The first algebraic tool that you learned about for distinguishing spaces is the fundamental group \( \pi_1(X) \). As you saw, this is already sufficient for distinguishing surfaces. But this tool has several drawbacks:

1. It fails to distinguish many spaces. For example, \( S^2 \) and \( S^3 \) are both simply-connected but are not homotopy equivalent.

2. It is in practice very difficult to calculate! You may be able to compute the group in terms of giving a presentation (listing generators and relations), but this does not mean you understand the group. Recall that in general given a group \( G \) with a given presentation, there is no algorithm to determine whether a given word represents the trivial element. Of course, for many particular group presentations there are perfectly good algorithms.

One remedy for (2) is to consider instead the abelianized fundamental group. As you saw before, this also suffices for the classification of surfaces. This is great since abelian groups are much easier to work with. For instance, we know that every finitely generated abelian group is a direct sum of cyclic groups. On the other hand, this is a coarser invariant and therefore fails even harder to distinguish spaces. With this tool, the torus \( S^1 \times S^1 \) and the figure eight \( S^1 \vee S^1 \) look the same.

One approach is to consider higher analogues of the fundamental group. Recall that the fundamental group is defined as

\[
\pi_1(X, x) \cong [S^1, (X, x)]_*
\]

where the brackets denote based homotopy classes of based maps. From this definition, it seems reasonable to define

\[
\pi_n(X, x) \cong [S^n, (X, x)]_*
\]

Note that in the case \( n = 0 \), based homotopy classes of maps from \( S^0 = \{-1, 1\} \) correspond precisely to unbased homotopy classes of maps from \( \{-1\} \) to \( X \), so that \( \pi_0(X, x) \) corresponds precisely to the path-components of \( X \).

When \( n = 1 \), we know we get a group, and we can ask what we get for \( n \geq 2 \). Recall that the group structure on \( \pi_1(X, x) \) can be defined using the pinch map \( S^1 \to S^1 \vee S^1 \) via

\[
[S^1, (X, x)]_* \times [S^1, (X, x)]_* \to [S^1, (X, x)]_*
\]

\[
(S^1 \xrightarrow{\alpha} X, S^1 \xrightarrow{\beta} X) \to (S^1 \xrightarrow{p} S^1 \vee S^1 \xrightarrow{(\alpha, \beta)} X)
\]

[ We also spent some time reviewing the fact that the wedge sum serves as the “coproduct in the category of based spaces”.

We can try to do the same for the \( \pi_n(X) \), starting from a pinch map for \( S^n \). If we recall that \( S^n \cong (S^1)^{\wedge n} \), then we see that pinching in each of the \( n \) coordinates leads to \( n \) different choices of pinch maps. In fact, these all provide the same multiplication by the following result

---

Date: December 8, 2016.
Proposition 1.1 (Eckmann-Hilton Argument). Let $X$ be a set with two binary operations, denoted $*_1$ and $*_2$, and a distinguished element $e \in X$, such that

1. $e$ is a unit element for both $*_1$ and $*_2$
2. $*_1$ and $*_2$ satisfy the “interchange law”: for all $w,x,y,z$ in $X$,
   $$(w *_1 x) *_2 (y *_1 z) = (w *_2 y) *_1 (x *_2 z).$$

Then in fact $*_1 = *_2$ and this operation is both associative and commutative.

Proof. We show that the operations agree and are commutative.

$$x *_2 y = (x *_1 e) *_2 (e *_1 y) = (x *_2 e) *_1 (e *_2 y) = x *_1 y$$

and

$$y *_2 x = (e *_1 y) *_2 (x *_1 e) = (e *_2 x) *_1 (y *_2 e) = x *_1 y.$$ 

These arguments are best visualized by thinking of $*_1$ as a “horizontal” multiplication and $*_2$ as a “vertical” multiplication. Then the interchange law says that you can either first multiply horizontally and then vertically or in the other order, and you get the same answer. ■

Applying the Eckmann-Hilton argument to the $n$-choices of pinch maps on $\pi_n(X)$ show that this is an abelian group if $n \geq 2$. The unit element is the constant map at the basepoint. To verify the interchange law holds, for example when $n = 2$, it suffices to see that the diagram

$$\begin{array}{ccc}
(S^1 \lor S^1) \land S^1 & \overset{\text{id} \land p}{\longrightarrow} & (S^1 \lor S^1) \land (S^1 \lor S^1) \\
S^2 \cong S^1 \land S^1 & \overset{\text{p} \land \text{id}}{\longrightarrow} & S^1 \land (S^1 \lor S^1) \\
\end{array}$$

commutes. But both composites along the sides of the diamond give $p \land p$, so we are done.

2. Fri, Aug. 28

Ok, great! We have a bunch of nice abelian groups $\pi_n(X)$. Can we compute these?

Back in Math 651, the first interesting example of a fundamental group was $\pi_1(S^1) \cong \mathbb{Z}$. In fact, this generalizes to the statement that $\pi_n(S^n) \cong \mathbb{Z}$ (we may prove this later). You also saw that $\pi_1(S^n) = 0$ if $n > 1$, and this also generalizes to the statement $\pi_k(S^n) = 0$ if $n > k$. So the “interesting” cases are $\pi_{n+k}(S^n)$.

When $n = 1$, there turns out to be nothing here. In fact, covering space theory can be used to show

Proposition 2.1. Let $p : E \longrightarrow B$ be a covering map. Then $p$ induces an isomorphism

$$p_* : \pi_n(E) \longrightarrow \pi_n(B)$$

for all $n \geq 2$.

We conclude that $\pi_n(S^1) \cong \pi_n(\mathbb{R}) = 0$, since $\mathbb{R}$ is contractible.

The next example to try is $\pi_{2+k}(S^2)$.

Example 2.2. For $X = S^2$, we know

$$\pi_1(S^2) = 0, \quad \pi_2(S^2) \cong \mathbb{Z}, \quad \pi_3(S^2) \cong \mathbb{Z} \quad \pi_4(S^2) \cong \pi_5(S^2) \cong \mathbb{Z}/2\mathbb{Z}, \quad \pi_6(S^2) \cong \mathbb{Z}/12\mathbb{Z}.\quad$$
But these homotopy groups $\pi_n(S^2)$ are only known up to $n = 64$, although it is known that (1) they are all finite, except for $\pi_2(S^2)$ and $\pi_3(S^2)$, and (2) infinitely many are nonzero. This was proved by J. P. Serre.

The situation is similar for the homotopy groups $\pi_k(S^n)$ in general. The homotopy groups of spheres are in some sense the “holy grail” of algebraic topology. They are a major driving force behind a great amount of research, though we know that we will never know all of the homotopy groups.

What this suggests is that if we try to use the homotopy groups $\pi_n(X)$ to distinguish spaces, we are not likely to get very far. Calculating homotopy groups is hard!!

Instead, we want a simpler invariant, from the point of view of computation. This is where homology enters the story.

Before we turn to homology, some language will be convenient. Last time, we discussed the fact that the wedge $X \vee Y$ plays the role of the “coproduct” of $X$ and $Y$ in the setting of based spaces. Here are some more examples

- In the setting of (unbased) spaces, the disjoint union $X \sqcup Y$ is the coproduct of $X$ and $Y$.
- In the setting of sets, the disjoint union $X \sqcup Y$ again is the coproduct of $X$ and $Y$.
- In the setting of vector spaces, the direct sum $V \oplus W$ plays the role of coproduct.
- In the setting of groups and homomorphisms, the free product $G \ast H$ is the coproduct.

There is also a dual notion of a product. The product $X \times Y$ is the “universal example of an object equipped with a pair of maps to $X$ and $Y$.” More precisely, if $W$ is any other such object, we can expect to have a unique map filling in the diagram

$$
\begin{array}{ccc}
W & \xrightarrow{f} & X \\
\downarrow{q_W} & & \downarrow{q_X} \\
Y & \xrightarrow{p_Y} & Y
\end{array}
$$

Here are some examples:

- In the setting of sets, the cartesian product $X \times Y$ satisfies the universal property.
- In the setting of (unbased) spaces, the product $X \times Y$ (given the product topology) satisfies this universal property.
- In the setting of based spaces, the product $X \times Y$, equipped with basepoint $(x_0, y_0)$, satisfies the universal property.
- In the setting of vector spaces, the direct sum $V \oplus W$ again plays the role of product.
- In the setting of groups, the direct product $G \times H$ is the product in the above sense.

**Proposition 2.3.** Suppose $W$ is a space with continuous maps $q_X : W \to X$ and $q_Y : W \to Y$ also satisfying the property of the product. Then $W$ is homeomorphic to $X \times Y$.

**Proof.** The universal property for $X \times Y$ gives us a map $f : W \to X \times Y$. 

$$
\begin{array}{ccc}
W & \xrightarrow{f} & X \\
\downarrow{q_W} & & \downarrow{q_X} \\
Y & \xrightarrow{p_Y} & Y
\end{array}
$$
But $W$ also has a universal property, so we get a map $\varphi : X \times Y \to W$ as well.

Now make Pacman eat Pacman!

We have a big diagram, but if we ignore all dotted lines, there is an obvious horizontal map $W \to W$ to fill in the diagram, namely the id$_W$. Since the universal property guarantees that there is a unique way to fill it in, we find that $\varphi \circ f = \text{id}_W$. Reversing the pacmen gives the other equality $f \circ \varphi = \text{id}_{X \times Y}$. In other words, $f$ is a homeomorphism, and $\varphi = f^{-1}$. ■

This argument may seem strange the first time you see it, but it is a typical argument that applies any time you define an object via a universal property. The argument shows that any two objects satisfying the universal property must be “the same”.
Categories and Functors

Before we delve into homology, we pause to introduce some convenient language that will appear many times throughout this course (and throughout your mathematical careers!). This is the language of categories, functors, and natural transformations.

