrow[dl, "\delta"] & C_n^{\mathrm{cell}}(X). \\
C_{n-1}^{\mathrm{cell}}(X) \arrow[r, equal] & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "\delta"] & \mathrm{H}_{n-1}(X^{n-1}) & &
\end{tikzcd}$$

But the two arrows surrounding $C_n^{\mathrm{cell}}(X)$ are part of the long exact sequence in homology for the pair (X^n, X^{n-1}) and therefore compose to zero. It follows that we have a chain complex, so that the following definition makes sense.

Definition 21.2. Given a CW structure on a space X, we define

$$\mathrm{H}_n^{\mathrm{cell}}(X) := \mathrm{H}_n(C_*^{\mathrm{cell}}(X)).$$

We can also introduce coefficients or consider a reduced theory, just as in the other versions of homology.

22. Mon, Oct. 19

Theorem 22.1. *For any CW complex X, we have*

$$\mathrm{H}_n^{\mathrm{cell}}(X) \cong \mathrm{H}_n(X).$$

Before we prove the theorem, it will be convenient to establish the following.

Lemma 22.2. (1) *For any $k < n$, the inclusion $X^n \hookrightarrow X$ induces an isomorphism* $\mathrm{H}_k(X^n) \cong \mathrm{H}_k(X)$.
(2) *For any $k > n$, we have* $\mathrm{H}_k(X^n) = 0$.

Proof. We only prove (i) in the case that X is finite-dimensional. See p. 138 of Hatcher for the general case. We have an exact sequence

$$\mathrm{H}_{k+1}(X^n, X^{n-1}) \xrightarrow{\delta} \mathrm{H}_k(X^n) \longrightarrow \mathrm{H}_k(X^{n-1}) \longrightarrow \mathrm{H}_k(X^n, X^{n-1}).$$

These outer two groups are zero if $k \notin \{n, n-1\}$. So if $k < n$, we conclude that $\mathrm{H}_k(X^n) \cong \mathrm{H}_k(X^{n+1}) \cong \dots \mathrm{H}_k(X)$. Similarly, if $k > n$, we have $\mathrm{H}_k(X^n) \cong \mathrm{H}_k(X^{n-1}) \cong \dots \mathrm{H}_k(X^0) = 0$. ■

Proof of Theorem 22.1. Consider the following diagram.

$$\begin{tikzcd}
 & & & 0 & \\
0 = \mathrm{H}_n(X^{n-1}) \arrow[dr] & & \mathrm{H}_n(X^{n+1}) \cong \mathrm{H}_n(X) \arrow[ur] & & \\
 & \mathrm{H}_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & & \\
\mathrm{H}_{n+1}(X^{n+1}, X^n) \arrow[ur, "\delta"] \arrow[rr, "\partial_{n+1}^{\mathrm{cell}}"] & & \mathrm{H}_n(X^n, X^{n-1}) \arrow[rr, "\partial_n^{\mathrm{cell}}"] \arrow[dr, "\delta"'] & & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}) \\
 & & & \mathrm{H}_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] & \\
 & & 0 = \mathrm{H}_{n-1}(X^{n-2}) \arrow[ur] & &
\end{tikzcd}$$

First, we have

$$\mathrm{H}_n(X) \cong \mathrm{H}_n(X^n) / \operatorname{im}(\delta).$$

Since j_n is injective, the latter quotient is identified with $\operatorname{im}(j_n) / \operatorname{im}(\partial_{n+1}^{\mathrm{cell}})$. But since the down-right sequence is exact, we can replace this with $\ker(\delta) / \operatorname{im}(\partial_{n+1}^{\mathrm{cell}})$. Finally, since j_{n-1} is injective, the latter is the same as the quotient

$$\ker(\partial_n^{\mathrm{cell}}) / \operatorname{im}(\partial_{n+1}^{\mathrm{cell}}) = \mathrm{H}_n^{\mathrm{cell}}(X).$$

■

Having established this theorem, we will now drop the decoration "cell" on cellular homology.

Example 22.3.

(1) T^2 has a CW structure with a single 0 cell, two 1-cells a and b, and a single 2-cell attached by the map $S^1 \longrightarrow S^1 \vee S^1$ represented by $aba^{-1}b^{-1}$. It follows that the coefficients in the differential $\partial_2 : C_2 = \mathbb{Z}\{e\} \longrightarrow C_1 = \mathbb{Z}\{a, b\}$ are both $1 + (-1) = 0$. So the cellular chain complex has no differentials!

23. Wed, Oct. 21

(2) The Klein bottle K has a CW structure with a single 0 cell, two 1-cells a and b, and a single 2-cell attached by the map $S^1 \longrightarrow S^1 \vee S^1$ represented by $abab^{-1}$. It follows that the differential $\partial_2 : C_2 = \mathbb{Z}\{e\} \longrightarrow C_1 = \mathbb{Z}\{a, b\}$ is $\partial_2(e) = (2a, 0)$.

(3) $\mathbb{RP}^n$ has a CW structure with a single cell in each dimension. The k-skeleton is $\mathbb{RP}^k$, and the attaching map $q : S^k \longrightarrow \mathbb{RP}^k$ for the $(k+1)$-cell is the defining double cover of $\mathbb{RP}^k$. To determine the degree of the composition

$$S^k \xrightarrow{q} \mathbb{RP}^k \longrightarrow \mathbb{RP}^k / \mathbb{RP}^{k-1} \cong S^k,$$

note that the cover q sends the equator S^{k-1} to $\mathbb{RP}^{k-1}$ and therefore gets collapsed in the next map. It follows that our map factors as

$$S^k \longrightarrow S^k / (S^{k-1}) \cong S^k \vee S^k \longrightarrow S^k.$$

Thinking now of $\mathbb{RP}^k$ as the quotient of the northern hemisphere of S^k, modulo a relation on the equator, we see that the degree of our map on the northern S^k is 1, whereas the degree on the southern S^k is the degree of the antipodal map, which is $(-1)^{k+1}$. If follows that the differential

$$\partial_{k+1} : C_{k+1}(\mathbb{RP}^n) \cong \mathbb{Z} \longrightarrow C_k(\mathbb{RP}^n) \cong \mathbb{Z}$$

is

$$\partial_{k+1}(e^{k+1}) = (1 + (-1)^{k+1})e^k = \begin{cases} 0 & k \text{ even} \\ 2 & k \text{ odd.} \end{cases}$$

So our cellular chain complex is

$$\ldots \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

If n is even, then the first differential is $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$, whereas if n is odd, then the first differential is $\mathbb{Z} \xrightarrow{0} \mathbb{Z}$. We read off

$$\mathrm{H}_k(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2\mathbb{Z} & k \text{ odd}, k < n \\ 0 & k \text{ even}, k \leq n \\ \mathbb{Z} & k = n \text{ odd} \\ 0 & k > n. \end{cases}$$

(4) We can build an infinite-dimensional CW complex $\mathbb{RP}^\infty$ as the union of the $\mathbb{RP}^n$'s. The homology of this space is then

$$\mathrm{H}_k(\mathbb{RP}^\infty) \cong \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2\mathbb{Z} & k \text{ odd} \\ 0 & \text{else.} \end{cases}$$

(5) $\mathbb{CP}^n$ has a CW structure with a single cell in every even dimension. There is no room for differentials, so we conclude that

$$\mathrm{H}_k(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & k \text{ even}, k \leq 2n \\ 0 & \text{else.} \end{cases}$$

(6) We can build an infinite-dimensional CW complex $\mathbb{CP}^\infty$ as the union of the $\mathbb{CP}^n$'s. The homology of this space is then

$$\mathrm{H}_k(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & k \text{ even} \\ 0 & \text{else.} \end{cases}$$

We long ago gave a description of $\mathrm{H}_0(X)$, but we have put off describing $\mathrm{H}_1(X)$. We do this now.

Theorem 23.1 (Hurewicz)**.** *Assume that X is a connected CW complex. Then*

$$H_1(X) \cong \pi_1(X)_{ab}.$$

Proof. First, note that cells in dimensions 3 or higher affect neither π_1 nor H_1. In other words, if X^2 is the 2-skeleton, then $\pi_1(X^2) \cong \pi_1(X)$ and $H_1(X^2) \cong H_1(X)$.

24. Fri, Oct. 23

By the van Kampen theorem, we know that $\pi_1(X^1) \twoheadrightarrow \pi_1(X^2)$ is surjective. Moreover, if we denote by $\beta_1, \dots, \beta_k$ the 2-cells of X (or really, their attaching maps, thought of as elements of $\pi_1(X^1)$), then the van Kampen theorem tells us that

$$\pi_1(X^2) \cong \pi_1(X^1)/\langle \beta_1, \dots, \beta_k \rangle.$$

Denote by $\tilde{X}^1$ the result of collapsing out a maximal tree in the graph X^1, and recall that the natural map $X^1 \longrightarrow \tilde{X}^1$ is a homotopy equivalence. The space $\tilde{X}^1$ is a wedge of circles $\tilde{X}^1 \cong \bigvee S^1$, each circle corresponding to a generator of $\pi_1(X^1)$. We now have

$$\pi_1(X^2) \cong \pi_1(\tilde{X}^1)/\langle \beta_1, \dots, \beta_k \rangle \cong F(\alpha_1, \dots, \alpha_n)/\langle \beta_1, \dots, \beta_k \rangle.$$

Let's now turn to homology. We know that $H_1(X)$ is computed as a quotient

$$C_2(X) \longrightarrow Z_1(X).$$

Lemma 24.1. *We have* $Z_1(X) = Z_1(X^1) = H_1(X^1) \cong H_1(\tilde{X}^1) = Z_1(\tilde{X}^1) = C_1(\tilde{X}_1)$.

The homology isomorphism follows from the fact that $X \longrightarrow \tilde{X}^1$ is a homotopy equivalence.

The lemma implies that $H_1(X)$ is the quotient

$$H_1(X) \cong \mathbb{Z}(\alpha_1, \dots, \alpha_n)/\langle \beta_1, \dots, \beta_k \rangle.$$

There is now an obvious surjection

$$\pi_1(X) \longrightarrow H_1(X)$$

induced by the abelianization map $F(\alpha_1, \dots, \alpha_n) \twoheadrightarrow \mathbb{Z}[\alpha_1, \dots, \alpha_n]$. The following lemma implies that the map $\pi_1(X) \longrightarrow \mathrm{H}_1(X)$ is also abelianization. ■

Lemma 24.2. *Let $\varphi : F \longrightarrow G$ be a surjection of groups with kernel N. Then the map $G = F/N \xrightarrow{\lambda} F_{ab}/N_{ab}$ induces an isomorphism $G_{ab} \cong F_{ab}/N_{ab}$.*

Proof. The map out of G_{ab} comes from the universal property of abelianization:

$$\begin{tikzcd}
G = F/N \arrow[rr, two heads, "\lambda"] \arrow[dr] & & F_{ab}/N_{ab} \\
& G_{ab} \arrow[ur, dashed, "\mu"'] &
\end{tikzcd}$$

Since λ is surjective, so is μ. To see that μ is injective, suppose that $\mu(gN) = 0$. This means that $g \in N \cdot [F, F] = [F, F] \cdot N$. But this is the commutator subgroup of F/N, so we are done. ■

There is also a statement in higher dimensions, assuming that all lower homotopy groups vanish. We state it without proof.

Theorem 24.3 (Hurewicz). *Assume that X is a CW complex satisfying $\pi_k(X) = 0$ for $k < n$ (we say that X is $(n-1)$*-**connected***), where $n \geq 2$. Define*

$$h_n : \pi_n(X) \longrightarrow H_n(X)$$

by

$$h_n(\alpha) = \alpha_*(x_n),$$

where $x_n \in H_n(S^n)$ is the class of the unique n-cell (in the minimal CW structure on S^n). Then h_n is an isomorphism of groups, known as the Hurewicz map.

Using induction and the fundamental group Hurewicz theorem, this implies the following result.

Corollary 24.4. *Suppose that X is a CW complex that is $(n-1)$-connected. Then $H_k(X) = 0$ for $0 < k < n$ as well.*

Note that the torus T^2 shows that Theorem 24.3 fails if we drop the connectivity hypothesis.

Homology of products

Our next goal will be to describe $\mathrm{H}_*(X \times Y)$ in terms of $\mathrm{H}_*(X)$ and $\mathrm{H}_*(Y)$. We will work with cellular homology and will therefore assume that X and Y are CW complexes. On your homework, you showed that $X \times Y$ has a CW structure in which the n-cells correspond to pairs of k-cells in X and j-cells in Y, where $k + j = n$.

In other words, we have a bijection

$$\{n\text{-cells in } X \times Y\} \cong \coprod_{k+j=n} \{k\text{-cells in } X\} \times \{j\text{-cells in } Y\}$$

Applying the free abelian group functor, we get that

$$C_n(X \times Y) \cong \bigoplus_{k+j=n} C_k(X) \otimes C_j(Y).$$

25. Mon, Oct. 26

Continuing our discussion from last time, recall that we had showed that

$$C_n(X \times Y) \cong \bigoplus_{k+j=n} C_k(X) \otimes C_j(Y).$$

We would like to say that we have an isomorphism of chain complexes, but we first need to discuss how to make the right side into a chain complex.

Definition 25.1. If C_* and D_* are chain complexes, define a chain complex $C_* \otimes D_*$ by

$$(C_* \otimes D_*)_n := \bigoplus_{k+j=n} C_k \otimes D_j$$

and where the differential $\partial_n^{C_* \otimes D_*}$ is defined by

$$\partial_n(x \otimes y) = \partial(x) \otimes y + (-1)^{\deg(x)} x \otimes \partial(y).$$

We need to check that this is in fact a complex, in the sense that $\partial_{n-1} \circ \partial_n = 0$. We have

$$\begin{aligned}
\partial_{n-1}(\partial_n(x \otimes y)) &= \partial_{n-1}\Big(\partial(x) \otimes y + (-1)^{\deg(x)} x \otimes \partial(y)\Big) \\
&= \partial(\partial(x)) \otimes y + (-1)^{\deg(\partial(x))}\partial(x) \otimes \partial(y) \\
&\qquad + (-1)^{\deg(x)}\partial(x) \otimes \partial(y) + (-1)^{2\deg(x)} x \otimes \partial(\partial(y)) \\
&= 0 + (-1)^{\deg(x)-1}\partial(x) \otimes \partial(y) + (-1)^{\deg(x)}\partial(x) \otimes \partial(y) + 0 = 0.
\end{aligned}$$

So $C_* \otimes D_*$ is in fact a chain complex.