Definition 3.1. A category $\mathcal{C}$ is a collection of “objects”, denoted $\text{Ob}(\mathcal{C})$, together with, for each pair of objects $X, Y \in \text{Ob}(\mathcal{C})$, a set $\text{Hom}_\mathcal{C}(X, Y)$ of “morphisms” which satisfies the following:

- For each $X, Y, Z \in \text{Ob}(\mathcal{C})$, there is a “composition” function
  \[ \circ : \text{Hom}_\mathcal{C}(Y, Z) \times \text{Hom}_\mathcal{C}(X, Y) \to \text{Hom}_\mathcal{C}(X, Z). \]
  We write $g \circ f$ or $gf$ for $\circ(g, f)$.
- For each $X \in \text{Ob}(\mathcal{C})$ there exists an “identity morphism” $\text{id}_X \in \text{Hom}_\mathcal{C}(X, X)$ such that for any $Y, Z \in \text{Ob}(\mathcal{C})$ and $f \in \text{Hom}_\mathcal{C}(Y, X)$, $g \in \text{Hom}_\mathcal{C}(X, Z)$ we have
  \[ \text{id}_X \circ f = f \quad \text{and} \quad g \circ \text{id}_X = g. \]
- Composition is associative, i.e., $h(gf) = (hg)f$.

Remark 3.2. We often write $\mathcal{C}(X, Y)$ for $\text{Hom}_\mathcal{C}(X, Y)$, and we often write $X \in \mathcal{C}$ for $X \in \text{Ob}(\mathcal{C})$.

Remark 3.3. A category $\mathcal{C}$ is called small if the collection $\text{Ob}(\mathcal{C})$ of objects forms a set.

Categories abound in mathematics. Here are just a few of the more common examples.

Example 3.4.

1. **Set**: the objects are sets and the morphisms are functions.
2. **FinSet**: the objects are finite sets and morphisms are functions.
3. **Vect_k**: where $k$ is a field: the objects are vector spaces over $k$ and morphisms are $k$-linear homomorphisms.
4. **Gp**: the objects are groups and the morphisms are homomorphisms.
5. **AbGp**: the objects are abelian groups and the morphisms are homomorphisms.
6. **Top**: the objects are topological spaces and the morphisms are continuous maps.
7. **Top_∗**: the objects are based topological spaces (spaces with a distinguished base point) and the morphisms are basepoint-preserving continuous maps.
8. **Ho(Top)**: the objects are topological spaces and the morphisms are homotopy classes of maps.
9. **Ho(Top_∗)**: the objects are based topological spaces and the morphisms are based homotopy classes of maps.

These are all “large” categories (many objects). Small categories also arise often, though in a different way.

Example 3.5.

10. $\bullet$ denotes a category with a single object and only an identity morphism.
11. $\bullet \longrightarrow \bullet$ denotes a category with two objects and one morphism connecting the two objects.
12. $\bullet \longrightarrow \bullet \longrightarrow \bullet$ denotes a category with three objects and two composable morphisms
13. $\bullet \longrightarrow \bullet \longrightarrow \bullet$ denotes a category with two objects and three parallel morphisms.

We defined categories so that we could talk about functors.
Definition 3.6. Let \( \mathcal{C} \) and \( \mathcal{D} \) be two categories. A (covariant) functor \( F : \mathcal{C} \to \mathcal{D} \) is the following data: for each \( C \in \mathcal{C} \) we have an object \( F(C) \in \mathcal{D} \), and for each arrow \( f \in \text{Hom}_\mathcal{C}(C,C') \) we have an arrow \( F(f) \in \text{Hom}_\mathcal{D}(F(C),F(C')) \) such that

\[
F(\text{id}_C) = \text{id}_{F(C)} \quad \text{and} \quad F(g \circ f) = F(g) \circ F(f).
\]

A contravariant functor \( F : \mathcal{C} \to \mathcal{D} \) is a functor that reverses the directions of the morphisms. If \( f : C \to C' \) is a morphism, then the contravariant functor \( F \) produces a morphism \( F(f) : F(C') \to F(C) \). We still require compatibility with composition, which now looks like \( F(g \circ f) = F(f) \circ F(g) \).

Remark 3.7. If \( F : \mathcal{C} \to \mathcal{D} \) is a covariant functor and \( f \) is an arrow in \( \mathcal{C} \), we often write \( f^* \) for \( F(f) \). If \( F \) is contravariant, we write \( f^* \) for \( F(f) \).

Example 3.8.

1. Functors \( \{ \bullet \to \bullet \} \to \text{Top} \) are given exactly by diagrams of shape \( X \xrightarrow{f} Y \) in \( \text{Top} \).
2. There is a functor \( \text{Top} \to \text{Ho(Top)} \) (and similarly in the based context) which does nothing on objects and which takes a map to its homotopy class.
3. The fundamental group defines a functor \( \pi_1 : \text{Top}^* \to \text{Gp} \) which assigns to a space \( X \) with basepoint \( x \) the fundamental group \( \pi_1(X,x) \). Given a basepoint-preserving map of based spaces \( f : X \to Y \), the homomorphism \( f_* : \pi_1(X,x) \to \pi_1(Y,f(x)) \) is defined by sending the class of a loop \( \alpha \) to the class of the loop \( f \circ \alpha \). The formulas

\[
(g \circ f)_* = g_* \circ f_* \quad \text{and} \quad (\text{id}_X)_* = \text{id}_{\pi_1(X)}
\]

say that \( \pi_1(-) \) is a functor. In fact, since the homomorphism \( f_* \) only depends on the homotopy class of \( f \), this functor factors as

\[
\begin{array}{ccc}
\text{Top} & \xrightarrow{\pi_1} & \text{Gp} \\
\downarrow & & \downarrow \\
\text{Ho(Top)} & \xrightarrow{\pi_1} & \text{Gp}
\end{array}
\]

4. Wed, Aug. 31

4. Abelianization defines a functor \( (-)_{ab} : \text{Gp} \to \text{AbGp} \). On objects, this is \( G \mapsto G_{ab} \). On morphisms, suppose that \( \varphi : H \to G \) is a homomorphism. Then \( \varphi_{ab} : H_{ab} \to G_{ab} \) is the induced morphism, defined using the universal property of quotients as in the diagram

\[
\begin{array}{ccc}
H & \xrightarrow{\varphi} & G \\
\downarrow & & \downarrow \\
H_{ab} & \xrightarrow{\varphi_{ab}} & G_{ab}
\end{array}
\]

Here the functor axioms are that

\[
(\varphi \circ \lambda)_{ab} = \varphi_{ab} \circ \lambda_{ab} \quad \text{and} \quad (\text{id}_G)_{ab} = \text{id}_{G_{ab}}.
\]

5. The free abelian group functor \( F : \text{Set} \to \text{AbGp} \) is defined on objects by

\[
F(X) = \bigoplus_{x \in X} \mathbb{Z}.
\]
An element of \( F(X) \) is a finite formal \( \mathbb{Z} \)-linear combination of elements of \( X \), and the group operation is defined by
\[
\left( \sum_{x \in X} n_x x \right) + \left( \sum_{x \in X} m_x x \right) := \sum_{x \in X} (n_x + m_x)x.
\]
Given a function \( f : X \to Y \), \( F(f) \) is defined by
\[
F(f) \left( \sum_{x \in X} n_x x \right) := \sum_{x \in X} n_x f(x).
\]

I skipped examples (6), (7), and (8) in class.

(6) **Represented functors**: Given a category \( \mathcal{C} \) and an object \( X \in \mathcal{C} \), define a functor
\[
F_X = \text{Hom}(X, -) : \mathcal{C} \to \text{Set}
\]
by
\[
Y \mapsto \text{Hom}_\mathcal{C}(X, Y).
\]
To see what this does on morphisms, given a morphism \( f : Y \to Z \) in \( \mathcal{C} \), we are required to have a function
\[
F_X(Y) = \text{Hom}_\mathcal{C}(X, Y) \to \text{Hom}_\mathcal{C}(X, Z) = F_X(Z).
\]
We define this to simply be composition with \( f \). You should check for yourself that this really defines a functor.

If we instead put \( X \) in the other slot, we get a functor
\[
H_X = \text{Hom}(-, X) : \mathcal{C}^{\text{op}} \to \text{Set}.
\]
This functor is contravariant, since if \( f : Y \to Z \) is morphism in \( \mathcal{C} \), then composition with \( f \) gives a function
\[
H_X(Z) = \text{Hom}_\mathcal{C}(Z, X) \to \text{Hom}_\mathcal{C}(Y, X) = H_X(Y).
\]

(7) If we consider the previous construction, taking \( \mathcal{C} = \text{Top} \) and \( X = * \), then the functor \( F_\ast : \text{Top} \to \text{Set} \) is the “underlying set” functor.

(8) If we consider the previous construction, taking \( \mathcal{C} = \text{Ho}(\text{Top}_*) \), then \( F_{S^1} \) is precisely the fundamental group functor! That this functor takes values in groups rather than just sets stems from the fact that \( S^1 \) has extra structure: it is a “cogroup object” in \( \text{Ho}(\text{Top}_*) \). Similarly, the represented functor \( F_{\ast\ast} \) is the functor \( \pi_n(-) \).