Proposition 25.2. *The above isomorphism extends to an isomorphism of chain complexes $C_*(X \otimes Y) \simeq C_*(X) \otimes C_*(Y)$.*

Proof. We know that $e^n_{\alpha,\beta} \in C_n(X \times Y)$ maps to $e^k_\alpha \otimes e^j_\beta \in C_k(X) \otimes C_j(Y)$, and that the differential on the latter is

$$\partial(e^k_\alpha \otimes e^j_\beta) = \partial(e^k_\alpha) \otimes e^j_\beta + (-1)^k e^k_\alpha \otimes \partial(e_\beta)^j.$$

So it remains to describe the differential $\partial(e^n_{\alpha,\beta})$.

By naturality, it suffices to consider the universal case, in which $X = I^k$, $Y = I^j$, and $X \times Y = I^k \times I^j \cong I^n$. We give the argument for $k = j = 1$ and $k = 1, j = 2$. For the general case, see Hatcher, section 3.B.

For $k = j = 1$, we want to compute $\partial(e^2)$ in $C_*(I^2)$. If we consider this 2-cell as being oriented counterclockwise, then the formula for $\partial(e^2)$ is

$$\partial(e^2) = -e^1_{0\times e_1} + e^1_{1\times e^1} + e^1_{0\times e^1} - e^1_{1\times e^1}.$$

And this exactly maps over to $\partial(e^1) \otimes e^1 - e^1 \otimes \partial(e^1) \in C_*(I^1) \otimes C_*(I^1)$.

For $k = 1$ and $j = 2$, we want to compute $\partial(e^3)$ in $C_*(I^3)$, where we are thinking of I^3 as $I^1 \times I^2$. Again, we orient each face of $\partial(I^3)$ with a counterclockwise orientation, looking from the outside of the cube. Then the formula for $\partial(e^3)$ is

$$\partial(e^3) = -e^2_{0\times e^2} + e^2_{1\times e^2} + e^2_{e^1\times 0\times e^1} - e^2_{e^1\times 1\times e^1} - e^2_{e^1\times e^1\times 0} + e^2_{e^1\times e^1\times 1}.$$

Again, this maps over exactly to $\partial(e^1) \otimes e^2 - e^1 \otimes \partial(e^2) \in C_*(I^1) \otimes C_*(I^2)$. ■

26. Wed, Oct. 28

It follows that the homology of $X \times Y$ is the homology of the complex $C_*(X) \otimes C_*(Y)$, and it remains to compute this latter homology. The answer is much simpler if we use field coefficients.

Proposition 26.1. *Let k be a field, and let C_* and D_* be chain complexes of k-vector spaces. Then*

$$\mathrm{H}_n(C_* \otimes_k D_*) \cong \bigoplus_{k+j=n} \mathrm{H}_k(C_*) \otimes_k \mathrm{H}_j(D_*).$$

Proof. There are several advantages to working with vector spaces. For one, every short exact sequence always splits (since every vector space is a free module). This implies that tensoring with a vector space will always preserve short exact sequences as well.

More generally, if C is a vector space and D_* is a chain complex of vector spaces, we will have $\mathrm{H}_n(C \otimes D_*) \cong C \otimes \mathrm{H}_n(D_*)$. In particular, we can take C to be any of the C_i. Now if C_* is a chain complex in which all differentials are zero, we are done.

Now consider a general complex C_*, and let $B_* \subseteq Z_* \subseteq C_*$ be the subcomplexes of boundaries and cycles, respectively. Then the complexes B_* and Z_* have no differentials, and moreover we have a short exact sequence of complexes

$$0 \longrightarrow Z_* \longrightarrow C_* \xrightarrow{\partial} B_* \longrightarrow 0.$$

Again, this will still be exact after tensoring with a complex D_*, so that we have

$$0 \longrightarrow Z_* \otimes D_* \longrightarrow C_* \otimes D_* \xrightarrow{\partial\otimes\mathrm{id}} B_* \otimes D_* \longrightarrow 0.$$

This short exact sequence gives rise to a long exact sequence in homology

$$\longrightarrow \mathrm{H}_n(Z_* \otimes D_*) \longrightarrow \mathrm{H}_n(C_* \otimes D_*) \longrightarrow \mathrm{H}_n(B_* \otimes D_*) \longrightarrow \mathrm{H}_{n-1}(Z_* \otimes D_*) \longrightarrow \dots$$

Tracing through, you can show that the connecting homomorphism $\mathrm{H}_n(B_* \otimes D_*) \longrightarrow \mathrm{H}_{n-1}(Z_* \otimes D_*)$ is simply induced by the inclusing of subcomplexes $B_* \hookrightarrow Z_*$.

Since B_* and Z_* are both complexes with trivial differentials, we can rewrite the sequence as

$$\longrightarrow (Z_* \otimes \mathrm{H}_*(D_*))_n \longrightarrow \mathrm{H}_n(C_* \otimes D_*) \longrightarrow (B_* \otimes \mathrm{H}_*(D_*))_n \longrightarrow (Z_* \otimes \mathrm{H}_*(D_*))_{n-1} \longrightarrow \dots$$

This now splits as a bunch of short exact sequences

$$0 \longrightarrow B_* \otimes \mathrm{H}_*(D_*) \longrightarrow Z_* \otimes \mathrm{H}_*(D_*) \longrightarrow \mathrm{H}_*(C_* \otimes D_*) \longrightarrow 0.$$

Again, since tensoring with $\mathrm{H}_*(D_*)$ preserves exact sequences, we conclude that $\mathrm{H}_*(C_* \otimes D_*) \cong \mathrm{H}_*(C_*) \otimes \mathrm{H}_*(D_*)$. ■

Corollary 26.2. *Let k be a field and X and Y CW complexes. Then*

$$\mathrm{H}_n(X \times Y; k) \cong \bigoplus_{k+j=n} \mathrm{H}_k(X;k) \otimes_k \mathrm{H}_j(Y;k).$$

Proof. This will follow from Proposition 26.1. We have

$$C_*(X) \otimes_{\mathbb{Z}} C_*(Y) \otimes_{\mathbb{Z}} k \cong C_*(X) \otimes_{\mathbb{Z}} C_*(Y) \otimes_{\mathbb{Z}} k \otimes_k k \cong (C_*(X) \otimes_{\mathbb{Z}} k) \otimes_k (C_*(Y) \otimes_{\mathbb{Z}} k).$$

Now just apply Proposition 26.1. ∎

27. Fri, Oct. 30

Example 27.1. Consider $X = Y = \mathbb{RP}^2$. We know that $\mathrm{H}_k(\mathbb{RP}^2; \mathbb{F}_2)$ is $\mathbb{F}_2$ when $k = 0, 1, 2$ and is zero in other degrees. The corollary gives us that

$$\dim_{\mathbb{F}_2} \mathrm{H}_k(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{F}_2) \cong \begin{cases} 1 & k = 0, 4 \\ 2 & k = 1, 3 \\ 3 & k = 2 \\ 0 & \text{else} \end{cases}.$$

If we try to compute this directly, we use the cellular chain complex for $\mathbb{RP}^2 \times \mathbb{RP}^2$, which takes the form

$$\begin{tikzcd}
C_4(\mathbb{RP}^2 \times \mathbb{RP}^2) \arrow[r, "\partial_4"] \arrow[d, equal] & C_3(\mathbb{RP}^2 \times \mathbb{RP}^2) \arrow[r, "\partial_3"] \arrow[d, equal] & C_2(\mathbb{RP}^2 \times \mathbb{RP}^2) \arrow[r, "\partial_2"] \arrow[d, equal] & C_1(\mathbb{RP}^2 \times \mathbb{RP}^2) \arrow[r, "\partial_1"] \arrow[d, equal] & C_0(\mathbb{RP}^2 \times \mathbb{RP}^2) \arrow[d, equal] \\
\mathbb{Z}\{e^4_{2,2}\} \arrow[r, swap, "\begin{pmatrix} 2 \\ 2 \end{pmatrix}"] & \mathbb{Z}\{e^3_{1,2}, e^3_{2,1}\} \arrow[r, swap, "\begin{pmatrix} 0 & 0 \\ 2 & -2 \\ 0 & 0 \end{pmatrix}"] & \mathbb{Z}\{e^2_{0,2}, e^2_{1,1}, e^2_{2,0}\} \arrow[r, swap, "\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}"] & \mathbb{Z}\{e^1_{0,1}, e^1_{1,0}\} \arrow[r, swap, "0"] & \mathbb{Z}\{e^0_{0,0}\}
\end{tikzcd}$$

If we tensor with $\mathbb{F}_2$, then all differentials become zero, and the homology is as given above.

On the other hand, the above example shows that Corollary 26.2 does not hold with $\mathbb{Z}$-coefficients. Recall that the integral homology of $\mathbb{RP}^2$ is $\mathbb{Z}$ in degree zero and $\mathbb{Z}/2\mathbb{Z}$ in degree 1. So if we just take tensor product of the homology, we don't get anything above degree two. But the above complex has a $\mathbb{Z}/2\mathbb{Z}$ in the homology in degree 3.

Looking back to the proof of Proposition 26.1, we can try to give the argument with integral chains and see where it breaks down. Since each cellular chain groups $C_n(X)$ is free abelian, and since $B_n \subseteq C_n(X)$ is a subgroup, it follows that B_n is also free abelian. This implies that every short exact sequence

$$0 \longrightarrow Z_n \hookrightarrow C_n(X) \longrightarrow B_{n-1} \longrightarrow 0$$

splits, so that tensoring with any group will again produce a short exact sequence. Free abelian groups are flat (i.e., tensoring with them preserves exact sequences) and the complexes Z_* and B_* have zero differentials, so it follows that

$$\mathrm{H}_n(Z_* \otimes D_*) \cong Z_* \otimes \mathrm{H}_n(D_*) \qquad \text{and} \qquad \mathrm{H}_n(B_* \otimes D_*) \cong B_* \otimes \mathrm{H}_n(D_*).$$

The spot where the argument breaks down is that although the connecting homomorphisms in the long exact sequence

$$\xrightarrow{\lambda \otimes \mathrm{id}} (Z_* \otimes \mathrm{H}_*(D_*))_n \longrightarrow \mathrm{H}_n(C_* \otimes D_*) \longrightarrow (B_* \otimes \mathrm{H}_*(D_*))_n \xrightarrow{\lambda \otimes \mathrm{id}} (Z_* \otimes \mathrm{H}_*(D_*))_{n-1} \longrightarrow \dots$$

are induced by the inclusion $\lambda_{n-1} : B_{n-1} \hookrightarrow Z_{n-1}$, we do not know that these are injective after tensoring with the groups $\mathrm{H}_n(D)$. The best we can say is that we have short exact sequences

$$0 \longrightarrow \operatorname{coker}(\lambda_n \otimes \mathrm{id}) \longrightarrow \mathrm{H}_n(C_* \otimes D_*) \longrightarrow \ker(\lambda_{n-1} \otimes \mathrm{id}) \longrightarrow 0.$$

But tensoring with any abelian group is right-exact, meaning that it preserves quotients. So $\mathrm{coker}(\lambda_n \otimes \mathrm{id}) \cong \mathrm{coker}(\lambda_n) \otimes \mathrm{H}_*(D) \cong \mathrm{H}_*(C) \otimes \mathrm{H}_*(D)$. So we have a short exact sequence

$$0 \longrightarrow (\mathrm{H}_*(C) \otimes \mathrm{H}_*(D))_n \longrightarrow \mathrm{H}_n(C_* \otimes D_*) \longrightarrow \ker(\lambda_{n-1} \otimes \mathrm{id}) \longrightarrow 0.$$

It remains to identify the kernel of $\lambda_{n-1} \otimes \mathrm{id}$.

28. Mon, Nov. 2

Definition 28.1. Let A be an abelian group. Then a **free resolution** of A is a exact sequence

$$\dots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$$

in which each group F_n is free abelian.

Proposition 28.2. *Any abelian group has a free resolution of length* 1*, meaning that* $F_n = 0$ *for* $n > 1$.

Proof. First pick any surjection $F_0 \xrightarrow{\varepsilon} A$, where F_0 is free abelian. This amounts to choosing a set of generators for A. Define $F_1 = \ker(\varepsilon)$. Then F_1 is a subgroup of a free abelian group and is therefore free abelian. ■

Definition 28.3. Let $F_1 \xrightarrow{\varphi} F_0 \xrightarrow{\varepsilon} A$ be a free resolution, and let B be an abelian group. Define

$$\mathrm{Tor}(A, B) := \ker(\varphi \otimes \mathrm{id}_B : F_1 \otimes B \longrightarrow F_0 \otimes B).$$

We need to show that this does not depend on the choice of resolution.

Lemma 28.4. *Any two free resolutions of A are chain-homotopy equivalent.*

Proof. Let

$$\begin{tikzcd}
F_1 \arrow[r, "\varphi"] \arrow[d, shift right, "f_1"'] & F_0 \arrow[dr, "\varepsilon"] \arrow[d, shift right, "f_0"'] & \\
G_1 \arrow[r, "\psi"'] \arrow[u, shift right, "g_1"'] & G_0 \arrow[r, "\delta"'] \arrow[u, shift right, "g_0"'] & A
\end{tikzcd}$$

be free resolutions. Since F_0 and G_0 are free, we can find maps f_0 and g_0 as in the diagram, and this induces factorizations f_1 and g_1. To see, for example, that $g_* f_* : F_* \longrightarrow F_*$ is chain-homotopic to the identity, we need a chain homotopy $h_0 : F_0 \longrightarrow F_1$ with

$$\begin{tikzcd}
F_1 \arrow[r, "f_1"] \arrow[d, "\varphi"'] & G_1 \arrow[r, "g_1"] \arrow[d, "\psi"] & F_1 \arrow[d, "\varphi"] \\
F_0 \arrow[r, "f_0"'] \arrow[urr, dashed, "h_0"] \arrow[dr, "\varepsilon"'] & G_0 \arrow[r, "g_0"'] \arrow[d, "\delta"] & F_0 \arrow[dl, "\varepsilon"] \\
& A &
\end{tikzcd}$$

$$g_0 f_0(x) - x = \varphi h_0(x) \qquad \text{and} \qquad g_1 f_1(x) - x = h_0 \varphi(x).$$

But

$$\varepsilon\Big(g_0 f_0(x) - x\Big) = \varepsilon g_0 f_0(x) - \varepsilon(x) = \varepsilon f_0(x) - \varepsilon(x) = 0,$$

so $g_0 f_0 - \mathrm{id}$ lands in the kernel of ε, which is F_1. That is, we have a factorization $F_0 \xrightarrow{h_0} F_1 \xrightarrow{\varphi} F_0$ of $g_0 f_0 - \mathrm{id}$. For the second equation, since φ is injective, it suffices to check it after applying φ. But

$$\varphi\Big(g_1 f_1(x) - x\Big) = \varphi g_1 f_1(x) - \varphi(x) = g_0 \psi f_1(x) - \varphi(x) = g_0 f_0 \varphi(x) - \varphi(x) = \varphi h_0 \varphi(x),$$

so we are done. ■

The ideas in Lemma 28.4 can be used to more generally prove

Proposition 28.5. *Suppose that $f_* : C_* \longrightarrow D_*$ is a quasi-isomorphism between chain complexes of free abelian groups. Then f_* is a chain homotopy-equivalence.*

We are now ready to prove

Proposition 28.6. *The group* $\mathrm{Tor}(A, B)$ *does not depend on the choice of free resolution of* A. *Moreover, this group can also be computed by choosing instead a free resolution for* B *rather than* A.