One concept that shows up in many branches of math is the notion of isomorphism. This is a sign that it should have a “categorical” definition.

**Definition 4.1.** A morphism \( f : X \to Y \) in a category \( \mathcal{C} \) is called an **isomorphism** if there exists a morphism \( g : Y \to X \) such that \( f \circ g = \text{id}_Y \) and \( g \circ f = \text{id}_X \).

**Example 4.2.**

(1) In \( \text{Set} \), an isomorphism is precisely a bijection.
(2) In \( \text{Gp} \), an isomorphism is a (group) isomorphism.
(3) In \( \text{Top} \), an isomorphism is a homeomorphism.
(4) In \( \text{Ho}(\text{Top}) \), an isomorphism is a homotopy equivalence.

What benefit do we draw from making the general categorical definition?
Proposition 4.3. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. If $\varphi$ is an isomorphism in $\mathcal{C}$, then $F(\varphi)$ is an isomorphism in $\mathcal{D}$.

As an application, since we saw that the fundamental group construction factors as $\text{Top} \rightarrow \text{Ho(Top)} \rightarrow \text{Gp}$, we get that if a based map $f$ is a homeomorphism, or even a homotopy equivalence, then $f_\ast$ is an isomorphism on homotopy groups.

**Homology**

There are several variants of homology, as we will see. Following Hatcher, we will start with "simplicial" homology. The input for this flavor of homology is what Hatcher calls a $\Delta$-complex. $\Delta^n$ is the usual notation for the standard $n$-simplex, which can be defined as

\[
\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1, \quad t_i \geq 0\}.
\]

We will denote by $v_i \in \Delta^n$ the vertex defined by $t_i = 1$ and $t_j = 0$ if $j \neq i$. Note that each "facet" of the simplex, in which we have restricted one of the coordinates to zero, is an $(n - 1)$-dimensional simplex. More generally, if we set $k$ of the coordinates equal to zero, we get a face which is an $(n - k)$-dimensional simplex.

$\Delta$-complexes are obtained by gluing together simplices along faces. We will need to keep track of orientations of simplices. In the standard $n$-simplex, we declare the ordering of vertices $v_0 \leq v_1 \leq \cdots \leq v_n$. All gluings performed in constructing a $\Delta$-complex are required to be orientation-preserving identifications. Thus if we want to glue an edge of $\Delta^2$ to an edge of $\Delta^3$, we first note the ordering of the vertices on each of the two edges, and we then glue together along the unique order-preserving linear isomorphism between the two edges.

5. Fri, Sept. 2

To match up with the notion of CW-complex that you saw in MA551/651, another way to view $\Delta$-complexes is as a pushout (gluing)

\[
\begin{array}{c}
\coprod_i \coprod F_i \Delta^{n_i} \\
\downarrow \\
\coprod_a \Delta^{n_a} \\
\downarrow \\
P \coprod \coprod \coprod X
\end{array}
\]

Here, each $F_i$ is a collection of $n_i$-dimensional faces (of various simplices) to be glued together. The variable $i$ runs over all of the glueings to be done. The variable $a$ runs over all of the (open) simplices of $X$.

**Remark 5.1.** This is a more convenient generalization of simplicial complex. A simplicial complex is also obtained by gluing together simplices, but there we require that each $n$-simplex has $n + 1$ distinct vertices and also that an $n$-simplex is uniquely specified by its vertices.

If you have not seen pushouts before, this is fancy (i.e. categorical) language for a glueing construction. In general, the pushout of a pair of morphisms $A \xrightarrow{f} X$ and $A \xrightarrow{g} Y$ is a universal object $X \cup_A Y$ equipped with compatible maps from $X$ and $Y$ in the sense of the following universal property: given an object $Z$ and maps as in the diagram, there exists a unique morphism $h$ as in the diagram:
Concretely, in topology this space is constructed as follows. Start with the set $X \amalg Y$ and impose the equivalence relation generated by $f(a) \sim g(a)$. Then $X \cup_A Y$ is defined to be $(X \amalg Y)/\sim$, equipped with the quotient topology.

Two familiar examples are

**Example 5.2.**

1. (Quotients) In the case that $X \amalg Y$ is the inclusion of a subspace and $Y = \ast$, then the pushout $\ast \cup_A X$ is precisely the quotient $X/A$.

2. (Attaching a disk) Let $X$ be a space, and consider the case where $g$ is the inclusion $S^1 \hookrightarrow D^2$. Then in the glueing, the boundary circle of $D^2$ is glued to $X$ according to the map $f$, but the interior of $D^2$ is untouched. So the space $D^2 \cup_{S^1} X$ looks like $X$ with a disk attached to it.

Ok, now let’s look at some $\Delta$-complexes.

**Example 5.3.**

1. $X = S^1$. This can be built as a $\Delta$-complex by starting with a 1-simplex $\Delta^1$ and then identifying the two faces together. Note that this $\Delta$-complex is not a simplicial complex. The pushout diagram in this case would be

\[
\begin{array}{ccc}
\Delta^0 \amalg \Delta^0 & \to & \Delta^1 \\
p & & p \\
\Delta^0 & \to & S^1.
\end{array}
\]

2. $X = S^1$. Another choice is to start with two simplices $\Delta^1$ and glue them together end-to-end. This is still not a simplicial complex, since the two 1-simplices have the same vertex set. Here, the pushout diagram in this case would be

\[
\begin{array}{ccc}
\coprod_{\mathbb{Z}/2} \Delta^0 & \to & \coprod_{\mathbb{Z}/2} \Delta^1 \\
p & & p \\
\coprod_{\mathbb{Z}/2} \Delta^0 & \to & S^1.
\end{array}
\]

3. $X = S^1$. To get a simplicial complex, we can start with three 1-simplices and glue together end-to-end. Here, the pushout diagram in this case would be

\[
\begin{array}{ccc}
\coprod_{\mathbb{Z}/3} \Delta^0 & \to & \coprod_{\mathbb{Z}/3} \Delta^1 \\
p & & p \\
\coprod_{\mathbb{Z}/3} \Delta^0 & \to & S^1.
\end{array}
\]
Let’s look at some surfaces.

Example 6.1.

1. \(X = S^2\), the sphere. We can obtain \(S^2\) by glueing together two 2-simplices \(\Delta^2\ \{a, b, c\}\) and \(\{x, y, z\}\). We first glue \(\{a, c\}\) to \(\{x, z\}\) to get a square. We then glue \(\{a, b\}\) to \(\{x, y\}\) and \(\{b, c\}\) to \(\{y, z\}\).

2. \(X = S^1 \times S^1\), the torus. We can obtain \(T^2\) by glueing together two 2-simplices \(\Delta^2\ \{a, b, c\}\) and \(\{x, y, z\}\). We first glue the edge \(\{a, c\}\) to the edge \(\{x, z\}\) to get a square. We then glue \(\{a, b\}\) to \(\{y, z\}\), and finally we glue \(\{b, c\}\) to \(\{x, z\}\). This is not a simplicial complex, since in the end we are left with a single vertex.

3. \(X = \mathbb{RP}^2\), the projective plane. We can also obtain this by glueing together 2-simplices \(\{a, b, c\}\) and \(\{x, y, z\}\). We first glue \(\{a, b\}\) to \(\{x, y\}\). We then glue \(\{b, c\}\) to \(\{x, z\}\) and \(\{a, c\}\) to \(\{y, z\}\).

4. \(X = K\), the Klein bottle. We can also obtain this by glueing together 2-simplices \(\{a, b, c\}\) and \(\{x, y, z\}\). We first glue \(\{a, b\}\) to \(\{x, z\}\). We then glue \(\{a, c\}\) to \(\{y, z\}\) and \(\{b, c\}\) to \(\{x, y\}\).

The Simplicial Chain Complex:

Given a \(\Delta\)-complex \(X\), let \(C_n^\Delta(X)\) be the free abelian group on the set of \(n\)-simplices of \(X\). An element of \(C_n^\Delta(X)\) is referred to as an (simplicial) \(n\)-chain on \(X\). Our goal is to assemble the \(C_n^\Delta(X)\), as \(n\) varies, into a “chain complex”

\[
\ldots \rightarrow C_3^\Delta(X) \rightarrow C_2^\Delta(X) \rightarrow C_1^\Delta(X) \rightarrow C_0^\Delta(X) \rightarrow 0.
\]

To say that this is a chain complex just means that composing two successive maps in the sequence gives 0. We wish to specify a homomorphism

\[\partial_n : C_n^\Delta(X) \rightarrow C_{n-1}^\Delta(X)\].

Since \(C_n^\Delta(X)\) is a free abelian group, the homomorphism \(\partial_n\) is completely specified by its value on each generator, namely each \(n\)-simplex. Let \(\sigma\) be an \(n\)-simplex of \(X\). Note that, since we have a chosen ordering of the vertices of \(\sigma\), the \(n\)-simplex \(\sigma\) determines a unique order-preserving map \(\sigma : \Delta^n \rightarrow X\), which restricts to an embedding of the open simplex.

There are \(n + 1\) standard inclusions \(d^i : \Delta^{n-1} \hookrightarrow \Delta^n\), given by inserting 0 in position \(i\) in \(\Delta^n\). Since no faces get collapsed down in the glueing performed to assemble \(X\), composing \(\sigma\) with an inclusion \(d^i\) gives an \((n - 1)\)-simplex of \(X\) (where the ordering is inherited from that of \(\sigma\)).

Definition 6.2. The simplicial boundary homomorphism

\[\partial_n : C_n^\Delta(X) \rightarrow C_{n-1}^\Delta(X)\]

is defined by

\[\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i [\sigma \circ d^i].\]

Example 6.3.

1. If \(\sigma\) is a 1-simplex (from \(v_0\) to \(v_1\)), then

\[\partial_1(\sigma) = [\sigma \circ d^1] - [\sigma \circ d^0] = [v_1] - [v_0].\]

2. If \(\sigma\) is a 2-simplex with vertices \(v_0, v_1,\) and \(v_2,\) and edges \(e_{01}, e_{02},\) and \(e_{12},\) then

\[\partial_2(\sigma) = [\sigma \circ d^0] - [\sigma \circ d^1] + [\sigma \circ d^2] = [e_{12}] - [e_{02}] + [e_{01}].\]
The claim is that this defines a chain complex. The signs have been inserted into the definition to make this work out.

**Proposition 6.4.** The boundary squares to zero, in the sense that $\partial_{n-1} \circ \partial_n = 0$.

**Proof.** We will use

**Lemma 6.5.** For $i > j$, the composite

$$\Delta^{n-2} \xrightarrow{d_i} \Delta^{n-1} \xrightarrow{d_j} \Delta^n$$

is equal to the composite

$$\Delta^{n-2} \xrightarrow{d_{i-1}} \Delta^{n-1} \xrightarrow{d_j} \Delta^n.$$

Consider the case $i = 3, j = 1, n = 4$. We have

$$d^3(d^1(t_1, t_2, t_3)) = d^3(t_1, 0, t_2, t_3) = (t_1, 0, t_2, 0, t_3) = d^1(t_1, t_2, 0, t_3) = d^1(d^2(t_1, t_2, t_3)).$$

This argument generalizes.

For the proposition,

$$\partial_{n-1} \left( \partial_n(\sigma) \right) = \partial_{n-1} \left( \sum_{i=0}^{n} (-1)^i [\sigma \circ d_i] \right)$$

$$= \sum_{i=0}^{n} (-1)^i \partial_{n-1}( [\sigma \circ d_i])$$

$$= \sum_{i=0}^{n} (-1)^i \sum_{j=0}^{n-1} (-1)^j [\sigma \circ d_i \circ d_j]$$

$$= \sum_{i=0}^{n} \sum_{j<i} (-1)^i (-1)^j [\sigma \circ d_i \circ d_j] + \sum_{i=0}^{n-1} \sum_{j<i} (-1)^i (-1)^j [\sigma \circ d_i \circ d_j]$$

(change bounds)

$$= \sum_{i=1}^{n} \sum_{j<i} (-1)^i (-1)^j [\sigma \circ d_i \circ d_j] + \sum_{i=0}^{n-1} \sum_{j<i} (-1)^i (-1)^j [\sigma \circ d_i \circ d_j]$$

(Lemma)

$$= \sum_{i=1}^{n} \sum_{j<i} (-1)^i (-1)^{i-1} [\sigma \circ d_i \circ d_j] + \sum_{i=0}^{n-1} \sum_{j<i} (-1)^i (-1)^j [\sigma \circ d_i \circ d_j]$$

$$= - \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} (-1)^j (-1)^{i-1} [\sigma \circ d_i \circ d_j] + \sum_{i=0}^{n-1} \sum_{j<i} (-1)^i (-1)^j [\sigma \circ d_i \circ d_j]$$

$$= 0.$$

7. FRI, SEPT. 9

We have shown that any two successive simplicial boundary homomorphisms compose to zero, so that we have a chain complex. What do we do with a chain complex? Take homology!

**Definition 7.1.** If

$$\ldots \xrightarrow{\partial_{n+1}} C_{n+1} \xrightarrow{\partial_n} C_n \xrightarrow{\partial_n} \ldots$$

is a chain complex, then we define the $n$th homology group $\text{H}_n(C_\ast, \partial_\ast)$ to be

$$\text{H}_n(C_\ast, \partial_\ast) := \ker \partial_n / \text{im} \partial_{n+1}.$$
Definition 7.2. Given a ∆-complex \( X \), we define the simplicial homology groups of \( X \) to be

\[
H^\Delta_n(X; \mathbb{Z}) := H_n(C^\Delta(X), \partial_*).
\]

Note that we only defined the groups \( C^\Delta_n(X) \) for \( n \geq 0 \). For some purposes, it is convenient to allow chain groups \( C_n \) for negative values of \( n \), so we declare that \( C^\Delta_n(X) = 0 \) for \( n < 0 \). This means that \( \ker \partial_0 = C^\Delta_0(X) \), so that \( H^\Delta_0(X) = C^\Delta_0(X) / \text{im} \partial_1 = \text{coker}(\partial_1) \). Similarly, if \( X \) has no simplices above dimension \( n \), then we see \( C^\Delta_k(X) = 0 \) for \( k > n \), which implies that \( H^\Delta_k(X) = 0 \). Also, \( \partial_{n+1} = 0 \), so that \( H^\Delta_n(X) = \ker \partial_n \).

**Terminology:** The group \( \ker \partial_n \) is also known as the group of \( n \)-cycles and sometimes written \( Z_n \). The group \( \text{im}(\partial_{n+1}) \) is also known as the group of boundaries and sometimes written \( B_n \).

**Remark 7.3.** It is worth noting that since each \( C^\Delta_n(X) \) is free abelian and \( \ker \partial_n \) and \( \text{im} \partial_{n+1} \) are both subgroups, they are necessarily also free abelian.

**Example 7.4.**

1. Consider \( X = S^1 \), built as a ∆-complex with a single 1-simplex \( e \), whose two vertices have been glued together. Thus we have a single 0-simplex. Our chain complex looks like

\[
\begin{align*}
C^\Delta_1(S^1) & \xrightarrow{\partial_1} C^\Delta_0(S^1) \\
\mathbb{Z}\{e\} & \xrightarrow{} \mathbb{Z}\{v\}
\end{align*}
\]

The differential is given by \( \partial_1(e) = [v] - [v] = 0 \). It follows that \( H^\Delta_1(S^1) = \mathbb{Z} \) and \( H^\Delta_0(S^1) = \mathbb{Z} \). Since all of the higher chain groups are zero, the same holds for the higher homology groups \( H^\Delta_n(S^1) \).

2. We had other constructions of \( S^1 \) as a ∆-complex. Our second construction had two 1-simplices \( e \) and \( f \) and two vertices \( x \) and \( y \), with \( \partial(e) = [y] - [x] \) and \( \partial(f) = [x] - [y] \). Now our chain complex looks like

\[
\begin{align*}
C^\Delta_1(S^1) & \xrightarrow{\partial_1} C^\Delta_0(S^1) \\
\mathbb{Z}\{e, f\} & \xrightarrow{} \mathbb{Z}\{x, y\}
\end{align*}
\]

Thus \( \ker \partial_1 = \mathbb{Z}\{e + f\} \) and \( \text{im} \partial_1 = \mathbb{Z}\{y - x\} \). It follows that \( H^\Delta_1(S^1) = \mathbb{Z} \) and \( H^\Delta_0(S^1) = \mathbb{Z} \).

3. \( X = S^2 \). We built this as a ∆-complex by gluing together two 2-simplices \( z_1 \) and \( z_2 \) along their boundaries. Our chain complex is

\[
\begin{align*}
C^\Delta_2(S^2) & \xrightarrow{\partial_2} C^\Delta_1(S^2) \xrightarrow{\partial_1} C^\Delta_0(S^2) \\
\mathbb{Z}\{z_1, z_2\} & \xrightarrow{} \mathbb{Z}\{y_1, y_2, y_3\} \xrightarrow{} \mathbb{Z}\{x_1, x_2, x_3\}
\end{align*}
\]

\[
\begin{pmatrix}
1 & -1 & 1 \\
-1 & -1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & -1 \\
1 & -1 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 & 1 \\
-1 & -1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]
8. MON, SEPT. 12

We see that the kernel of \( \partial_2 \) is \( \mathbb{Z}\{z_1 - z_2\} \), so that \( H^1_2(S^2) \cong \mathbb{Z} \).

The image of \( \partial_2 \) is \( \mathbb{Z}\{y_1 - y_2 + y_3\} \), which is also seen to be the kernel of \( \partial_1 \). Thus \( H_0^1(S^2) = 0 \).

The third column of \( \partial_1 \) is the difference of the first two, so that the image of \( \partial_1 \) is \( \mathbb{Z}\{x_2 - x_1, x_3 - x_1\} \). It follows that
\[
H^0_0(S^2) = \mathbb{Z}\{x_1, x_2, x_3\}/\langle x_2 - x_1, x_3 - x_1 \rangle \cong \mathbb{Z}\{x_1\}.
\]

(4) \( X = T^2 \). The torus was similarly built by gluing two 2-simplices. The chain complex we obtain from our gluing data pictured to the right is

\[
\begin{array}{cccc}
C_2^\Lambda(T^2) & \xrightarrow{\partial_2} & C_1^\Lambda(T^2) & \xrightarrow{\partial_1} & C_0^\Lambda(T^2) \\
\mathbb{Z}\{z_1, z_2\} & \xrightarrow{(1 - 1, 1 - 1)} & \mathbb{Z}\{y_1, y_2, y_3\} & \xrightarrow{(0, 0, 0)} & \mathbb{Z}\{x\}
\end{array}
\]

The \( \partial_2 \) is the same as for \( S^2 \), so we again find \( H_2(T^2) \cong \mathbb{Z} \). But now \( \ker \partial_1 = \mathbb{Z}\{y_1, y_2, y_3\} \), so that
\[
H^1_1(T^2) = \mathbb{Z}\{y_1, y_2, y_3\}/\langle y_1 - y_2 + y_3 \rangle \cong \mathbb{Z}\{y_1, y_3\}.
\]

Since \( \text{im} \partial_1 = 0 \), we see that \( H^0_0(T^2) \cong \mathbb{Z} \).