Proof. By Lemma 28.4, any two resolutions are chain homotopy-equivalent. But chain homotopy-equivalences are preserved by tensoring with B, so it follows that $\mathrm{Tor}(A, B)$ is independent of the choice of resolution.

Now let $F_* \xrightarrow{\varepsilon} A$ and $G_* \xrightarrow{\delta} B$ be free resolutions. Note that we can think of ε and δ as quasi-isomorphisms of chain complexes. Then we have a zig-zag of chain maps

$$F_* \otimes B \xleftarrow{\mathrm{id}\otimes\delta} F_* \otimes G_* \xrightarrow{\varepsilon\otimes\mathrm{id}} A \otimes G_*.$$

By a problem on your homework, these are both quasi-isomoprhisms (since F_* and G_* are complexes of free abelian groups). By Proposition 28.5, these are both chain homotopy equivalences, so that composing $\varepsilon \otimes \mathrm{id}$ with a homotopy inverse for $\mathrm{id} \otimes \delta$ gives the desired result. ∎

29. Wed, Nov. 4

We introduced Tor in order to extend an exact sequence, and we have

Proposition 29.1. *If* $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *is a short exact of abelian groups and* D *is an abelian group, there is an exact sequence*

$$0 \longrightarrow \mathrm{Tor}(A, D) \longrightarrow \mathrm{Tor}(B, D) \longrightarrow \mathrm{Tor}(C, D) \longrightarrow A \otimes D \longrightarrow B \otimes D \longrightarrow C \otimes D \longrightarrow 0.$$

Proof. Take a resolution $F_1 \longrightarrow F_0 \longrightarrow D$ of D. Since each F_i is free abelian, we know that the rows in the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \otimes F_1 \arrow[r] \arrow[d] & B \otimes F_1 \arrow[r] \arrow[d] & C \otimes F_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \otimes F_0 \arrow[r] & B \otimes F_0 \arrow[r] & C \otimes F_0 \arrow[r] & 0
\end{tikzcd}$$

are exact. But we can think of this as a short exact sequence of (vertical) chain complexes. This gives rise to a long exact sequence in homology, which is precisely the claimed exact sequence. ∎

Going back to the reason we introduced Tor, recall that we saw the group

$$\ker\left(B_{n-1} \otimes \mathrm{H}_j(D) \xrightarrow{\lambda_{n-1}\otimes\mathrm{id}} Z_{n-1} \otimes \mathrm{H}_j(D)\right)$$

showing up in an exact sequence. Since $\mathrm{coker}(\lambda_{n-1}) \cong \mathrm{H}_i(C)$, it follows that the kernel in question is precisely $\mathrm{Tor}(\mathrm{H}_{n-1}(C), \mathrm{H}_j(D))$. We have now proved

Theorem 29.2. *[Künneth] For CW complexes* X *and* Y, *there is an exact sequence*

$$0 \longrightarrow \mathrm{H}_*(X; \mathbb{Z}) \otimes \mathrm{H}_*(Y; \mathbb{Z}) \longrightarrow \mathrm{H}_*(X \times Y; \mathbb{Z}) \longrightarrow \mathrm{Tor}(\mathrm{H}_{*-1}(X), \mathrm{H}_*(Y)) \longrightarrow 0.$$

In fact this sequence is always split, so that there is an isomorphism

$$\mathrm{H}_n(X \times Y; \mathbb{Z}) \cong \left(\bigoplus_{i+j=n} \mathrm{H}_i(X; \mathbb{Z}) \otimes \mathrm{H}_j(Y; \mathbb{Z})\right) \oplus \left(\bigoplus_{i+j=n} \mathrm{Tor}\left(\mathrm{H}_{i-1}(X; \mathbb{Z}), \mathrm{H}_j(Y; \mathbb{Z})\right)\right).$$

Example 29.3. We turn back to $X = \mathbb{RP}^2 \times \mathbb{RP}^2$. Using the Künneth theorem and remembering that $\mathbb{RP}^2$ only has nontrivial homology in degree 0 and 1, we get

$$\begin{aligned}
\mathrm{H}_0(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z}) &\cong \mathrm{H}_0(\mathbb{RP}^2; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{H}_0(\mathbb{RP}^2; \mathbb{Z}) \cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z} \\
\mathrm{H}_1(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z}) &\cong \mathrm{H}_1(\mathbb{RP}^2; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{H}_0(\mathbb{RP}^2; \mathbb{Z}) \oplus \mathrm{H}_0(\mathbb{RP}^2; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{H}_1(\mathbb{RP}^2; \mathbb{Z}) \\
&\qquad \oplus \mathrm{Tor}(\mathrm{H}_0(\mathbb{RP}^2; \mathbb{Z}), \mathrm{H}_0(\mathbb{RP}^2; \mathbb{Z})) \\
&\cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}) \\
\mathrm{H}_2(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z}) &\cong \mathrm{H}_1(\mathbb{RP}^2; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{H}_1(\mathbb{RP}^2; \mathbb{Z}) \oplus \mathrm{Tor}(\mathrm{H}_0(\mathbb{RP}^2; \mathbb{Z}), \mathrm{H}_1(\mathbb{RP}^2; \mathbb{Z})) \\
&\qquad \oplus \mathrm{Tor}(\mathrm{H}_1(\mathbb{RP}^2; \mathbb{Z}), \mathrm{H}_0(\mathbb{RP}^2; \mathbb{Z})) \\
&\cong \mathbb{Z}/2\mathbb{Z} \oplus \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \\
\mathrm{H}_3(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z}) &\cong \mathrm{Tor}(\mathrm{H}_1(\mathbb{RP}^2; \mathbb{Z}), \mathrm{H}_1(\mathbb{RP}^2; \mathbb{Z})) \\
&\cong \mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})
\end{aligned}$$

There are three Tor groups to compute. Using the free resolutions $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}$ and $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$, we see that these groups are

$$\mathrm{Tor}(\mathbb{Z}, \mathbb{Z}) = 0 \qquad \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = 0, \qquad \mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0, \qquad \mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$

It follows that

$$\mathrm{H}_0(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z}) \cong \mathbb{Z}, \qquad \mathrm{H}_1(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z},$$
$$\mathrm{H}_2(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}, \qquad \mathrm{H}_3(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$

This is the same answer that comes from the chain complex we wrote down in Example 27.1.

Cohomology

We have now developed quite a bit of machinery, so let's try to answer the following question:

Problem: Show that $\mathbb{CP}^2$ is not homotopy equivalent to $S^2 \vee S^4$.

The first tool we learned about for distinguishing homotopy types is the fundamental group, but both of these spaces are simply-connected (the 2-skeleton of both spaces is S^2). The next tool we learned about was homology, but the homology of both of these spaces is $\mathbb{Z}$ in dimensions $0, 2, 4$ and trivial in other dimensions. So we need something else! Cohomology will allow us to distinguish these spaces.

In defining homology, we always worked with chain complexes. Cohomology starts with cochain complexes.

Definition 29.4. A **cochain complex** C^* is a sequence C^n of abelian groups, together with differentials $\partial^n : C^n \longrightarrow C^{n+1}$, such that $\partial^{n+1} \circ \partial^n = 0$. Given a cochain complex C^*, we define its **cohomology groups** to be

$$\mathrm{H}^n(C^*) := \ker(\partial^n) / \operatorname{im}(\partial^{n-1}).$$

There is a canonical way to obtain a cochain complex from a chain complex, simply by dualizing. Namely, if C^* is a chain complex, we define the dual cochain complex by

$$C^n := \mathrm{Hom}(C_n, \mathbb{Z}),$$

with differential given by $\partial^n = \mathrm{Hom}(\partial_{n+1}, \mathbb{Z})$. More precisely, if $f \in \mathrm{Hom}(C_n, \mathbb{Z})$, then $\partial^n(f) \in \mathrm{Hom}(C_{n+1}, \mathbb{Z})$ is defined by

$$\partial^n(f)(x) = -(-1)^n f(\partial_{n+1}(x)), \tag{29.5}$$

where the sign arises from the Koszul sign rule. The "extra" negative sign out front appears from the general formula $\partial(f) = \partial \circ f - (-1)^{\deg(f)} f \circ \partial$.

Since $\mathrm{Hom}(-, \mathbb{Z})$ is only left-exact, the cohomology groups are not simply the duals of the homology groups, as we will see in examples below.

Definition 29.6. We define the **cohomology of a space** X by

$$\mathrm{H}^n(X; \mathbb{Z}) := \mathrm{H}^n(\mathrm{Hom}(C_*(X), \mathbb{Z})).$$

We can define this in any setting in which we defined homology before.

30. Fri, Nov. 6

Example 30.1.

(1) $X = S^1$. If we dualize the cellular chain complex, $\mathbb{Z} \xrightarrow{0} \mathbb{Z}$, we get the cochain complex $\mathbb{Z} \xleftarrow{0} \mathbb{Z}$, so that the cohomology groups agree with the homology groups in this case.

(2) $X = T^2$. If we dualize the cellular chian complex $\mathbb{Z} \xrightarrow{0} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z}$, we get the cochain complex

$$\mathbb{Z} \xleftarrow{0} \mathbb{Z}^2 \xleftarrow{0} \mathbb{Z},$$

so that again the cohomology groups are the same as the homology groups.

(3) $X = \mathbb{RP}^2$. If we dualize the cellular chian complex $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$, we get the cochain complex

$$\mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z},$$

so that we have

$$\mathrm{H}^n(\mathbb{RP}^2; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2\mathbb{Z} & n = 2 \\ 0 & \text{else.} \end{cases}$$

This finally gives us an answer which differs from homology.

We can also compute the cohomology using coefficients in $\mathbb{F}_2$. If we start the (integral) cellular chain complex into $\mathbb{F}_2$, we get the cochain complex of $\mathbb{F}_2$-vector spaces

$$\mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2.$$

The cohomology groups are

$$\mathrm{H}^n(\mathbb{RP}^2; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & n = 0 \\ 0 & \text{else.} \end{cases}$$

These agree with the mod 2 homology groups $\mathrm{H}_n(\mathbb{RP}^2; \mathbb{F}_2)$.

So we see that, sometimes the cohomology groups of a space agree with the homology groups, but not always. Let's now determine the precise relationship.

We will again work in the general context of a chain complex C_* of free abelian groups, and we will let M be an arbitrary abelian group of coefficients. Like in the proof of the Künneth theorem, we have the short exact sequence of chain complexes

$$0 \longrightarrow Z_* \longrightarrow C_* \longrightarrow B_{*-1} \longrightarrow 0.$$

Here B_{*-1} is the chain complex with $(B_{*-1})_n = B_{n-1}$. Since B_{*-1} is a complex of free abelian groups, this sequence splits. This means that applying $\mathrm{Hom}(-, M)$ will produce a (split) short exact sequence of cochain complexes. Taking cohomology then gives a long exact sequence in cohomology

$$\mathrm{H}^n(\mathrm{Hom}(B_{*-1}, M)) \longrightarrow \mathrm{H}^n(\mathrm{Hom}(C_*, M)) \longrightarrow \mathrm{H}^n(\mathrm{Hom}(Z_*, M)) \xrightarrow{\delta} \mathrm{H}^{n+1}(\mathrm{Hom}(B_{*-1}, M)) \longrightarrow \dots$$

Now the complexes B_* and Z_* have trivial differentials, so this remains true after applying $\mathrm{Hom}(-M)$. The above long exact sequence then becomes

$$\mathrm{Hom}^n(B_{*-1}, M) \longrightarrow \mathrm{H}^n(\mathrm{Hom}(C_*, M)) \longrightarrow \mathrm{Hom}^n(Z_*, M)) \xrightarrow{\delta} \mathrm{Hom}^{n+1}(B_{*-1}, M) \longrightarrow \dots$$

Note that $(B_{*-1})_{n+1} = B_{(n+1)-1} = B_n$, so that $\mathrm{Hom}^{n+1}(B_{*-1}, M) = \mathrm{Hom}^n(B_*, M)$. The connecting homomorphism

$$\mathrm{Hom}^n(Z_*, M) \longrightarrow \mathrm{Hom}^n(B_*, M)$$

is $\mathrm{Hom}(\iota, M)$, where $\iota : B_* \hookrightarrow Z_*$ is the inclusion. It follows that our long exact sequence splits into a bunch of short exact sequences

$$0 \longrightarrow \mathrm{coker}(\mathrm{Hom}(\iota, M))^{n-1} \longrightarrow \mathrm{H}^n(\mathrm{Hom}(C_*, M)) \longrightarrow \ker(\mathrm{Hom}(\iota, M))^n \longrightarrow 0.$$

We have a short exact sequence

$$0 \longrightarrow B_* \longrightarrow Z_* \longrightarrow \mathrm{H}_*(C_*) \longrightarrow 0.$$

By HW 11, $\mathrm{Hom}(-, M)$ is left exact, so that $\ker(\mathrm{Hom}(\iota, M))^n = \mathrm{Hom}(\mathrm{H}_n(C_*), M)$. We have a short exact sequence

$$0 \longrightarrow \mathrm{coker}(\mathrm{Hom}(\iota, M))^{n-1} \longrightarrow \mathrm{H}^n(\mathrm{Hom}(C_*, M)) \longrightarrow \mathrm{Hom}(\mathrm{H}_n(C_*), M) \longrightarrow 0.$$

Like in the proof of the Künneth theorem, this sequence splits, and we are left with an "error" term to understand.

Definition 30.2. Let $F_1 \longrightarrow F_0 \longrightarrow A$ be a free resolution and let M be an abelian group. We define

$$\mathrm{Ext}(A, M) := \mathrm{coker}\left(\mathrm{Hom}(F_0, M) \longrightarrow \mathrm{Hom}(F_1, M)\right).$$

Proposition 30.3. *The group* $\mathrm{Ext}(A, M)$ *does not depend on the choice of resolution of* A.