(5) \( X = \mathbb{RP}^2 \). The projective plane was built from two simplices as in the picture to the right. This produces the chain complex

\[
\begin{array}{cccc}
C_2^\Lambda(\mathbb{RP}^2) & \xrightarrow{\partial_2} & C_1^\Lambda(\mathbb{RP}^2) & \xrightarrow{\partial_1} & C_0^\Lambda(\mathbb{RP}^2) \\
\mathbb{Z}\{z_1, z_2\} & \xrightarrow{(1 - 1, 1 - 1)} & \mathbb{Z}\{y_1, y_2, y_3\} & \xrightarrow{(0, 0, 0)} & \mathbb{Z}\{x_1, x_2\}
\end{array}
\]

In this case, \( \ker \partial_2 = 0 \), so that \( H^1_2(\mathbb{RP}^2) = 0 \).

For \( H^0_1 \), we see that \( \ker \partial_1 = \mathbb{Z}\{y_1, y_2 - y_3\} \). The image of \( \partial_2 \) is \( \mathbb{Z}\{y_1 - y_2 + y_3, y_1 + y_2 - y + 3\} \). Thus the quotient is
\[
H^1_1(\mathbb{RP}^2) = \mathbb{Z}\{y_1, y_2 - y_3\}/\langle y_1 - y_2 + y_3, y_1 + y_2 - y - 3 \rangle 
\cong \mathbb{Z}\{y_1\}/(2y_1) \cong \mathbb{Z}/2\mathbb{Z}.
\]

Finally, the image of \( \partial_1 \) is \( \mathbb{Z}\{x_2 - x_1\} \), so that
\[
H^0_0(\mathbb{RP}^2) \cong \mathbb{Z}\{x_1, x_2\}/\langle x_2 - x_1 \rangle \cong \mathbb{Z}\{x_1\}.
\]
Remark 8.1. In general, homology groups can be computed by finding the Smith normal form for the differentials. For example, in the $X = T^2$ case, the SNF for $\partial_2$ is $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, from which we read off that the kernel is 1-dimensional. The differential $\partial_1$ is simply zero, and up to a change of basis, the differential $\partial_2$ hits a generator. It follows that a rank two group survives to give $H_1^\Delta$.

Similarly, for $X = \mathbb{R}P^2$, the SNF for $\partial_2$ is $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, which shows that $H_2^\Delta = 0$. The SNF for $\partial_1$ is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so that the kernel is rank 2. Up to change of basis, $\partial_1$ hits one generator and twice the other, so that $H_1^\Delta$ is $\mathbb{Z}/2\mathbb{Z}$.

9. Wed, Sept. 14

Now that we have computed some examples, we want to develop the machine some more, so that we don’t need to compute by hand every time. The first question we will address is how homology behaves with respect to disjoint unions.

Proposition 9.1. Let $X$ and $Y$ be $\Delta$-complexes. There is then a canonical $\Delta$-complex structure on $X \sqcup Y$, and we have

$$H_n^\Delta(X \sqcup Y) \cong H_n^\Delta(X) \oplus H_n^\Delta(Y)$$

for all $n$.

Proof. The point is that we already have a direct sum decomposition on the level of chain complexes. Namely, if we write $\Delta_n(X)$ for the set of $n$-simplices of $X$, then

$$\Delta_n(X \sqcup Y) = \Delta_n(X) \sqcup \Delta_n(Y),$$

so that

$$C_n^\Delta(X \sqcup Y) = \mathbb{Z}\{\Delta_n(X \sqcup Y)\} \cong \mathbb{Z}\{\Delta_n(X)\} \oplus \mathbb{Z}\{\Delta_n(Y)\} = C_n^\Delta(X) \oplus C_n^\Delta(Y).$$

Moreover, the differential is compatible with this splitting, in the sense that we have the commutative diagram

$$\begin{array}{ccc}
C_n^\Delta(X \sqcup Y) & \xrightarrow{\partial_n} & C_{n-1}^\Delta(X \sqcup Y) \\
\cong & & \cong \\
C_n^\Delta(X) \oplus C_n^\Delta(Y) & \xrightarrow{\partial_n \oplus \partial_n} & C_{n-1}^\Delta(X) \oplus C_{n-1}^\Delta(Y)
\end{array}$$

This shows that $H_n^\Delta(X \sqcup Y) \cong H_n^\Delta(X) \oplus H_n^\Delta(Y)$ for all $n$. 

Another way we might think of this result is that we have the two inclusions $\iota_X : X \hookrightarrow X \sqcup Y$ and $\iota_Y : Y \hookrightarrow X \sqcup Y$. We might expect each of these maps to induce a map on homology, such as $H_*(\iota_X) : H_*(X) \longrightarrow H_*(X \sqcup Y)$, and that the isomorphism of Proposition 9.1 is simply the sum $H_*(\iota_X) + H_*(\iota_Y)$. This raises the question:

Question 9.2. Is homology a functor?

The answer depends on how you interpret the question. So far, we have only defined homology of $\Delta$-complexes. So we can ask if each $H_n^\Delta$ defines a functor

$$H_n^\Delta : \Delta\text{Top} \longrightarrow \text{AbGp}$$

for some suitable category $\Delta\text{Top}$ of $\Delta$-complexes. The morphisms in this category, which we will call the $\Delta$-maps, are maps satisfying the following condition: for each simplex $\sigma : \Delta^n \longrightarrow X$ of $X$, the composition $\Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y$ is an $n$-simplex of $Y$. Note that when we say “is an $n$-simplex”, we also mean with its given orientation. Now by the definition of a $\Delta$-map, $f$ will induce a function

$$\hat{f} : \Delta^n(X) \longrightarrow \Delta^n(Y)$$
for each \( n \) and therefore also a homomorphism

\[
f_* : C_n^\Delta(X) \rightarrow C_n^\Delta(Y)
\]

for each \( n \). We would like to say that this gives rise to homomorphisms on homology. In order to conclude this, we need to know how \( f_* \) interacts with the differential (boundary operator).

Note that if \( d^i : \Delta^{n-1} \hookrightarrow \Delta^n \) is the \( i \)th face inclusion, the composition with \( d^i \) induces a function \( d_i : \Delta^n(X) \rightarrow \Delta^{n-1}(X) \). Since \( d_i \) and \( \hat{f} \) are given by composition with \( d^i \) and \( f \), respectively, we conclude that the diagram

\[
\begin{array}{ccc}
\Delta^n(X) & \xrightarrow{\hat{f}} & \Delta^n(Y) \\
\downarrow d_i & & \downarrow d_i \\
\Delta^{n-1}(X) & \xrightarrow{f} & \Delta^{n-1}(Y)
\end{array}
\]

commutes for each \( n \). This implies that the diagram

\[
\begin{array}{ccc}
C_n^\Delta(X) & \xrightarrow{f_*} & C_n^\Delta(Y) \\
\downarrow d_i & & \downarrow d_i \\
C_{n-1}^\Delta(X) & \xrightarrow{f_*} & C_{n-1}^\Delta(Y)
\end{array}
\]

commutes for each \( n \). This is precisely the notion of a map of chain complexes.

**Definition 9.3.** Let \((C_*, \partial_C^*)\) and \((D_*, \partial_D^*)\) be chain complexes. Then a **chain map** \( f_* : (C_*, \partial_C^*) \rightarrow (D_*, \partial_D^*)\) is a sequence of homomorphisms \( f_n : C_n \rightarrow D_n \), for each \( n \), such that each diagram

\[
\begin{array}{ccc}
C_n & \xrightarrow{f_n} & D_n \\
\downarrow \partial_C^* & & \downarrow \partial_D^* \\
C_{n-1} & \xrightarrow{f_{n-1}} & D_{n-1}
\end{array}
\]

commutes for each \( n \).

We set up this definition in order to get

**Proposition 9.4.** A chain map \( f_* : (C_*, \partial_C^*) \rightarrow (D_*, \partial_D^*)\) induces homomorphisms \( f_n : H_n(C_*, \partial_C^*) \rightarrow H_n(D_*, \partial_D^*)\) for each \( n \).

**Proof.** Let \( x \in C_n \) be a cycle, meaning that \( \partial_C(x) = 0 \). Then \( \partial_D(f_n(x)) = f_{n-1}(\partial_C(x)) = f_{n-1}(0) = 0 \), so that \( f_n(x) \) is a cycle in \( D_n \). In order to get a well-defined map on homology, we need to show that if \( x \) is in the image of \( \partial_{C, n+1} \), then \( f_n(x) \) is in the image of \( \partial_{D, n+1} \). But if \( x = \partial_{C, n+1}(y) \), then \( f_n(x) = f_n(\partial_{C, n+1}(y)) = \partial_{D, n+1}f_{n+1}(y) \), which shows that \( f_n(x) \) is a boundary. \( \blacksquare \)
There is an obvious way to compose chain maps, so that chain complexes and chain maps form a category $\text{Ch}_{\geq 0}(\mathbb{Z})$.

**Proposition 10.1.** The assignment $X \mapsto (C^\Delta_*(X), \partial_*)$ and $f \mapsto f_*$ defines a functor

$$C^\Delta_* : \Delta \text{Top} \rightarrow \text{Ch}_{\geq 0}(\mathbb{Z}).$$

Given the above discussion, it only remains to show that this construction takes identity morphisms to identity morphisms and that it preserves composition. We leave this as an exercise.

Note that the sequence of homology groups $H_i(C_*(X), \partial_C)$ of a chain complex is not quite a chain complex, since there are no differentials between the homology groups. You can think of this as a degenerate case of a chain complex, in which all differentials are zero. But it is more common to simply call this a **graded abelian group**. If $X_*$ and $Y_*$ are graded abelian groups, then a graded map $f_* : X_* \rightarrow Y_*$ is simply a collection of homomorphisms $f_n : X_n \rightarrow Y_n$. Graded maps compose in the obvious way, so that we get a category $\text{GrAb}$ of graded abelian groups. Then Proposition 9.4 is the main step in proving

**Proposition 10.2.** Homology defines a functor

$$H_* : \text{Ch}_{\geq 0}(\mathbb{Z}) \rightarrow \text{GrAb}.$$

The composition of two functors is always a functor. Thus Proposition 10.1 and Proposition 10.2 combine to yield

**Proposition 10.3.** Simplicial homology defines a functor

$$H^\Delta_* : \Delta \text{Top} \rightarrow \text{GrAb}.$$

This means that simplicial homology is a reasonably well-behaved construction.

**Example 10.4.** Consider the $\Delta$-map depicted by the figure.

![Diagram](image)

Note that there is a unique $\Delta$-map compatible with these $\Delta$-structures depicted. Calling the map $\varphi$, we must have $\varphi(e) = \varphi(f) = g$ and $\varphi(x) = \varphi(y) = v$. The induced chain map is

$$\begin{bmatrix}
  Z\{e, f\} & \text{(1)} \\
  \text{(1)} & \text{(1)}
\end{bmatrix} \rightarrow \begin{bmatrix}
  Z\{g\} \\
  0
\end{bmatrix}$$

We see that the induced map on homology $H_i(S^1) \rightarrow H_i(S^1)$ sends a generator to twice a generator when $i = 1$, but sends a generator to a generator when $i = 0$.

Still, the notion of $\Delta$-map is quite restrictive. For instance, there is no $\Delta$-map in the other direction in the above example. Moreover, if $X$ is a $\Delta$-complex with at least one simplex that is
not 0-dimensional, then there is no $\Delta$-map $X \to \ast$. It would be great to have functoriality with respect to a larger collection of maps between spaces.

There is another variant of homology that is more convenient when working with based spaces. Thus let $X$ be a $\Delta$-complex, with a particular 0-simplex $x_0$ identified as the basepoint. Then the inclusion $\{x_0\} \hookrightarrow X$ is a $\Delta$-map, so that we get a well-defined homomorphism $H_*(\{x_0\}) \to H_*(X)$.

**Definition 10.5.** We define the **reduced homology** groups $\tilde{H}_*(X)$ of $(X, x_0)$ to be the cokernel of this map $H_*(\{x_0\}) \to H_*(X)$.
11. Mon, Sept. 19

Since $H_n(\{x_0\}) = 0$ if $n > 0$, the reduced homology groups are the same as the ordinary homology groups, except in degree 0. We have simply reduced away the subgroup of $H_0(X)$ generated by the basepoint. In fact, this subgroup is infinite. To see this, consider the chain maps

$$C^\Delta_\ast(\{x_0\}) \xrightarrow{\varepsilon_0} C^\Delta_\ast(X) \xrightarrow{\varepsilon} C^\Delta_\ast(\{x_0\}),$$

where $\varepsilon_0$ is the homomorphism that sends every 0-simplex to the generator $x_0$. To see that this makes $\varepsilon$ into a chain map, it suffices to see that

$$\begin{array}{ccc}
C^\Delta_1(X) & \xrightarrow{\varepsilon_1} & C^\Delta_1(\{x_0\}) = 0 \\
\downarrow \partial_1 & & \downarrow \partial_1 \\
C^\Delta_0(X) & \xrightarrow{\varepsilon_0} & C^\Delta_0(\{x_0\}) = \mathbb{Z}\{x_0\}
\end{array}$$

commutes. But if $e$ is a 1-simplex from $v_0$ to $v_1$, then $\varepsilon\partial_1(e) = \varepsilon(v_1 - v_0) = x_0 - x_0 = 0$ as desired. Since $\varepsilon \circ \iota_\ast = \text{id}_{C^\Delta_\ast(\{x_0\})}$, the same must be true after passage to homology (by Prop. 10.2), giving a splitting

$$\mathbb{Z} \cong H^\Delta_0(\{x_0\}) \longrightarrow H^\Delta_0(X) \longrightarrow H^\Delta_0(\{x_0\}).$$

Thus we have

$$H^\Delta_0(X) \cong \tilde{H}^\Delta_0(X) \oplus \mathbb{Z}.$$ 

Let us try to understand some of the homology group functors more closely.

**Proposition 11.1.** For any $\Delta$-complex $X$, the group $H^\Delta_0(X)$ is (isomorphic to) the free abelian group on the set $\pi_0(X)$ of path components of $X$. In particular, for any path-connected space, this group is just $\mathbb{Z}$.

**Proof.** Let $X' \subseteq X$ be the union of all 1-simplices in $X$, and let $\iota : X' \hookrightarrow X$ be the inclusion.

**Lemma 11.2.** The inclusion induces a bijection $\iota_* : \pi_0(X') \cong \pi_0(X)$.

**Proof.** We define $r : \pi_0(X) \longrightarrow \pi_0(X')$ as follows: for any $x \in X$, pick a simplex $\sigma$ containing $x$. The define $r(x)$ to be the path-component in $X'$ of any point $y$ lying in a 1-dimensional face of $\sigma$. This does not depend on the choice of $y$ since the union of the 1-dimensional faces of $\sigma$ is path-connected. It also does not depend on the choice of $\sigma$, since if $\sigma'$ is another such choice, then $\sigma \cap \sigma'$ is a simplex containing $x$, and we can pick our $y$ from this intersection.

It is clear that $r \circ \iota_\ast$ is the identity on $\pi_0(X')$. On the other hand, if $x \in X$ then any representative $y$ for $r(x)$ must lie in some simplex $\sigma$ in $X$ that also contains $x$. Since $\sigma$ is path-connected, this implies that $i \circ r$ is the identity of $\pi_0(X).$ \hfill \blacksquare

Note that the inclusion $\iota : X' \hookrightarrow X$ also induces isomorphisms $C_i^\Delta(X') \cong C_i^\Delta(X)$ for $i = 0, 1,$ which is all that is relevant for calculation of $H_0$. Thus, by the above lemma, we may without loss of generality replace $X$ by $X'$.

Recall that $H^\Delta_0(X) = C^\Delta_0(X)/\text{im}(\partial_1)$. Let $p : \Delta^0(X) \longrightarrow \pi_0(X)$ be the function that sends each vertex of $X$ to its path-component. This induces a homomorphism $p_* : C^\Delta_0(X) \longrightarrow \mathbb{Z}\{\pi_0(X)\}$, since the free abelian group construction is a functor. If $e \in \Delta^1(X)$ is a 1-simplex in $X$, then both endpoints of $e$ lie in the same path component of $X$, since $e$ is precisely a path from one endpoint to the other. It follows that $p_*(\partial_1(e)) = 0$ in $\mathbb{Z}\{\pi_0(X)\}$. This shows that $p_*$ induces a homomorphism

$$p_* : H^\Delta_0(X; \mathbb{Z}) \longrightarrow \mathbb{Z}\{\pi_0(X)\}.$$
Note that each path-component of \( X \) must contain a vertex, since if \( x \in X \), then \( x \) must lie in some 1-simplex \( \sigma \) of \( X \). But there is a straight-line path in the simplex \( \sigma \) from \( x \) to either endpoint of \( \sigma \), showing that the vertex lies in the same path-component as \( x \). This shows that \( p_\ast \) is surjective.

Making a choice of 0-simplex in each path-component of \( X \) provides a function \( s : \pi_0(X) \to \Delta^0(X) \) and therefore a function

\[
s_\ast : \mathbb{Z}\{\pi_0(X)\} \to C_0^\ast(X) \to H_0(X;\mathbb{Z}).
\]

It remains to show that the composition

\[
H_0^\ast(X;\mathbb{Z}) \xrightarrow{p_\ast} \mathbb{Z}\{\pi_0(X)\} \xrightarrow{s_\ast} H_0^\ast(X;\mathbb{Z})
\]

is the identity. For any 0-chain \( \sum n_i x_i \) in \( X \), the composition produces the 0-chain \( \sum n_i s(x_i) \), so it suffices to show these two 0-chains agree modulo the image of \( \partial_1 \). It suffices to show that \( x_i - s(x_i) \) is in the image of \( \partial_1 \). But \( x_i \) and \( s(x_i) \) are both 0-simplices lying in the same component of \( X \), so that there must be a path between them which is a finite union of 1-simplices (since paths are compact). Applying \( \partial_1 \) to the corresponding finite sum of 1-simplices produces the difference \( x_i - s(x_i) \).