This follows from Lemma 28.4. A proof very similar to that of Proposition 29.1 gives

Proposition 30.4. *If* $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *is a short exact of abelian groups and* D *is an abelian group, there is an exact sequence*

$$0 \longrightarrow \mathrm{Hom}(C, D) \longrightarrow \mathrm{Hom}(B, D) \longrightarrow \mathrm{Hom}(A, D) \longrightarrow \mathrm{Ext}(C, D) \longrightarrow \mathrm{Ext}(B, D) \longrightarrow \mathrm{Ext}(A, D) \longrightarrow 0.$$

To summarize, we have

Theorem 30.5 (Universal Coefficients)**.** *For any chain complex* C_* *of free abelian groups and any abelian group* M*, we have isomorphisms*

$$\mathrm{H}^n(\mathrm{Hom}(C_*, M)) \cong \mathrm{Hom}(\mathrm{H}_n(C_*), M) \oplus \mathrm{Ext}(\mathrm{H}_{n-1}(C_*), M).$$

When applied to the cohomology of a space, this theorem reads as

Theorem 30.6 (Universal Coefficients)**.** *For any space* X *and any abelian group* M*, we have isomorphisms*

$$\mathrm{H}^n(X; M) \cong \mathrm{Hom}(\mathrm{H}_n(X; \mathbb{Z}), M) \oplus \mathrm{Ext}(\mathrm{H}_{n-1}(X; \mathbb{Z}), M).$$

31. Mon, Nov. 9

[We spent time in class discussing HW 9.]

There is also a Universal Coefficients Theorem for homology. It reads

Theorem 31.1 (Universal Coefficients, Homology)**.** *For any space X and abelian group M, there are isomorphisms*

$$\mathrm{H}_n(X;M) \cong \big(\mathrm{H}_n(X;\mathbb{Z}) \otimes_{\mathbb{Z}} M\big) \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(X;\mathbb{Z}), M).$$

On your homework, you showed that $\mathrm{Tor}(\mathbb{Z}, A) = 0$ and that $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, A)$ is the n-torsion subgroup of A.

Proposition 31.2. $\mathrm{Ext}(\mathbb{Z}, A) = 0$ *and* $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, A) \cong A/nA$*.*

Proof. The first statement is immediate since $\mathbb{Z}$ has a free resolution of length 0. The second follows immediately from the free resolution $\mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$. ■

Note that it follows that, unlike Tor, the groups $\mathrm{Ext}(A, M)$ are not symmetric in A and M.

Example 31.3. Starting from the integral homology of $\mathbb{RP}^2$, which is $\mathrm{H}_0 \cong \mathbb{Z}$ and $\mathrm{H}_1 \cong \mathbb{Z}/2\mathbb{Z}$, we can deduce the integral cohomology, as well as the mod 2 homology and cohomology. The integral cohomology is as found in example 30.1 because $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$ and $\mathrm{Ext}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. The mod 2 homology and cohomology are found similarly.

32. Wed, Nov. 11

Cohomology as a functor

We defined the cohomology of a space by dualizing a chain complex $C_*(X)$ and then passing to cohomology of the cochain complex. If we start with a chain functor $C_*(-) : \mathbf{Top} \longrightarrow \mathbf{Ch}_{\geq 0}(\mathbb{Z})$, like singular chains, then it follows that the resulting cohomology theory is also a functor on spaces. However, in the process of dualizing, we turn a covariant functor into a contravariant functor, so that we have

Proposition 32.1. *Singular cohomology defines a contravariant functor*

$$\mathrm{H}^*(-;\mathbb{Z}) : \mathbf{Top}^{op} \longrightarrow \mathbf{GrAb}.$$

Just as for homology, simplicial cohomology is only functorial with respect to Δ-maps. We did not previously discuss functoriality of cellular homology.

Definition 32.2. Let X and Y be CW complexes. We say that $f : X \longrightarrow Y$ is **cellular** if, for each $n \geq 0$, we have $f(X^n) \subseteq Y^n$.

In other words, f should map the n-skeleton of X into the n-skeleton of Y. A composition of two cellular maps is again cellular, and the identity map of any CW complex is cellular. This means that the following definition is valid.

Definition 32.3. Let $\mathbf{CW_{cell}}$ denote the category whose objects are CW complexes and whose morphisms are cellular maps.

Proposition 32.4. *Cellular homology and cohomology determine functors*

$$\mathrm{H}_*^{cell} : \mathbf{CW_{cell}} \longrightarrow \mathbf{GrAb}, \qquad \mathrm{H}^*_{cell} : (\mathbf{CW_{cell}})^{op} \longrightarrow \mathbf{GrAb}.$$

The point is that you need the assumption that f is cellular in order to make sense of an induced map $C_*^{cell}(X) \xrightarrow{f_*} C_*^{cell}(Y)$. The formula for f_* is given in much the same way as the cellular differential. For an n-cell e^n_α of X, then we set

$$f_*(e^n_\alpha) := \sum_{\beta\ n\text{-cell of } Y} n^f_{\alpha,\beta} e^n_\beta,$$

where $n^f_{\alpha,\beta}$ is the degree of

$$S^n_\alpha \hookrightarrow \bigvee S^n \cong X^n/X^{n-1} \xrightarrow{f} Y^n/Y^{n-1} \cong \bigvee S^n \longrightarrow S^n_\beta.$$

The middle map only makes sense if f is assumed to be cellular.

It is certainly a deficiency in cellular (co)homology that it is only functorial with respect to cellular maps. For example, a famously noncellular map is the diagonal $X \longrightarrow X \times X$, for any space X. On the other hand, we can always use the following to replace an arbitrary map by a cellular one.

Theorem 32.5 (Cellular approximation, Theorem 4.8 of Hatcher)**.** *Let $f : X \longrightarrow Y$ be a map between CW complexes. Then f is homotopic to a cellular map $\widehat{f} : X \longrightarrow Y$. Furthermore, any two such cellular replacements for f are cellularly homotopic to each other, meaning that the homotopy $h : X \times I \longrightarrow Y$ is cellular.*

This means that if we denote by $\mathbf{Ho}(\mathbf{CW})$ the category whose objects are CW complexes and whose morphisms are homotopy classes of (arbitrary) maps, then we have the following result.

Proposition 32.6. *Cellular homology and cohomology determine functors*

$$\mathrm{H}_*^{cell} : \mathbf{Ho}(\mathbf{CW}) \longrightarrow \mathbf{GrAb}, \qquad \mathrm{H}^*_{cell} : (\mathbf{Ho}(\mathbf{CW}))^{op} \longrightarrow \mathbf{GrAb}.$$

There is a similar story for simplicial (co)homology, using

Theorem 32.7 (Simplicial approximation, Theorem 2C.1 of Hatcher)**.** *Let $f : X \longrightarrow Y$ be a map between Δ-complexes. If X is a finite complex, then f is homotopic to a Δ-map after applying barycentric subdivision to X finitely many times.*

There is also an Eilenberg-Steenrod characterization of cohomology. We won't give the full statement. It is nearly identical to what we described for homology. We have not yet defined relative cohomology groups, but these are defined in the same way as relative homology. The point is that, as discussed in Proposition 14.3, we have a split exact sequence of chain complexes

$$0 \longrightarrow C_*(A) \longrightarrow C_*(X) \longrightarrow C_*(X)/C_*(A) \longrightarrow 0,$$

which implies that

$$0 \longrightarrow C^*(X, A; M) = \mathrm{Hom}(C_*(X, A), M) \longrightarrow C^*(X; M) \longrightarrow C^*(A; M) \longrightarrow 0$$

is (split) exact. The relative cohomology groups are then defined as the cohomology of the cochain complex $C^*(X, A; M)$.

Getting back to the axioms, the only changes from those for homology are that

(1) Cohomology is a contravariant functor
(2) The connecting homomorphism should be $h^n(A, \varnothing) \xrightarrow{\delta} h^{n+1}(X, A)$.
(3) The additivity condition for a disjoint union $(X, A) \cong \coprod_i (X_i, A_i)$ should now read

$$\prod_i h_n(X_i, A_i) \cong h_n(X, A).$$

Again, if what we have discussed above were the full story, it would not be clear why anybody would both with cohomology, since it is not so different from homology. But cohomology has an important extra feature.

Cup products

It turns out that, for any space X and any commutative ring R of coefficients, $\mathrm{H}^*(X;R)$ will be a graded ring. To say it is a graded ring means that

(1) The unit 1 is in degree 0 and
(2) If x and y are in degree n and k, respectively, then $x \cdot y$ is in degree $n+k$.

33. Fri, Nov. 13

The unit is quite easy to describe: define $u \in C^0(X;R) = \mathrm{Hom}(C_0(X), R)$ to be the function which takes value 1 on each basis element.

Lemma 33.1. *u is a cocycle and therefore determines a cohomology class.*

Proof. In any of our three versions of homology, the differential $\delta_1 : C_1(X) \longrightarrow C_0(X)$ is given by $\delta_1(e) = e_1 - e_0$. Since $u(e_1) = 1 = u(e_0)$, we conclude that $\delta^0(u)(e) = 0$ for all e, so that $\delta^0(u) = 0$. ■

Note that since there is no δ^{-1} coming into $\mathrm{C}^0(X;R)$, it follows that u is a nontrivial cohomology class, and this will play the role of the unit.

We are left with specifying the multiplication

$$\mathrm{H}^n(X;R) \otimes \mathrm{H}^k(X;R) \longrightarrow \mathrm{H}^{n+k}(X;R).$$

There are several ways to do this. One way is to first write down an "external" product

$$\mathrm{H}^n(X;R) \otimes \mathrm{H}^k(Y;R) \xrightarrow{\times} \mathrm{H}^{n+k}(X \times Y;R).$$

This is also known as the **cross product**.

Let's consider first cellular cohomology. Recall that we have an isomorphism $C_*(X) \otimes C_*(Y) \cong C_*(X \times Y)$. Let φ be the composition

$$\begin{aligned} C^*(X;R) \otimes C^*(Y;R) = \mathrm{Hom}(C_*(X), R) \otimes \mathrm{Hom}(C_*(Y), R) &\longrightarrow \mathrm{Hom}(C_*(X) \otimes C_*(Y), R \otimes R) \\ \cong \mathrm{Hom}(C_*(X \times Y), R \otimes R) &\longrightarrow \mathrm{Hom}(C_*(X \times Y), R), \end{aligned}$$

where the last map is simply induced by the multiplication $R \otimes R \longrightarrow R$ in the ring R. Then we define the external product as

$$\mathrm{H}^*(X;R) \otimes \mathrm{H}^*(Y;R) \longrightarrow \mathrm{H}^*(C^*(X;R) \otimes C^*(Y;R)) \xrightarrow{\mathrm{H}^*(\varphi)} \mathrm{H}^*(X \times Y;R).$$

Finally, the **cup product** in cellular cohomology is defined as the composition

$$\mathrm{H}^*(X;R) \otimes \mathrm{H}^*(X;R) \longrightarrow \mathrm{H}^*(X \times X;R) \xrightarrow{\Delta^*} \mathrm{H}^*(X;R).$$

However, recall that, as we discussed last time, the diagonal $\Delta : X \longrightarrow X \times X$ is **not** a cellular map, so in order to actually compute the cup product, a cellular approximation of the diagonal must be used.

Proposition 33.2. *The cup product makes $\mathrm{H}^*(X;R)$ into a graded ring.*

Proof. We must check that the cup product is associative and unital. To show that u is a left unit, we first note that u can also be described as $u = c^*(1)$, where $c : X \longrightarrow *$. Note also that

$$X \xrightarrow{\Delta} X \times X \xrightarrow{c \times \mathrm{id}} * \times X = X$$

is the identity map of X. Then the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^0(*;R) \otimes \mathrm{H}^n(X;R) \arrow[r, "c^*\otimes \mathrm{id}"] \arrow[d] & \mathrm{H}^0(X) \otimes \mathrm{H}^n(X;R) \arrow[d] \\
\mathrm{H}^n(* \times X;R) \arrow[r, "(c\times \mathrm{id})^*"] \arrow[dr] & \mathrm{H}^n(X \times X;R) \arrow[d] \\
& \mathrm{H}^n(X;R)
\end{tikzcd}$$

shows that $u \cdot x = x$. A similar argument shows that $x \cdot u = x$. Associativity similarly follows from the space-level commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & X \times X \arrow[d, "\mathrm{id}\times\Delta"] \\
X \times X \arrow[r, "\Delta\times\mathrm{id}"'] & X \times X \times X.
\end{tikzcd}$$

■

34. Mon, Nov. 16

Even better, this is a commutative ring, but we mean this in the graded sense.

Definition 34.1. A graded ring A^* is said to be (graded-)commutative if

$$x \cdot y = (-1)^{ab} y \cdot x,$$

where $a = \deg(x)$ and $b = \deg(y)$.

Proposition 34.2. *The cohomology ring is graded commutative.*

Proof. This follows from a combination of topological and algebraic results. The topological result is that the diagram

$$\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[dr, "\Delta"'] & X \times X \arrow[d, "t"] \\
& X \times X
\end{tikzcd}$$

commutes, where t is the transposition. The algebraic result is that the square

$$\begin{tikzcd}
C_n(X) \otimes C_k(Y) \arrow[r] \arrow[d, "\tau"'] & C_{n+k}(X \times Y) \arrow[d, "t_*"] \\
C_k(Y) \otimes C_n(X) \arrow[r] & C_{n+k}(Y \times X)
\end{tikzcd}$$

commutes, where $\tau(x \otimes y) = (-1)^{nk} y \otimes x$. The reason for the sign $(-1)^{nk}$ is as follows. Say e^n_α is an n-cell in X and e^k_β is a k-cell in Y. We wish to know what is the coefficient of $e^{n+k}_{\beta\times\alpha}$ in $t_*(e^{n_k}_{\alpha\times\beta})$.

Recall that this coefficient is the degree of the map

$$S^{n+k} \hookrightarrow \bigvee S^{n+k} \cong (X \times Y)^n/(X \times Y)^{n-1} \xrightarrow{t} (Y \times X)^n/(Y \times X)^{n-1} \cong \bigvee S^{n+k} \longrightarrow S^{n+k}.$$

But this map is the permutation of coordinates

$$S^{n+k} = S^n \wedge S^k \cong S^k \wedge S^n = S^{n+k},$$

which has degree $(-1)^{nk}$ since it can be expressed as nk iterations of a twist $S^1 \wedge S^1 \cong S^1 \wedge S^1$. ■

Proposition 34.3. *The cup product is natural.*

Example 34.4. $X = S^1$. This is not a very interesting example, since there is no room for a nontrivial product. If x is a generator in degree 1, then x^2 must be zero since $\mathrm{H}^2(S^1) = 0$. It follows that the cohomology ring is

$$\mathrm{H}^*(S^1; \mathbb{Z}) \cong \mathbb{Z}[x]/x^2.$$

This is often called an **exterior algebra**.

Example 34.5. For a similar reason, we see that

$$\mathrm{H}^*(S^n; \mathbb{Z}) \cong \mathbb{Z}[x_n]/x_n^2,$$

where x_n has degree n.

Example 34.6. $X = T^2 = S^1 \times S^1$. We know that the cohomology is free abelian on generators w_0, x_1, y_1, and z_2, where the subscript indicates the degree of the class. Thus the only question about the ring structure is what are the products x_1^2, y_1^2, and $x_1 y_1$.

Let $p_i : T^2 \longrightarrow S^1$, for $i = 1, 2$ be the projection maps. These induce ring homomorphisms

$$p_i^* : \mathrm{H}^*(S^1) \longrightarrow \mathrm{H}^*(T^2).$$

Since the projection is cellular, we can calculate these maps explicitly. We claim that $p_1^*(v_1) = x_1$ and $p_1^*(v_1) = y_1$. To see this, note that we can take v_1 to be the dual basis element to the 1-cell of S^1, so that $v_1(e_1) = 1$. Similarly, we take x_1 to be dual to $e^1_{1,0}$ and y_1 to be dual to $e^1_{0,1}$. Then

$$p_1^*(v_1)(ie^1_{1,0} + je^1_{0,1}) = v_1(i(p_1)_*(e^1_{1,0}) + j(p_1)_*(e^1_{0,1})) = v_1(ie_1 + j0) = i,$$

so that $p_1^*(v_1) = x_1$.

Now since the p_i are ring homomorphisms and $v_1^2 = 0$ in $\mathrm{H}^*(S^1)$, we conclude that x_1 and y_1 both square to zero in $\mathrm{H}^*(T^2)$. It only remains to determine the product $x_1 \cdot y_1$.

35. Wed, Nov. 18

Recall that, by definition, $x_1 \cdot y_1 = \Delta^*(x_1 \times y_1)$. Here $x_1 \times y_1 \in \mathrm{H}^2(T^2 \times T^2)$. In order to calculate the cup product, we must take a cellular approximation of the diagonal on T^2. Since $T^2 = S^1 \times S^1$, we can start with a cellular approximation $\tilde{\Delta}_{S^1}$ of the diagonal on S^1 and then define our approximation on T^2 to be

$$\tilde{\Delta}_{T^2} : T^2 = S^1 \times S^1 \xrightarrow{\tilde{\Delta}_{S^1} \times \tilde{\Delta}_{S^1}} S^1 \times S^1 \times S^1 \times S^1 \xrightarrow{\mathrm{id} \times t \times \mathrm{id}} S^1 \times S^1 \times S^1 \times S^1 = T^2 \times T^2.$$

The approximation $\tilde{\Delta}_{S^1}$ can be taken from an approximation on I, and we see that the induced map on chains is $e^1 \mapsto e^1_{1,0} + e^1_{0,1}$. Recalling that $t : S^1 \times S^1 \longrightarrow S^1 \times S^1$ induces the map

$$e^1_{1,0} \mapsto -e^1_{0,1}, \qquad e^1_{0,1} \mapsto -e^1_{1,0}$$

on chains, it follows that $\tilde{\Delta}_{T^2}$ induces the map

$$e^2_{1,1} \mapsto -e^2_{1,1,0,0} - e^2_{0,1,1,0} - e^2_{1,0,0,1} - e^2_{0,0,1,1}$$

on C_2. Now we have

$$\begin{aligned}(x_1 \cdot y_1)(e^2_{1,1}) &:= (x_1 \times y_1)(-e^2_{1,1,0,0} - e^2_{0,1,1,0} - e^2_{1,0,0,1} - e^2_{0,0,1,1}) \\ &= x_1(e^1_{0,1})y_1(e^1_{1,0}) + x_1(e^1_{1,0})y_1(e^1_{0,1}) = 0 \cdot 0 + 1 \cdot 1 = 1.\end{aligned}$$

It follows that $x_1 \cdot y_1 = z_2$.

36. Fri, Nov. 20

Another (easier) way to think about the above example is using the Künneth theorem. First, as we indicated in the previous example, the projections p_X and p_Y induce ring maps

$$p_X^* : \mathrm{H}^*(X) \longrightarrow \mathrm{H}^*(X \times Y), \qquad p_Y^* : \mathrm{H}^*(Y) \longrightarrow \mathrm{H}^*(X \times Y).$$

Proposition 36.1. *Let $R \xrightarrow{f} T$ and $S \xrightarrow{g} T$ be ring homomorphisms (all rings are assumed to be commutative). Then there is a unique ring homomorphism making the following diagram commute:*

$$\begin{tikzcd} R \arrow[dr, "\eta_1"'] \arrow[drr, bend left, "f"] & & \\ & R \otimes S \arrow[r, dashed] & T \\ S \arrow[ur, "\eta_2"] \arrow[urr, bend right, "g"'] & & \end{tikzcd}$$

In other words, $R \otimes S$ is the coproduct in the category of commutative rings. Here, $\eta_1(r) = r \otimes 1$ and $\eta_2(s) = 1 \otimes s$. The multiplication on $R \otimes S$ is given on simple tensors by

$$(r_1 \otimes s_1) \cdot (r_2 \otimes s_2) := r_1 r_2 \otimes s_1 s_2$$

and then extended linearly to all of $R \otimes S$. The unit is $1 \otimes 1$.

Proof. Given f and g, then $\varphi : R \otimes S \longrightarrow T$ may be defined on simple tensors by the formula

$$\varphi(r \otimes s) = f(r)g(s).$$

This clearly makes the diagram commute, and it is simple to check that this is a ring homomorphism. ■

Note that if R^* and S^* are graded rings, the same result holds, but signs must be introduced appropriately. For instance, the multiplication on $R^* \otimes S^*$ is given by

$$(r_1 \otimes s_1) \cdot (r_2 \otimes s_2) := (-1)^{\deg(s_1)\deg(r_2)} r_1 r_2 \otimes s_1 s_2.$$

Applying the previous result to the ring maps p_X^* and p_Y^* defines a ring homomorphism

$$\mathrm{H}^*(X) \otimes \mathrm{H}^*(Y) \longrightarrow \mathrm{H}^*(X \times Y).$$

A cohomological version of the Künneth theorem is

Theorem 36.2 (Theorem 3.16 of Hatcher)**.** *Suppose that the groups $\mathrm{H}^k(Y;\mathbb{Z})$ are finitely generated free abelian groups for all k. Then the cross product*

$$\mathrm{H}^*(X;\mathbb{Z}) \otimes_Z \mathrm{H}^*(Y;\mathbb{Z}) \longrightarrow \mathrm{H}^*(X \times Y;\mathbb{Z})$$

is an isomorphism of rings.

Of course, by symmetry the hypothesis on $\mathrm{H}^*(Y;\mathbb{Z})$ could equally well be placed on $\mathrm{H}^*(X;\mathbb{Z})$ instead.

Example 36.3. Turning back to $X = T^2$, this result tells us that

$$\mathrm{H}^*(T^2;\mathbb{Z}) \cong (\mathbb{Z}[x_1]/x_1^2) \otimes_{\mathbb{Z}} (\mathbb{Z}[y_1]/y_1^2) \cong \mathbb{Z}[x_1, y_1]/(x_1^2, y_1^2).$$

In particular, $x_1 y_1 \neq 0$ in this ring.

Example 36.4. $X = \mathbb{RP}^2$. With coefficients in $\mathbb{Z}$, there is no room for a multiplication, so this is uninteresting. However, with coefficients in $\mathbb{F}_2$, we have a one-dimensional vector space in degrees 0, 1, and 2. Letting the generators (i.e. nonzero elements) be x_0, x_1, x_2, we have either

$$x_1 \cup x_1 = 0 \qquad \text{or} \qquad x_1 \cup x_1 = x_2.$$

We could try to compute this directly as we did for the torus, but this would involve a cellular approximation of the diagonal, which already for the minimal CW structure on $\mathbb{RP}^2$ is not so easy to understand.

Here is another approach. We will need to use a relative version of the cup product, which takes the form

$$\mathrm{H}^n(X, A) \otimes \mathrm{H}^k(X, B) \longrightarrow \mathrm{H}^{n+k}(X, A \cup B).$$

This is deduced from a relative cross product

$$\mathrm{H}^n(X, A) \otimes \mathrm{H}^k(Y, B) \longrightarrow \mathrm{H}^{n+k}(X \times Y, A \times Y \cup X \times B).$$

Recall that points in $\mathbb{RP}^2$ can be thought of as equivalence classes $[x_0 : x_1 : x_2]$ under scalar multiplication. If we consider the subspace defined by $x_2 = 0$, this gives an embedded $\mathbb{RP}^1 \cong S^1$ inside $\mathbb{RP}^2$, which we will call $\mathbb{RP}^1_\ell$. Similarly, setting $x_0 = 0$ defines $\mathbb{RP}^1_r$. Let $U_\ell = \mathbb{RP}^2 - \mathbb{RP}^1_\ell$ and $U_r = \mathbb{RP}^2 - \mathbb{RP}^1_r$. Then $U_r \cong U_\ell \cong \mathbb{R}^2 \simeq *$. We see that $\mathbb{RP}^1_\ell \cap \mathbb{RP}^1_r = \{1\}$, where $1 = [0 : 1 : 0]$. It follows that $U_\ell \cup U_r = \mathbb{RP}^2 - \{1\}$. Moreover, if we let $\mathbb{RP}^1_m$ be defined by $x_1 = 0$, then $\mathbb{RP}^2 - \{1\}$ deformation retracts onto $\mathbb{RP}^1_m$.

Now in the diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathbb{RP}^2) \otimes \mathrm{H}^1(\mathbb{RP}^2) \arrow[r, "\cup"] & \mathrm{H}^2(\mathbb{RP}^2) & \\
\mathrm{H}^1(\mathbb{RP}^2, U_\ell) \otimes \mathrm{H}^1(\mathbb{RP}^2, U_r) \arrow[u, "\cong"] \arrow[d, "\cong"] \arrow[r, "\cup"] & \mathrm{H}^2(\mathbb{RP}^2, \mathbb{RP}^2 - \{1\}) \arrow[u, "\cong"] \arrow[d, "\cong"] \arrow[r, "\cong"] & \mathrm{H}^2(\mathbb{RP}^2, \mathbb{RP}^1_m) \arrow[ul, "\cong"] \\
\mathrm{H}^1(\mathbb{R}^2, \mathbb{R}^2 - \mathbb{R}^1_\ell) \otimes \mathrm{H}^1(\mathbb{R}^2, \mathbb{R}^2 - \mathbb{R}^1_r) \arrow[d, "\cong"] \arrow[r, "\cup"] & \mathrm{H}^2(\mathbb{R}^2, \mathbb{R}^2 - \{0\}) & \\
\mathrm{H}^1(\mathbb{R}^1, \mathbb{R}^1 - \{0\}) \otimes \mathrm{H}^1(\mathbb{R}^1, \mathbb{R}^1 - \{0\}) \arrow[ur, "\times"] & &
\end{tikzcd}$$

the top square commutes by naturality of the cup product. The top left vertical map is an isomorphism since $U_r \cong U_\ell \simeq *$. The right horizontal map is an isomorphism since $\mathbb{RP}^2 - \{1\}$ deformation retracts onto $\mathbb{P}^1_m$, and the right diagonal map is an isomorphism by the long exact sequence for the pair. The lower square again commutes by naturality, and the vertical maps are isomorphisms by excision. The bottom diagonal map is an isomorphism by Theorem 36.2. Tracing through the diagram, it follows that $x_1 \cup x_1 \neq 0$, so that we must have $x_1 \cup x_1 = x_2$. In other words, we have shown

$$\mathrm{H}^*(\mathbb{RP}^2; \mathbb{F}_2) \cong \mathbb{F}_2[x_1]/(x_1^3).$$

A similar argument shows that

$$\mathrm{H}^*(\mathbb{RP}^n; \mathbb{F}_2) \cong \mathbb{F}_2[x_1]/(x_1^{n+1})$$

and

$$\mathrm{H}^*(\mathbb{RP}^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[x_1]$$

37. Mon, Nov. 23

It is also possible to describe the cup product for singular or simplicial cohomology. To do this, we introduce some notation. Given an n-simplex $\sigma : \Delta^n \longrightarrow X$ and some $0 \leq i \leq n$, let

$$d_l^i(\sigma) := \sigma \circ d^n \circ d^{n-1} \circ \cdots \circ d^{n-i+1} = \sigma_{|_{[v_0,\dots,v_i]}}$$

be the "left" i-dimensional face and similarly

$$d_r^i(\sigma) := \sigma \circ d^0 \circ \cdots \circ d^0 = \sigma_{|_{[v_{n-i},\dots,v_n]}}$$

be the "right" i-dimensional face. Then given $\alpha \in \mathrm{H}^n(X;R)$ and $\beta \in \mathrm{H}^k(X;R)$, we define $\alpha \cup \beta$ on an $(n+k)$-simplex σ by

$$(\alpha \cup \beta)(\sigma) := (-1)^{nk}\alpha(d_l^n(\sigma)) \cdot \beta(d_r^k(\sigma)).$$

Proposition 37.1. *The above cup product defines a chain map*

$$C^*(X;R) \otimes C^*(X;R) \longrightarrow C^*(X;R),$$

where $C^*(X;R)$ means either singular or simplicial cochains.