12. Wed, Sept. 21

Proposition 11.1 is not stated optimally, in the sense that it does not say to what extent this depends on \( X \). That is, both \( H_0(-;\mathbb{Z}) \) and \( \mathbb{Z}\{\pi_0(-)\} \) can be viewed as functors \( \Delta\text{Top} \to \text{AbGp} \). A stronger version of the proposition would say that these are isomorphic as functors. This brings up the question of what should be the notion of a “morphism between functors”.

**Natural Transformations**

**Definition 12.1.** Let \( F, G : \mathcal{C} \to \mathcal{D} \) be functors. A natural transformation \( \eta : F \to G \) is a collection of maps \( \eta_C : F(C) \to G(C) \), one for each \( C \in \mathcal{C} \), such that for any \( C, C' \in \mathcal{C} \) and any \( f \in \text{Hom}_{\mathcal{C}}(C, C') \), the following diagram commutes:

\[
\begin{array}{ccc}
F(C) & \xrightarrow{F(f)} & F(C') \\
\downarrow{\eta_C} & & \downarrow{\eta_C'} \\
G(C) & \xrightarrow{G(f)} & G(C')
\end{array}
\]

The morphism \( \eta_C \) is sometimes called the component of \( \eta \) at the object \( C \).

**Example 12.2.**

1. We previously described abelianization as a functor \( (-)_{ab} : \text{Gp} \to \text{AbGp} \). Now \( \text{AbGp} \) includes in \( \text{Gp} \) as a subcategory, so we can think of abelianization as giving a functor \( (-)_{ab} : \text{Gp} \to \text{Gp} \). The identity functor \( \text{Id}_{\text{Gp}} : \text{Gp} \to \text{Gp} \) is another functor with the same domain and codomain. For any group \( G \), the abelianization \( G_{ab} \) is defined as a quotient of \( G \), so that there is a quotient homomorphism \( \eta : G \to G_{ab} \). This homomorphism is “natural in \( G \)”, in the sense that there is a natural transformation \( \eta : \text{Id}_{\text{Gp}} \to (-)_{ab} \) whose components are \( \eta_G \). In other words, for each group homomorphism \( \varphi : H \to G \), the diagram

\[
\begin{array}{ccc}
H & \xrightarrow{\varphi} & G \\
\downarrow{\eta_H} & & \downarrow{\eta_G} \\
H_{ab} & \xrightarrow{\varphi_{ab}} & G_{ab}
\end{array}
\]
commutes. If you look back at Example 3.8(4), this was precisely the diagram used to define the morphism $\varphi_{ab}$.

(2) Recall that for any based $\Delta$-complex $(X, x_0)$, we have a quotient homomorphism

$$H^n_\Delta(X) \to \tilde{H}^n_\Delta(X, x_0).$$

This is a natural transformation of functors $\Delta \text{Top} \to \text{AbGp}$. In order to make sense of this claim, we first need to discuss the functoriality of reduced homology. Let $f : X \to Y$ be a based $\Delta$-map. Then the induced map on reduced homology is defined to be the dashed arrow coming from the universal property of the quotient:

$$\begin{array}{ccc}
H^n_\Delta(x_0) & \to & H^n_\Delta(y_0) \\
\downarrow & & \downarrow \\
H^n_\Delta(X) & \xrightarrow{f_*} & H^n_\Delta(Y) \\
\downarrow & & \downarrow \\
\tilde{H}^n_\Delta(X, x_0) & \to & \tilde{H}^n_\Delta(Y, y_0).
\end{array}$$

Note the the commutativity of the bottom square is precisely the statement that the quotient $H^n_\Delta \to \tilde{H}^n_\Delta$ is a natural transformation.

(3) Let $k$ be a field. For any vector space $V$ over $k$, we define the dual vector space $V^* := \text{Hom}_k(V, k)$.

This is the vector space of linear functionals on $V$. In fact the assignment $V \mapsto V^*$ determines a contravariant functor $(-)^* : \text{Vect}_k \to \text{Vect}_k$. Composing this functor with itself gives a covariant functor $(-)^{**} : \text{Vect}_k \to \text{Vect}_k$ which sends a vector space to its double dual. Because we will need this below, we note that if $\phi : V \to W$ is a linear map, then the induced linear map $\phi^{**} : V^{**} \to W^{**}$ is given by $\phi^{**}(X)(\lambda) = X(\lambda \circ \phi)$.

Now fix $v \in V$. We define a function $\text{eval}_v : V^* \to k$ by $\text{eval}_v(\lambda) = \lambda(v)$. This is in fact $k$-linear and so determines an element of $(V^*)^*$. But now the assignment $v \mapsto \text{eval}_v$ can also be seen to be $k$-linear, so we have a homomorphism $\text{eval}_V : V \to V^{**}$. This map is an isomorphism if $V$ is finite dimensional. Moreover, the homomorphisms $V \to V^{**}$ fit together to determine a natural transformation of functors $\text{Id} \to (-)^{**}$. Again, this means that for every linear map $\phi : V \to W$, the diagram

$$\begin{array}{ccc}
V & \xrightarrow{\phi} & W \\
\text{eval}_V \bigg\downarrow & & \bigg\downarrow \text{eval}_W \\
V^{**} & \xrightarrow{\phi^{**}} & W^{**}
\end{array}$$

commutes. To see this, let $\lambda : W \to k$ be an element of $W^*$. Then

$$[\text{eval}_W \circ \phi](v)(\lambda) = \lambda(\phi(v)) = \text{eval}_V(v)(\lambda \circ \phi) = \phi^{**}(\text{eval}_V(v))(\lambda) = [\phi^{**} \circ \text{eval}_V](v)(\lambda)$$

This is a precise version of the statement that a finite-dimensional vector space is canonically isomorphic to its double dual.

**Remark 12.3.** For finite-dimensional vector spaces, it is also true that $V$ is isomorphic to $V^*$, but to construct such an isomorphism one must first choose a basis for $V$. Thus the isomorphism $V \cong V^*$ cannot be natural.
We saw that if we restrict ourselves to \((\mathbf{Vect})_{k,d}\), then \text{eval} determines a natural transformation \(\text{Id} \to (-)^{**}\) in which each component \(V \to V^{**}\) is an isomorphism. More generally, a natural transformation \(\eta : F \to G\) between functors \(F, G : \mathcal{C} \to \mathcal{D}\) is called a \textbf{natural isomorphism} if \(\eta_C : F(C) \to G(C)\) is an isomorphism for each \(C \in \mathcal{C}\). This is equivalent to asking that there be a natural transformation \(\delta : G \to F\) such that \(\delta \circ \eta = \text{id}_F\) and \(\eta \circ \delta = \text{id}_G\).

**Proposition 13.1.** The isomorphisms of Proposition 11.1 assemble together to yield a natural isomorphism of functors \(H^0_\Delta (-; \mathbb{Z}) \cong \mathbb{Z} \{ \pi_0(-) \}\).

**Proof.** We must show that for each \(\Delta\)-map of \(\Delta\)-complexes \(f : X \to Y\), the square

\[
\begin{array}{ccc}
H^0_\Delta(X; \mathbb{Z}) & \xrightarrow{f_*} & H^0_\Delta(Y; \mathbb{Z}) \\
\downarrow & & \downarrow \\
\mathbb{Z}\{\pi_0(X)\} & \xrightarrow{\mathbb{Z}\{\pi_0(f)\}} & \mathbb{Z}\{\pi_0(Y)\}
\end{array}
\]

commutes. The vertical maps are induced by maps out of \(C^0_\Delta\), so that it suffices to check that

\[
\begin{array}{ccc}
C^0_\Delta(X; \mathbb{Z}) & \xrightarrow{f_*} & C^0_\Delta(Y; \mathbb{Z}) \\
\downarrow & & \downarrow \\
\mathbb{Z}\{\pi_0(X)\} & \xrightarrow{\mathbb{Z}\{\pi_0(f)\}} & \mathbb{Z}\{\pi_0(Y)\}
\end{array}
\]

commutes. Starting with a 0-chain \(\sum_i n_i x_i\), either composition gives the element \(\sum_i n_i f(x_i)\). \(\blacksquare\)

We have now given a description of the \textbf{functor} \(H^0_\Delta (-; \mathbb{Z})\). What about \(H^1_\Delta\) (or higher homology)? There is a nice answer for \(H_1\), but it is more convenient to address using a different model for homology.

**Singular Homology**

Simplicial homology is great because, as we have seen, it is very computable. On the other hand, it has the serious defect that it is only defined on \(\Delta\)-complexes (and \(\Delta\)-maps). We introduce here a variant that is defined on all spaces.

The basic idea is this: in defining simplicial homology, we took the chains to be free abelian on the set \(\Delta^n(X)\) of simplices of \(X\), which we noted could be thought of as maps \(\Delta^n \to X\). If you look at the formula for the differential, it only uses the formulation as maps from simplices to \(X\).

**Definition 13.2.** Given a space \(X\), define a \textbf{singular} \(n\)-\textbf{simplex} of \(X\) to be any continuous map \(\Delta^n \to X\). We define the group of \textbf{singular} \(n\)-\textbf{chains} on \(X\) to be

\[C_n(X) := \mathbb{Z}\{\textbf{Top}(\Delta^n, X)\}.
\]

We sometimes write \(\text{Sing}_n(X) := \text{Top}(\Delta^n, X)\). Again, the formula for the differential in Definition 6.2 makes just as much sense in the singular context.

**Definition 13.3.** Given a space \(X\), we define the \textbf{singular homology groups} of \(X\) to be the homology groups of the chain complex \((C_*(X), \partial)\).