Proof. We must check the formula

$$\partial(\alpha \cup \beta) = \partial(\alpha)\beta + (-1)^n \alpha\partial(\beta)$$

if $\alpha \in C^n(X;R)$ and $\beta \in C^k(X;R)$. Recall from (29.5) that $\partial(\alpha) = (-1)^{n+1}\alpha \circ \partial$. For simplicity, we consider the case $n = 2$ and $k = 1$. Then

$$\begin{aligned}
\partial^3(\alpha \cup \beta)(\sigma) &= (\alpha \cup \beta)(\partial_4(\sigma)) = (\alpha \cup \beta)(\sigma \circ d^0 - \sigma \circ d^1 + \sigma \circ d^2 - \sigma \circ d^3 + \sigma \circ d^4) \\
&= (\alpha \cup \beta)(\sigma_{|_{[v_1,v_2,v_3,v_4]}} - \sigma_{|_{[v_0,v_2,v_3,v_4]}} + \sigma_{|_{[v_0,v_1,v_3,v_4]}} - \sigma_{|_{[v_0,v_1,v_2,v_4]}} + \sigma_{|_{[v_0,v_1,v_2,v_3]}}) \\
&= \alpha(\sigma_{|_{[v_1,v_2,v_3]}})\beta(\sigma_{|_{[v_3,v_4]}}) - \alpha(\sigma_{|_{[v_0,v_2,v_3]}})\beta(\sigma_{|_{[v_3,v_4]}}) + \alpha(\sigma_{|_{[v_0,v_1,v_3]}})\beta(\sigma_{|_{[v_3,v_4]}}) \\
&\qquad - \alpha(\sigma_{|_{[v_0,v_1,v_2]}})\beta(\sigma_{|_{[v_2,v_4]}}) + \alpha(\sigma_{|_{[v_0,v_1,v_2]}})\beta(\sigma_{|_{[v_2,v_3]}})
\end{aligned}$$

On the other hand,

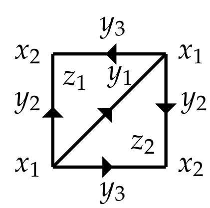
$$\begin{aligned}
[\partial^2(\alpha)\beta](\sigma) &= -\partial^2(\alpha)(\sigma_{|_{[v_0,v_1,v_2,v_3]}})\beta(\sigma_{|_{[v_3,v_4]}}) = \alpha(\partial_3(\sigma_{|_{[v_0,v_1,v_2,v_3]}}))\beta(\sigma_{|_{[v_3,v_4]}}) \\
&= \alpha(\sigma_{|_{[v_1,v_2,v_3]}})\beta(\sigma_{|_{[v_3,v_4]}}) - \alpha(\sigma_{|_{[v_0,v_2,v_3]}})\beta(\sigma_{|_{[v_3,v_4]}}) + \alpha(\sigma_{|_{[v_0,v_1,v_3]}})\beta(\sigma_{|_{[v_3,v_4]}}) - \alpha(\sigma_{|_{[v_0,v_1,v_2]}})\beta(\sigma_{|_{[v_3,v_4]}})
\end{aligned}$$

and

$$\begin{aligned}
[\alpha\partial^1(\beta)](\sigma) &= \alpha(\sigma_{|_{[v_0,v_1,v_2]}})\partial^1(\beta)(\sigma_{|_{[v_2,v_3,v_4]}}) = \alpha(\sigma_{|_{[v_0,v_1,v_2]}})\beta(\partial_2(\sigma_{|_{[v_2,v_3,v_4]}})) \\
&= \alpha(\sigma_{|_{[v_0,v_1,v_2]}})\beta(\sigma_{|_{[v_3,v_4]}}) - \alpha(\sigma_{|_{[v_0,v_1,v_2]}})\beta(\sigma_{|_{[v_2,v_4]}}) + \alpha(\sigma_{|_{[v_0,v_1,v_2]}})\beta(\sigma_{|_{[v_2,v_3]}})
\end{aligned}$$

■

Example 37.2. $X = \mathbb{RP}^2$. Recall that the projective plane was built from two simplices as in the picture to the right. Taking coefficients in $\mathbb{F}_2$, this gives the chain complex



$$\begin{array}{ccccc}
C_2^\Delta(\mathbb{RP}^2) \otimes \mathbb{F}_2 & \xrightarrow{\partial_2} & C_1^\Delta(\mathbb{RP}^2) \otimes \mathbb{F}_2 & \xrightarrow{\partial_1} & C_0^\Delta(\mathbb{RP}^2) \\
\| & & \| & & \| \\
\mathbb{F}_2\{z_1, z_2\} & \xrightarrow[\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}]{} & \mathbb{F}_2\{y_1, y_2, y_3\} & \xrightarrow[\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}]{} & \mathbb{F}_2\{x_1, x_2\}
\end{array}$$

and therefore the cochain complex

$$\begin{tikzcd}
C^2_\Delta(\mathbb{RP}^2;\mathbb{F}_2) \arrow[d, equal] & C^1_\Delta(\mathbb{RP}^2;\mathbb{F}_2) \arrow[l, "\partial^2"'] \arrow[d, equal] & C^0_\Delta(\mathbb{RP}^2;\mathbb{F}_2) \arrow[l, "\partial^1"'] \arrow[d, equal] \\
\mathbb{F}_2\{z_1^*, z_2^*\} & \mathbb{F}_2\{y_1^*, y_2^*, y_3^*\} \arrow[l, "\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}"] & \mathbb{F}_2\{x_1^*, x_2^*\}. \arrow[l, "\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}"]
\end{tikzcd}$$

Representatives for the nonzero cohomology classes are

$$\alpha_0 = [x_1^* + x_2^*], \qquad \alpha_1 = [y_1^* + y_2^*], \qquad \alpha_2 = [z_1^*] = [z_2^*].$$

We want to establish that $\alpha_1^2 = \alpha_2$, or, equivalently, that $\alpha_1^2 \neq 0$. We have

$$\alpha_1^2(z_1) := \alpha_1(y_1)\alpha_1(y_3) = 0$$

and

$$\alpha_1^2(z_2) := \alpha_1(y_1)\alpha_1(y_2) = 1.$$

It follows that $\alpha_1^2 = \alpha_2$.

38. Mon, Nov 30

Earlier, we saw that the $\mathbb{F}_2$-cohomology of $\mathbb{RP}^n$ is a truncated polynomial ring on a generator in degree 1. A similar argument can be used to show that the $\mathbb{Z}$-cohomology of $\mathbb{CP}^n$ is a truncated polynomial ring on a degree 2 generator. We will see a simpler argument for this, using Poincaré duality. But first, let us see why knowing the cohomology ring can be useful.

Example 38.1. We can use the cohomology ring structure to show that $\mathbb{CP}^2$ is not homotopy equivalent to $S^2 \vee S^4$. We know they have the same cohomology groups, but a homotopy equivalence also induces an isomorphism of cohomology rings, so it suffices to show that

$$\mathrm{H}^*(S^2 \vee S^4;\mathbb{Z}) \not\cong \mathrm{H}^*(\mathbb{CP}^2;\mathbb{Z}) \cong \mathbb{Z}[z_2]/z_2^3.$$

The long exact sequence for the pair $(S^2 \vee S^4, S^2)$ shows that the restriction $\mathrm{H}^2(S^2 \vee S^4;\mathbb{Z}) \xrightarrow{\cong} \mathrm{H}^2(S^2;\mathbb{Z})$ is an isomorphism.

Write $\mathrm{H}^2(S^2 \vee S^4;\mathbb{Z}) \cong \mathbb{Z}\{y_2\}$ and $\mathrm{H}^4(S^2 \vee S^4;\mathbb{Z}) \cong \mathbb{Z}\{y_4\}$. Note that we have a retraction $S^2 \xrightarrow{\iota} S^2 \vee S^4 \xrightarrow{p} S^2$. By functoriality, we also get a retraction on cohomology,

$$\begin{tikzcd}
\mathrm{H}^*(S^2;\mathbb{Z}) \arrow[r, "p^*"] \arrow[d, "\cong"'] \arrow[rr, bend left, "\mathrm{id}"] & \mathrm{H}^*(S^2 \vee S^4;\mathbb{Z}) \arrow[r, "\iota^*"] \arrow[d, "\cong"'] & \mathrm{H}^*(S^2;\mathbb{Z}) \arrow[d, "\cong"'] \\
\mathbb{Z}[x_2]/x_2^2 \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & \mathbb{Z}\{y_2, y_4\} \arrow[r] & \mathbb{Z}[x_2]/x_2^2
\end{tikzcd}$$

Since ι^* is an isomorphism on H^2, it follows that the same is true for p^*. In particular $p^*(x_2) = \pm y_2$. It follows that

$$y_2^2 = (p^*(x_2))^2 = p^*(x_2^2) = 0.$$

It follows that $\mathrm{H}^*(S^2 \vee S^4;\mathbb{Z}) \not\cong \mathrm{H}^*(\mathbb{CP}^2;\mathbb{Z})$.

Note that this shows that the attaching map $S^3 \xrightarrow{\eta} S^2$ for the 4-cell in $\mathbb{CP}^2$ is not null-homotopic. If η were null-homotopic, this would give a homotopy equivalence $\mathbb{CP}^2 \simeq S^2 \vee S^4$.

The ideas in the previous example show more generally that the inclusions of the wedge summands induce a ring isomorphism

$$\widetilde{\mathrm{H}}^*(X \vee Y) \cong \widetilde{\mathrm{H}}^*(X) \times \widetilde{\mathrm{H}}^*(Y).$$

One classical application is to the study of division algebras over $\mathbb{R}$. We emphasize that we do not assume the division algebras to be associative.

Proposition 38.2. *If $\mathbb{R}^n$ is an $\mathbb{R}$-division algebra, then n must be a power of 2.*

Proof. Let $\mu : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an $\mathbb{R}$-division algebra multiplication. Then μ is linear in each variable, so that we get an induced map

$$\varphi : \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1} \longrightarrow \mathbb{RP}^{n-1}.$$

Passing to $\mathbb{F}_2$-cohomology, this gives a ring homomorphism

$$\mathbb{F}_2[z_1]/z_1^n \longrightarrow \mathbb{F}_2[x_1, y_1]/(x_1^n, y_1^n).$$

It follows that $\varphi^*(z_1) = n_x x_1 + n_y y_1$ for some $n_x, n_y \in \mathbb{F}_2$.

To determine the coefficients n_x and n_y, let $u : * \hookrightarrow \mathbb{R}$ denote the inclusion of 1. Then the composition

$$* \times \mathbb{R} \xrightarrow{u \times \mathrm{id}} \mathbb{R} \times \mathbb{R} \xrightarrow{\mu} \mathbb{R}$$

is the identity. Passing to cohomology, it follows that $n_y = 1$. Using the unit in the other variable shows that n_x. We have shown that $\varphi^*(z_1) = x_1 + y_1$. We then get

$$0 = \varphi(z_1^n) = \varphi(z_1)^n = (x_1 + y_1)^n = \sum_{k=0}^{n} \binom{n}{k} x_1^k y_1^{n-k} = \sum_{k=1}^{n-1} \binom{n}{k} x_1^k y_1^{n-k}.$$

Since the monomials $x_1^k y_1^{n-k}$, for various k, are linearly independent, it follows that each $\binom{n}{k}$ must be zero.

Lemma 38.3 (Lucas's Theorem)**.** *Let $n = a_0 + a_1 2 + a_2 2^2 + \cdots + a_i 2^i$ and $k = b_0 + b_1 2 + b_2 2^2 + \cdots + b_j 2^j$ be the 2-adic expansions. Then*

$$\binom{n}{k} \equiv \prod_i \binom{a_i}{b_i} \pmod 2.$$

Since a_i and b_i are in $\{0, 1\}$, we see that

$$\binom{a_i}{b_i} = \begin{cases} 0 & a_i \equiv 0 \ \& \ b_i \equiv 1 \\ 1 & \text{else.} \end{cases}.$$

Now we want to have $\binom{n}{k} \equiv 0$ for all $0 < k < n$. By the above, this means that for some i we must have $a_i \equiv 0$ and $b_i \equiv 1$. But if n is not a power of 2, it is possible to find a k that violates this condition. For instance, taking $n = 5 = 1 + 2^2$, we can take $k = 2^2$. ■

In fact, the statement can be improved to show that the only possible values for n are 1, 2, 4, and 8, but this requires more advanced techniques (K-theory!). This was proved in 1958 by Kervaire and Milnor, but is often attributed to Adams, since it follows from his Hopf Invariant One Theorem (1960). It had already been known since the 1920's that the only real **normed** division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ (Hurwitz's Theorem) and since the 19th century that $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ are the only associative division algebras (Frobenius's Theorem).

Orientations

When we restrict our attention to manifolds, we can say quite a bit more about cohomology. We start by recalling

Definition 39.1. A (topological) n**-manifold** M is a Hausdorff, second-countable space such that each point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

Last semester, you discussed orientability for surfaces (2-manifolds), and we can now give a general, rigorous treatment. The two main properties we would want of an orientation are

(1) an orientation should be determined by a coherent family of "local" orientations around each point $x \in M$
(2) an orientation of $\mathbb{R}^n$ should be preserved by a rotation but reversed by a reflection.

Since a manifold is locally like $\mathbb{R}^n$, we should first define an orientation of $\mathbb{R}^n$. There are many ways to do this, but it will be convenient for us to give a definition in terms of homology. With that in mind, we note that for any $x \in \mathbb{R}^n$, the relative homology group $\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}; \mathbb{Z})$ is isomorphic to $\mathbb{Z}$. We then define an orientation of $\mathbb{R}^n$ at x to be a choice of generator of $\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}; \mathbb{Z}) \cong \mathbb{Z}$. Note that since rotations have degree 1 and reflections have degree -1, our definition satisfies condition (2).

Since a manifold M is locally like $\mathbb{R}^n$, this allows us to define local orientations on any M. The key is that excision shows that

$$\mathrm{H}_n(M, M - \{x\}; \mathbb{Z}) \cong \mathrm{H}_n(U, U - \{x\}; \mathbb{Z}) \cong \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}; \mathbb{Z}) \cong \mathbb{Z}.$$

In fact, it will be convenient for us to consider a general commutative ring R as the coefficient group.

Definition 39.2. Let R be a commutative ring and M an n-manifold. Then, for any $x \in M$, a **local** R**-orientation** at x is a choice μ_x of (R-module) generator of $\mathrm{H}_n(M, M - \{x\}; R)$.

This gives us the local definition. Now we want to say that M is R-orientable if there is a compatible family of orientations.

Definition 39.3. An R**-orientation** of M is an open cover $\mathcal{U} = \{U\}$ of M together with a homology class $\mu_U \in \mathrm{H}_n(M, M - U; R)$ for each $U \in \mathcal{U}$ such that for each $x \in U$, μ_U restricts to a (R-module) generator under $\mathrm{H}_n(M, M - U; R) \longrightarrow \mathrm{H}_n(M, M - \{x\}; R) \cong R$. We also require that if $U \cap V \neq \emptyset$ for $U, V \in \mathcal{U}$, then μ_U and μ_V determine the same element of $\mathrm{H}_n(M, M - (U \cap V); R)$. We say that M is R**-orientable** if there exists an R-orientation.

An equivalent definition is to say that an R-orientation is a collection μ_x of local orientations such that each point x has a neighborhood U and class μ_U restricting to each μ_y for all $y \in U$.

The two choices of R of primary interest are $R = \mathbb{Z}$ and $R = \mathbb{F}_2$. In the case $R = \mathbb{Z}$, we simply say "orientable" without referencing the coefficients.

Proposition 39.4 (Hatcher, Prop 3.25). *For a connected M, there is an orientation double-covering $\widetilde{M} \longrightarrow M$, and M is orientable if and only if $\widetilde{M}$ is* **not** *connected, in which case $\widetilde{M} \cong M \coprod M$.*

Examples in the non-orientable case are $S^2 \longrightarrow \mathbb{RP}^2$ and $T^2 \longrightarrow K$. Covering space theory then gives.

Corollary 39.5. *If $\pi_1(M)$ has no subgroup of index two, then M is orientable.*

Note that a $\mathbb{Z}$-orientation μ of M determines an R-orientation of M for any R, using the ring homomorphism $\mathbb{Z} \longrightarrow R$, $1 \mapsto 1$. However, this is not an if and only if.

Proposition 39.6. *Any manifold has a (unique) $\mathbb{F}_2$-orientation.*

Proof. The point is that orientablity is about being able to make consistent choices of generators. But there is always a canonical choice of generator of a 1-dimensional $\mathbb{F}_2$-vector space: the (unique) nonzero element. ■

Recall that a **closed** manifold is one that is compact and without boundary.

Theorem 39.7. *Let M be a connected, closed n-manifold. Then either*

(1) *M is orientable and $\mathrm{H}_n(M;\mathbb{Z}) \longrightarrow \mathrm{H}_n(M, M-\{x\};\mathbb{Z}) \cong \mathbb{Z}$ is an isomorphism for all $x \in M$ OR*
(2) *M is nonorientable and $\mathrm{H}_n(M;\mathbb{Z}) = 0$.*

Working with $\mathbb{F}_2$-coefficients, it turns out that $\mathrm{H}_n(M;\mathbb{F}_2) \cong \mathbb{F}_2$ for any M, corresponding to the fact that every manifold is $\mathbb{F}_2$-orientable. See [Hatcher, Theorem 3.26] for the statement over an arbitrary coefficient ring. In the orientable case, a generator of $\mathrm{H}_n(M;\mathbb{Z})$ is called a **fundamental class** or **orientation class** for M. Note that there are two such classes (the two choices of generator).

The key step in the proof is to show that for connected *noncompact* n-manifolds N, we have $\widetilde{H}_n(N;\mathbb{Z}) = 0$. Applying this in the case $N = M - \{x\}$, we get that

$$\mathrm{H}_n(M;\mathbb{Z}) \longrightarrow \mathrm{H}_n(M, M-\{x\};\mathbb{Z}) \cong \mathbb{Z}$$

is injective. This already shows that $\mathrm{H}_n(M;\mathbb{Z})$ must be either $\mathbb{Z}$ or 0.

Example 39.8. On Oct. 21, we computed that

$$\mathrm{H}_n(\mathbb{RP}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}.$$

It follows that $\mathbb{RP}^n$ is orientable if and only if n is odd.

Poincaré Duality

Our last main topic for the course is duality, given as the following result.

Theorem 39.9 (Poincaré Duality)**.** *Let M be a closed, orientable n-manifold. Then there is an isomorphism*

$$D : \mathrm{H}^k(M;\mathbb{Z}) \cong \mathrm{H}_{n-k}(M;\mathbb{Z})$$

for all k.

Under this isomorphism, the unit $1 \in \mathrm{H}^0(M)$ corresponds to the fundamental class $\mu \in \mathrm{H}_n(M)$. The map D can be described in terms of the *cap product*.

40. Fri, Dec. 4

Definition 40.1. The **cap product** in the singular/simplicial theories is a map

$$\mathrm{H}^p(X;\mathbb{Z}) \otimes \mathrm{H}_q(X;\mathbb{Z}) \xrightarrow{\cap} \mathrm{H}_{q-p}(X;\mathbb{Z}).$$

On the level of cochains, the formula is

$$\alpha \cap \sigma = \alpha(\sigma_{[v_0,\dots,v_p]})\sigma_{v_p,\dots,v_q}.$$

Of course, it must be checked that this formula on the level of chains/cochains is compatible with differentials and therefore gives a well-defined cap product. More precisely, we need to verify

$$\partial^p(\alpha) \cap \sigma + (-1)^p \alpha \cap \partial_q(\sigma) = \alpha(\sigma_{[v_0,\dots,v_p]})\partial_{q-p}(\sigma_{[v_p,\dots,v_q]}).$$

We do this in the case $p = 1$ and $q = 2$. We have

$$\begin{aligned}
\partial^1(\alpha) \cap \sigma &= \partial^1(\alpha)(\sigma_{[v_0,v_1,v_2]})[v_2] = \alpha(\partial_2(\sigma_{[v_0,v_1,v_2]}))[v_2] \\
&= \alpha(\sigma_{[v_1,v_2]})[v_2] - \alpha(\sigma_{[v_0,v_2]})[v_2] + \alpha(\sigma_{[v_0,v_1]})[v_2]
\end{aligned}$$

$$\begin{aligned}
\alpha \cap \partial_2(\sigma) &= \alpha \cap \sigma_{[v_1,v_2]} - \alpha \cap \sigma_{[v_0,v_2]} + \alpha \cap \sigma_{[v_0,v_1]} \\
&= \alpha(\sigma_{[v_1,v_2]})[v_2] - \alpha(\sigma_{[v_0,v_2]})[v_2] + \alpha(\sigma_{[v_0,v_1]})[v_1]
\end{aligned}$$

and

$$\alpha(\sigma_{[v_0,v_1]})\partial_1(\sigma_{[v_1,v_2]}) = \alpha(\sigma_{[v_0,v_1]})[v_2] - \alpha(\sigma_{[v_0,v_1]})[v_1]$$

Putting these together gives

$$\partial^1(\alpha) \cap \sigma - \alpha \cap \partial_2(\sigma) = \partial_1(\alpha \cap \sigma)$$

as desired.

We can also define the cap product in the cellular theory. Again, this requires a cellular approximation $\widetilde{\Delta}$ of the diagonal (boooo!!) Δ. Given such an approximation, the cap product is induced from

$$C^*(X) \otimes C_*(X) \xrightarrow{\mathrm{id}\otimes\widetilde{\Delta}_*} C^*(X) \otimes C_*(X \times X) \cong C^*(X) \otimes C_*(X) \otimes C_*(X) \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} \mathbb{Z} \otimes C_*(X) \cong C_*(X).$$

Here $\mathrm{ev} : C^*(X) \otimes C_*(X) \longrightarrow \mathbb{Z}$ is the evaluation map, defined by $\mathrm{ev}(\alpha \otimes \sigma) = \alpha(\sigma)$. The evaluation is also often written using brackets, so that

$$\langle \alpha, \sigma \rangle := \alpha(\sigma) = \mathrm{ev}(\alpha \otimes \sigma).$$

There is an important relation of the cap product to the cup product, which comes immediately from the definitions:

Proposition 40.2. *For $\alpha \in \mathrm{H}^p(X)$, $\beta \in \mathrm{H}^q(X)$, and $\sigma \in \mathrm{H}_{p+q}(X)$, we have*

$$\langle \alpha \cup \beta, \sigma \rangle = (-1)^{pq} \langle \beta \cup \alpha, \sigma \rangle = \langle \alpha, \beta \cap \sigma \rangle \in \mathbb{Z}.$$

Now that we have defined the cap product, we can define the map D of Theorem 39.9. We assume that M is closed and orientable, so that according to Theorem 39.7 it has a fundamental class $\mu_M \in \mathrm{H}_n(M;\mathbb{Z})$. Then we define

$$D(\alpha) := \alpha \cap \mu_M \in \mathrm{H}_{n-k}(M;\mathbb{Z}).$$

Although we are really interested in the case of M compact, we will consider the more general case in which M is not necessarily compact. However, in the more general case the cohomology groups do not agree with the homology groups. For example, the noncompact 1-manifold $N = \mathbb{R}$ does not satisfy the Poincaré duality formula. In order to deal with the noncompact case, we need a new idea, that of **compactly supported cohomology**. The idea, at least in the simplicial/singular context, is to consider only cochains which are "compactly supported" meaning they are nonzero on only finitely many simplices.

For a compact subspace $K \subset N$, we can consider the cohomology group $\mathrm{H}^p(N, N-K; R)$. We think of this as cohomology supported on K. Now if $K \subseteq L$, we have $(N-L) \subseteq (N-K)$ and therefore a homomorphism

$$\mathrm{H}^p(N, N-K; R) \xrightarrow{f_{K,L}} \mathrm{H}^p(N, N-L; R).$$

The idea of compactly supported cohomology is to take the "union" of these groups as K varies over the compact subsets of N.

Definition 40.3. We define the **compactly supported cohomology group** by

$$\mathrm{H}_c^p(N;R) := \varinjlim_{K} \mathrm{H}^p(N, N-K; R).$$

Here the symbol $\varinjlim_K$ means "direct limit". This can be characterized by a universal property (the universal target of all of the groups $\mathrm{H}^p(N, .N-K; R)$). More concretely, this can be described as the quotient of the direct sum $\bigoplus_K \mathrm{H}^p(N, N-K; R)$ by elements of the form $\alpha - f_{K,L}(\alpha)$.

Some key properties of compactly supported cohomology are

(1) If N is compact, then $\mathrm{H}_c^p(N;R) \cong \mathrm{H}^p(N;R)$ since N is a maximal element of the K's.
(2) There is an identification
$$\mathrm{H}_c^p(N;R) \cong \widetilde{\mathrm{H}}^p(\widehat{N};R),$$
where $\widehat{N}$ is the one-point compactification.

This gives, for example, that

$$\mathrm{H}_c^p(\mathbb{R}^n;R) \cong \widetilde{\mathrm{H}}^p(S^n;R) \cong \begin{cases} R & p=n \\ 0 & \text{else.} \end{cases}$$

41. Mon, Dec. 7

Theorem 41.1 (Generalized Poincaré Duality)**.** *Let N be an R-oriented n-manifold. Then there is an isomorphism*

$$D : \mathrm{H}_c^k(N;R) \longrightarrow \mathrm{H}_{n-k}(M;R)$$

for all k.

Sketch. The general strategy is as follows

(1) Prove the theorem in the case $N = \mathbb{R}^n$. We have already seen above that the groups are abstractly isomorphic in this csae.
(2) Use a Mayer-Vietoris argument to deduce the result for $U \cup V$ assuming it holds for U, V, and $U \cap V$. This is the most difficult part of the argument. See [Hatcher, Lemma 3.36].
(3) Show that if $\{U_i\}$ is a collection of *nested* open sets and the result holds for each, then it holds for the union.
(4) Use the previous results to show the theorem holds for any open subset of $\mathbb{R}^n$
(5) Use Zorn's Lemma to do the general case. Let V be a maximal subset for which the theorem holds and let $x \in N - V$. Then x has a neighborhood U homeomorphic to $\mathbb{R}^n$, so the theorem holds on U. But then it must also hold on $V \cup U$, contradicting maximality. So V must be all of N.

■

Corollary 41.2. *A closed, odd-dimensional manifold M has Euler characteristic $\chi(M) = 0$.*

Proof. Since any manifold is $\mathbb{F}_2$-orientable, we apply the Poincaré Duality theorem with $\mathbb{F}_2$-coefficients. Recall that $\chi(M)$ can be calculated as

$$\chi(M) = \sum_i (-1)^i \operatorname{rank}(\mathrm{H}_i(M;\mathbb{Z})) = \sum_i (-1)^i \operatorname{rank}(C_i(M))$$

by Proposition 20.2. But since the groups $C_i(M)$ are free abelian, the latter sum agrees with $\sum_i (-1)^i \dim_{\mathbb{F}_2}(C_i(M) \otimes \mathbb{F}_2)$. By an argument similar to that given in the proof of Proposition 20.2, this agrees with $\sum_i (-1)^i \dim_{\mathbb{F}_2} \mathrm{H}_i(M;\mathbb{F}_2)$.

But now by combining duality and universal coefficients, we have

$$\dim_{\mathbb{F}_2} \mathrm{H}_i(M;\mathbb{F}_2) = \dim_{\mathbb{F}_2} \mathrm{H}^{n-i}(M;\mathbb{F}_2) = \dim_{\mathbb{F}_2} \mathrm{H}_{n-i}(M;\mathbb{F}_2).$$

Since n is odd it follows that $\dim_{\mathbb{F}_2} \mathrm{H}_i(M;\mathbb{F}_2)$ will always cancel $\dim_{\mathbb{F}_2} \mathrm{H}_{n-i}(M;\mathbb{F}_2)$ in the formula for $\chi(M)$. ∎

If M is closed and R-orientable, then consider the mapping

$$\mathrm{H}^k(M;R) \otimes \mathrm{H}^{n-k}(M;R) \longrightarrow R$$

defined by $(\alpha,\beta) \mapsto \langle \alpha \cup \beta, \mu_M \rangle$. This defines a bilinear pairing on the cohomology groups. Recall that, a bilinear pairing $A \otimes_R B \longrightarrow R$ is called **nonsingular** if the adjoint maps $A \longrightarrow \mathrm{Hom}_R(B,R)$ and $B \longrightarrow \mathrm{Hom}_R(A,R)$ are isomorphisms. The following result is a consequence of the Poincaré duality theorem.

Proposition 41.3. *Taking $R = \mathbb{F}$ a field, the above pairing is nonsingular (again assuming that M is closed and $\mathbb{F}$-orientable).*

Proof. Let $\alpha \neq 0 \in \mathrm{H}^k(M;\mathbb{F})$. We need to know that there is a $\beta \in \mathrm{H}^{n-k}(M;\mathbb{F})$ such that $\langle \alpha \cup \beta, \mu_M \rangle \neq 0$. But recall that

$$\langle \alpha \cup \beta, \mu_M \rangle = \langle \alpha, \beta \cap \mu_M \rangle = \langle \alpha, D(\beta) \rangle.$$

Since $\alpha \neq 0$ and the evaluation pairing $\mathrm{H}^k(M;\mathbb{F}) \otimes_{\mathbb{F}} \mathrm{H}_k(M;\mathbb{F}) \longrightarrow \mathbb{F}$ is nonsingular by the homework, there must be some homology class $\gamma \in \mathrm{H}_k(M;\mathbb{F})$ such that $\langle \alpha, \gamma \rangle \neq 0$. But since the duality map is an isomorphism, we can write $\gamma = D(\beta)$ for some β, which gives the result. ∎

The same result holds for $R = \mathbb{Z}$ if we quotient the homology and cohomology by their torsion subgroups.

Example 41.4. $M = \mathbb{RP}^n$. We have already determined the cup product structure on $\mathrm{H}^*(\mathbb{RP}^n;\mathbb{F}_2)$, but this was not so easy. We can instead obtain the cup product structure immediately from the preceding results (recall that every manifold is $\mathbb{F}_2$-orientable). In the case of $\mathbb{RP}^2$, the previous result says that the cup product

$$\mathrm{H}^1(\mathbb{RP}^2;\mathbb{F}_2) \otimes \mathrm{H}^1(\mathbb{RP}^1;\mathbb{F}_2) \longrightarrow \mathrm{H}^2(\mathbb{RP}^2;\mathbb{F}_2)$$

cannot be zero, which was the only nontrivial step in determining the cohomology ring.

In the case of $\mathbb{RP}^3$, we learn that

$$\mathrm{H}^1(\mathbb{RP}^3;\mathbb{F}_2) \otimes \mathrm{H}^2(\mathbb{RP}^3;\mathbb{F}_2) \longrightarrow \mathrm{H}^3(\mathbb{RP}^2;\mathbb{F}_2)$$

is nonzero. The only remaining question is whether $x_2 = x_1^2$. But we can determine this by restricting along the inclusion $\mathbb{RP}^2 \hookrightarrow \mathbb{RP}^3$. An induction proof now easily shows that

$$\mathrm{H}^*(\mathbb{RP}^n;\mathbb{F}_2) \cong \mathbb{F}_2[x_1]/(x_1^{n+1}).$$

By restricting to finite skeleta, it now follows that

$$\mathrm{H}^*(\mathbb{RP}^\infty;\mathbb{F}_2) \cong \mathbb{F}_2[x_1].$$

Example 41.5. $M = \mathbb{CP}^n$. Since $\mathbb{CP}^n$ is simply-connected, it is $\mathbb{Z}$-orientable, so that Poincaré Duality applies. Also, we know that all homology and cohomology is torsion-free. The preceding result then tells us that

$$\mathrm{H}^2(\mathbb{CP}^2;\mathbb{Z}) \otimes \mathrm{H}^2(\mathbb{CP}^2;\mathbb{Z}) \longrightarrow \mathrm{H}^4(\mathbb{CP}^2;\mathbb{Z})$$

is nonzero and further that there exists $i \in \mathbb{Z}$ such that $z_2 \cup iz_2$ is a generator for H^4. Certainly i must be ± 1, so that z_2^2 is a generator. Now a similar argument as above shows that z_2^k is a generator in $\mathrm{H}^{2k}(\mathbb{CP}^n;\mathbb{Z})$ whenever $k \leq n$. We get

$$\mathrm{H}^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[x_2]/(x_2^{n+1})$$

and

$$\mathrm{H}^*(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}[x_2].$$

42. Wed, Dec. 9

Last time, we mentioned the cohomology rings $\mathrm{H}^*(\mathbb{RP}^n;\mathbb{F}_2)$ and $\mathrm{H}^*(\mathbb{CP}^n;\mathbb{Z})$. There is a similar answer for quaternionic projective space:

$$\mathrm{H}^*(\mathbb{HP}^n;\mathbb{Z}) \cong \mathbb{Z}[x_4]/(x_4^{n+1}),$$

where x_4 is in degree 4.

Example 42.1. Lens spaces. Recall that in Homework 2 you computed the homology of the 3-dimensional lens space $M = S^3/C^3$ to be

$$\mathrm{H}_i(M;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0,3 \\ \mathbb{Z}/3\mathbb{Z} & i = 1 \\ 0 & \text{else.} \end{cases}$$

By Universal Coefficients, we compute

$$\mathrm{H}^0(M;\mathbb{F}_3) \cong \mathrm{Hom}(\mathbb{Z},\mathbb{F}_3) \cong \mathbb{F}_3,$$

$$\mathrm{H}^1(M;\mathbb{F}_3) \cong \mathrm{Hom}(\mathbb{F}_3,\mathbb{F}_3) \oplus \mathrm{Ext}(\mathbb{Z},\mathbb{F}_3) \cong \mathbb{F}_3,$$

$$\mathrm{H}^2(M;\mathbb{F}_3) \cong \mathrm{Hom}(0,\mathbb{F}_3) \oplus \mathrm{Ext}(\mathbb{F}_3,\mathbb{F}_3) \cong \mathbb{F}_3,$$

$$\mathrm{H}^3(M;\mathbb{F}_3) \cong \mathrm{Hom}(\mathbb{Z},\mathbb{F}_3) \oplus \mathrm{Ext}(0,\mathbb{F}_3) \cong \mathbb{F}_3,$$

and

$$\mathrm{H}^k(M;\mathbb{F}_3) \cong 0, k > 3.$$

On your most recent homework, you saw that $x_1^2 = 0$. But Poincaré Duality gives that $x_1 \cup x_2$ is a generator for H^3. It follows that

$$\mathrm{H}^*(M;\mathbb{F}_3) \cong \mathbb{F}_3[x_1,x_2]/(x_1^2,x_2^2).$$

If we consider the higher dimensional lens spaces S^{2n-1}/C_3, the answer turns out to be

$$\mathrm{H}^*(S^{2n-1}/C^3;\mathbb{F}_3) \cong \mathbb{F}_3[x_1,x_2]/(x_1^2,x_2^n).$$

The same is true for any odd prime: we get

$$\mathrm{H}^*(S^{2n-1}/C^p;\mathbb{F}_p) \cong \mathbb{F}_p[x_1,x_2]/(x_1^2,x_2^n).$$

The same results hold when $n = \infty$, so that we have

$$\mathrm{H}^*(S^\infty/C^p;\mathbb{F}_p) \cong \mathbb{F}_p[x_1,x_2]/(x_1^2).$$

The Realization problem

We have seen a few examples of cohomology rings. We also know how to combine examples to form new ones (by the Kunneth theorem, $\mathrm{H}^*(X \times Y) \cong \mathrm{H}^*(X) \otimes \mathrm{H}^*(Y)$, at least up to torsion information). A reasonable to question to ask is

Question 42.2. *Which rings can arise as the cohomology ring of a space?*

We have some restrictions already: we know the cohomology ring of a space is always graded-commutative. For instance, working with coefficients in $\mathbb{Z}$, there is the following result, which we will address next class.

Proposition 42.3. *If* $\mathrm{H}^*(X;\mathbb{Z}) \cong \mathbb{Z}[x_n]$ *for some* n*, then* n *is either* 2 *or* 4.

This was proved by Steenrod in 1960. If one allows multiple polynomial generators, there are more examples:

Example 42.4.

(1) Let $\mathrm{Gr}_n(\mathbb{C}^\infty)$ denote the Grassmannian of n-planes inside of $\mathbb{C}^\infty$. This space is also known as $BU(n)$, the classifying space for the (unitary) Lie group $U(n)$. Then
$$\mathrm{H}^*(\mathrm{Gr}_n(\mathbb{C}^\infty);\mathbb{Z}) \cong \mathbb{Z}[x_2, x_4, \ldots, x_{2n}].$$

(2) Let $\mathrm{Gr}_n(\mathbb{H}^\infty)$ denote the Grassmannian of n-planes inside of $\mathbb{H}^\infty$. This space is also known as $BSp(n)$, the classifying space for the (symplectic) Lie group $Sp(n)$. Then
$$\mathrm{H}^*(\mathrm{Gr}_n(\mathbb{H}^\infty);\mathbb{Z}) \cong \mathbb{Z}[x_4, x_8, \ldots, x_{4n}].$$

(3) There is also a classifying space $BSU(n)$ for the special unitary group, and
$$\mathrm{H}^*(BSU(n);\mathbb{Z}) \cong \mathbb{Z}[x_4, \ldots, x_{2n}].$$

It turns out that these are essentially all of the examples. More precisely,

Theorem 42.5 (Andersen-Grodal, 2008)**.** *If $\mathrm{H}^*(X;\mathbb{Z})$ is a finitely generated polynomial algebra over $\mathbb{Z}$, then $\mathrm{H}^*(X;\mathbb{Z})$ is a tensor product of the above examples.*

This theorem had been known since roughly 1980 with some additional hypotheses on the ring. In fact, Andersen-Grodal give a complete characterization of possible (even) degrees of polynomial generators for any coefficients R. For example, over $\mathbb{F}_3$, the polynomial algebra $\mathbb{F}_3[x_4, x_{12}]$ is realizable, although this is not true over $\mathbb{Z}$. Similarly, $\mathbb{F}_5[x_8]$ is realizable over $\mathbb{F}_5$ but not $\mathbb{Z}$.

In the case where R is a field of characteristic zero, it had already been proved by Serre in his 1951 thesis that every polynomial algebra on even degree generators can be realized as the cohomology of a space.

Of course, the more general question of which rings can arise is much more difficult and is an open problem in general.

Cohomology Operations

We saw that one benefit of the ring structure on cohomology is that it allowed us to distinguish, for example, $\mathbb{CP}^2$ from $S^2 \vee S^4$, even though they have the same homology. However, if we suspend both spaces, we run into trouble.

Proposition 42.6. *For any space X, the cohomology ring $\mathrm{H}^*(\Sigma X)$ is trivial, in the sense that all cup products of classes in positive degrees vanish.*

The point is that the cup product is defined using the diagonal. But the diagonal map $S^1 \longrightarrow S^1 \wedge S^1 \cong S^2$ is null-homotopic. This implies that the diagonal on $\Sigma X = S^1 \wedge X$ is also null-homotopic.

On the other hand, we do not expect suspension to somehow detach the top cell of $\mathbb{CP}^2$, and we would like to see this somehow reflected in cohomology. This can be done by considering *cohomology operations*.

Example 42.7. For any p, consider the short exact sequence $\mathbb{Z}/p\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}$. This induces a long exact sequence in cohomology
$$\longrightarrow \mathrm{H}^n(X;\mathbb{F}_p) \longrightarrow \mathrm{H}^n(X;\mathbb{Z}/p^2\mathbb{Z}) \longrightarrow \mathrm{H}^n(X;\mathbb{F}_p) \xrightarrow{\delta} \mathrm{H}^{n+1}(X;\mathbb{F}_p) \longrightarrow \ldots.$$
Thus the connecting homomorphism is a natural transformation $\mathrm{H}^n(-;\mathbb{F}_p) \longrightarrow \mathrm{H}^{n+1}(-;\mathbb{F}_p)$. It is often called the **Bockstein homomorphism** and denoted by β. It can be shown that this has some important properties, for instance

(1) For any degree 1 class x, we have $\beta(x) = x^2$.
(2) β commutes with suspension, meaning that $\beta(\Sigma x) = \Sigma\beta(x)$.

43. Fri, Dec. 11

In fact, cohomology operations are well understood.

Theorem 43.1 (Steenrod). *There are natural (Steenrod) cohomology operations*

$$\mathrm{Sq}^n : \mathrm{H}^k(-;\mathbb{F}_2) \longrightarrow \mathrm{H}^{k+n}(-;\mathbb{F}_2)$$

for all n and k such that

(1) *If x is of degree n then $\mathrm{Sq}^n(x) = x^2$*
(2) *If x is of degree $< n$, then $\mathrm{Sq}^n(x) = 0$*
(3) *The Sq^n commute with suspension*
(4) Sq^0 *is the identity and* Sq^1 *is the Bockstein*
(5) *The Cartan formula holds:*

$$\mathrm{Sq}^n(x \cup y) = \sum_i \mathrm{Sq}^i(x) \cup \mathrm{Sq}^{n-i}(y).$$

We can use cohomology operations in $\mathbb{F}_2$-cohomology to distinguish $\Sigma\mathbb{CP}^2$ from $S^3 \vee S^5$. Both spaces have classes x_3 and x_5 in degrees 3 and 5, respectively. It is easy to see that in $S^3 \vee S^5$, we have $\mathrm{Sq}^2(x_3) = 0$, since the class x_3 is pulled back from the collapse map $S^3 \vee S^5 \longrightarrow S^3$.

However, in $\Sigma\mathbb{CP}^2$, we have

$$\mathrm{Sq}^2(x_3) = \mathrm{Sq}^2(\Sigma x_2) = \Sigma\mathrm{Sq}^2(x_2) = \Sigma x_2^2 = \Sigma x_4 = x_5.$$

It follows that we cannot have a homotopy equivalence between $\Sigma\mathbb{CP}^2$ and $S^3 \vee S^5$.

We can ask what happens when we compose two or more cohomology operations. The main formula used to understand these compositions is the **Adém relation**

$$\mathrm{Sq}^n \circ \mathrm{Sq}^k = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{k-j-1}{n-2j} \mathrm{Sq}^{n+k-j} \circ \mathrm{Sq}^j$$

for $n < 2k$. For instance, these relations give $\mathrm{Sq}^1\mathrm{Sq}^1 = 0$, $\mathrm{Sq}^1\mathrm{Sq}^2 = \mathrm{Sq}^3$, and $\mathrm{Sq}^2\mathrm{Sq}^2 = 0$. In fact, it can be shown that

Proposition 43.2. *The operation Sq^n is indecomposable if and only if n is a power of 2.*

For instance, we have the relation

$$\mathrm{Sq}^6 = \mathrm{Sq}^5\mathrm{Sq}^1 + \mathrm{Sq}^2\mathrm{Sq}^4.$$

Corollary 43.3. *If $\mathrm{H}^*(X;\mathbb{F}_2) \cong \mathbb{F}_2[x_n]/x_n^r$, where $r \in \{3, 4, \dots, \infty\}$, then n must be a power of 2.*

Proof. If n is not a power of 2, then we can decompose Sq^n into a linear combination of nontrivial compositions of squaring operations. But we must have $\mathrm{Sq}^n(x_n) = x_n^2$, which would imply that x_n^2 is a sum of classes, each of which is an operations applied to a class in degree strictly between n and $2n$. But there are no such classes by assumption. ■

There are also operations in $\mathrm{H}^*(-;\mathbb{F}_p)$.

Theorem 43.4 (Steenrod). *There are natural (Steenrod) cohomology operations*

$$\mathrm{P}^n : \mathrm{H}^k(-;\mathbb{F}_p) \longrightarrow \mathrm{H}^{k+2n(p-1)}(-;\mathbb{F}_p)$$

for all n and k such that

(1) *If x is of degree $2n$ then $\mathrm{P}^n(x) = x^p$*

(2) *If x is of degree $< 2n$, then* $\mathrm{P}^n(x) = 0$
(3) *The* P^n *commute with suspension*
(4) P^0 *is the identity*
(5) *The Cartan formula holds:*
$$\mathrm{P}^n(x \cup y) = \sum_i \mathrm{P}^i(x) \cup \mathrm{P}^{n-i}(y).$$

There is a similar Adém relation:
$$P^n P^k = \sum_{j=0}^{\lfloor a/p \rfloor} \binom{(p-1)(k-j)-1}{n-pj} P^{n+k-j} P^j,$$
and it can be used to show that P^n is indecomposable if and only if n is a power of p.

Corollary 43.5. *If* $\mathrm{H}^*(X; \mathbb{F}_3) \cong \mathbb{F}_3[x_n]/x_n^r$, *where* $r \in \{p+1, p+2, \ldots, \infty\}$, *then n must be of the form* $n = 2p^j m$, *where* $m \mid (p-1)$.

Proof. Note first that, as we mentioned last time, n must be even. Suppose $n = 2k$. Now we must have $P^k(x_n) = x_n^p \neq 0$. If k is not a power of p, then we can decompose P^k, so that some $P^{p^i}(x_n)$ must be nonzero for some $p^i < k$. This class lives in degree $n + 2p^i(p-1)$. But the nonzero cohomology of X lies in degrees that are multiples of n, so n must divide $2p^i(p-1)$. Since n is even and p is odd, we conclude that n is of the stated form. ■

Proof of Proposition 42.3. If $\mathrm{H}^*(X; \mathbb{Z})$ is polynomial on a class x_n, the same is true after passage to $\mathbb{F}_2$ or $\mathbb{F}_3$ coefficients. The $\mathbb{F}_2$-case tells us that n must be a power of 2. The $\mathbb{F}_3$-case tells us that n is either of the form $2 \cdot 3^j$ or $4 \cdot 3^j$. It follows that $n = 2$ and $n = 4$ are the only possibilities. ■