If \(X\) is a \(\Delta\)-complex, then any simplex of \(X\) may be thought of as a singular simplex. This gives natural maps \(C^*_\Delta(X) \to C_*(X)\) of chain complexes and therefore natural maps of graded groups \(H^*_\Delta(X) \to H_*(X)\). We will see later that these are isomorphisms.
Notice that the groups \( C_\ast(X) \) are much bigger than the groups \( C^\Delta_\ast(X) \). For a \( \Delta \)-complex with finitely many simplices, the latter groups all have finite rank, whereas this is almost never the case for the groups \( C_\ast(X) \).

**Example 13.4.** Consider \( X = \ast \). Then \( C_n(\{\ast\}) = \mathbb{Z}\{\text{Top}^n(\ast, \{\ast\})\} \cong \mathbb{Z} \) for all \( n \). The differential \( \partial_n : C_n(\{\ast\}) \to C_{n-1}(\{\ast\}) \) takes the (constant) singular \( n \)-simplex \( c_n \) to the alternating sum

\[
\sum_i (-1)^i c_{n-1} = \begin{cases} 
  c_{n-1} & \text{n even} \\
  0 & \text{n odd}
\end{cases}.
\]

In other words, the chain complex is

\[
\ldots \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z},
\]

so that the only nonzero homology group is \( H_0(\ast) \cong \mathbb{Z} \).

But already for \( X = \Delta^1 \), the chain groups are infinite rank, and computing becomes impractical. On the other hand, the singular homology groups have much better properties.

**Proposition 13.5.** Singular homology defines a functor

\[
H_\ast : \text{Top} \to \text{GrAb}.
\]

**Proof.** The proof strategy is the same as for Proposition 10.3. The main point is that, for any continuous map \( f : X \to Y \), composition with \( f \) defines a function \( \hat{f} : \text{Sing}_n(X) \to \text{Sing}_n(Y) \). The rest of the argument is the same. \( \blacksquare \)

14. Mon, Sept. 21

This implies, for instance, that homeomorphic spaces have isomorphic singular homology groups. But now that we’ve been given an inch, we want a whole yard. We will show that homology factors through the homotopy category.

It is not true that the singular chains functors \( C_\ast(-) : \text{Top} \to \text{Ch}_{\geq 0}(\mathbb{Z}) \) factor through the homotopy category, so a new idea is needed, that of a chain homotopy between chain maps of chain complexes.

**Definition 14.1.** Let \( f, g : C_\ast \Rightarrow D_\ast \) be chain maps. Then a chain homotopy \( h \) is a sequence of homomorphisms \( h_n : C_n \to D_{n+1} \) satisfying

\[
\partial_{n+1}^D h_n(c) + h_{n-1}(\partial_{n}^C c) = g(c) - f(c).
\]

**Remark 14.2.** It is probably not apparent why this notion deserves the name “chain homotopy”. A homotopy in topology means a map \( I \times X \to Y \), and it turns out that there is a chain complex \( I_\ast \) such that a chain homotopy in the sense given above is the same as a chain map \( I_\ast \otimes X_\ast \to Y_\ast \), where here \( \otimes \) means the tensor product of chain complexes.

**Proposition 14.3.** Let \( h : X \times I \to Y \) be a homotopy between \( f = h_0 \) and \( g = h_1 \). Then there exists a chain homotopy \( h_\ast^C \) between \( C_\ast(f) \) and \( C_\ast(g) \).

We give the full proof below, but let’s first sketch it out in low dimensions. We start with \( n = 0 \). If \( x \in X \) is a singular 0-simplex (in other words, a point), we define \( h_0(x) := h_x \), the path in \( Y \) traced out by the homotopy \( h \) at \( x \). We then have

\[
\partial^Y_1(h_0(x)) + h_{-1}(\partial^X_0 x) = h_x(1) - h_x(0) + 0 = g(x) - f(x)
\]
as desired. Now we try $n = 1$. So let $\sigma$ be a path in $X$, say from $x$ to $x'$. Then $h_1(\sigma)$ should be a linear combination of two-simplices in $Y$. On the path $\sigma$, the homotopy $h$ traces out a square in $Y$, which we can decompose into 2-simplices as in the picture

We then define $h_1(\sigma) = h'' - h'$ and check

$$\partial_2^Y(h_1(\sigma)) + h_0(\partial_1^X\sigma) = \partial_2^Y(h'' - h') + h_0(x' - x)$$

$$= g(\sigma) - d + h_x - [h_{x'} - d + f(\sigma)] + h_{x'} - h_x = g(\sigma) - f(\sigma)$$

as we wanted.

15. WED, SEPT. 28

Proof. If $\sigma$ is a singular $n$-simplex of $X$, then $h$ gives the composite

$$\Delta^n \times I \xrightarrow{\sigma \times \text{id}} X \times I \xrightarrow{h} Y.$$ 

Lemma 15.1. The product $\Delta^n \times I$ has a canonical $\Delta$-complex structure with $n + 1$ (simplicial) $(n + 1)$-simplices.

Proof. We sketch this structure for $n = 1$ and 2.

Vertices of the simplices of $\Delta^n \times I$ are labelled by pairs $(j, k)$, where $0 \leq j \leq n$ and $0 \leq k \leq 1$. The $(n + 1)$-simplices each include a single “vertical” 1-simplex with endpoints $(i, 0)$ and $(i, 1)$. We denote by $p_i : \Delta^{n+1} \hookrightarrow \Delta^n \times I$ the inclusion of the simplex which includes the vertical edge at $(i, 0)$.

We abuse notation and write $p_i(\sigma)$ for the composition

$$\Delta^{n+1} \xrightarrow{p_i} \Delta^n \times I \xrightarrow{\sigma \times \text{id}} X \times I.$$ 

We then define

$$h_n^C(\sigma) = \sum_{i=0}^{n} (-1)^i h p_i(\sigma).$$

To verify that this is a chain homotopy as claimed, we make several observations:
(1) The $\Delta$-complex $\Delta^n \times I$ has $n$ “internal” $n$-simplices, with vertices
$$(0,0), (1,0), \ldots, (i,0), (i+1,1), \ldots, (n,1).$$

When calculating $\partial_{n+1}(h_n^C(\sigma))$, this $n$-simplex shows up as both $p_i(\sigma) \circ d^i$ and $p_{i+1}(\sigma) \circ d^i$. Since $p_i(\sigma)$ and $p_{i+1}(\sigma)$ appear with opposite signs in $h_n^C(\sigma)$, these two will cancel out in $\partial_{n+1}(h_n^C(\sigma))$.

Thus the only terms that remain in $\partial_{n+1}(h_n^C(\sigma))$ are the “external” $n$-simplices, which contain a vertical edge, as well as the “horizontal” $n$-simplices $g(\sigma)$ and $f(\sigma)$.

(2) Each of the external $n$-simplices occurs as the face of a single $n+1$-simplex and thus appears only once in $\partial_{n+1}(h_n^C(\sigma))$. Moreover, each of these can be written in the form $p_i(\sigma \circ d^i)$ and therefore appears in $h_{n-1}^C(\partial_n(\sigma))$. In fact, every term of $h_{n-1}^C(\partial_n(\sigma))$ arises in this way.

\[\square\]

**Proposition 15.2.** If $f, g : C_* \Rightarrow D_*$ are chain-homotopic, then $H_*(f) = H_*(g)$.

**Proof.** It suffices to show that for any $n$-cycle $c$, the difference $g(c) - f(c)$ is in the image of the boundary map. But this comes directly from the definition of chain-homotopy, since $h_{n-1}(\partial_n^C(c)) = h_{n-1}(0) = 0$.

Combining propositions 14.3 and 15.2 gives

**Proposition 15.3** (Homotopy invariance). If $f, g : X \Rightarrow Y$ are homotopic, then $H_*(f) = H_*(g)$.

**Corollary 15.4.** If $X \simeq Y$, then $H_*(X) \cong H_*(Y)$.

So the homology of any contractible space agrees with the homology of a point. Said differently, the reduced homology of any contractible space is zero.

16. Fri, Sept. 30

**Coefficients**

Recall that when we originally introduced homology, we wrote $H_*(X; \mathbb{Z})$. We know how to let $X$ vary, but the notation suggests that we should also be able to substitute for the $\mathbb{Z}$ as well.

**Definition 16.1.** Given an abelian group $M$, we define the group of singular chains with coefficients in $M$ to be
$$C_*(X; M) := \bigoplus_{\Delta^\text{int}(X)} M.$$ If you know about tensor products, another description of this is $C_*(X; M) \cong C_*(X) \otimes_{\mathbb{Z}} M$. The singular homology groups with coefficients in $M$ are then defined by
$$H_*(X; M) := H_*(C_*(X; M)).$$

Similarly, the simplicial homology groups with coefficients in $M$ are defined by
$$H^\Delta_*(X; M) := H_*(C^\Delta_*(X; \mathbb{Z}) \otimes M).$$

This simply means that when we write an $n$-chain as a linear combination $\sum_{i} n_i \sigma_i$, each $n_i$ should be in $M$ rather than $\mathbb{Z}$. The

The most common choices for $M$, other than $\mathbb{Z}$, are the fields $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{F}_p$. 

24
Example 16.2. $X = S^1$. If we take the $\Delta$-complex having a single 0-simplex and single 1-simplex, then the chain complex with coefficients in $M$ is just

$$
\begin{array}{c}
C_1^\Delta(S^1;M) \xrightarrow{\partial_1} C_0^\Delta(S^1;M) \\
M\{e\} \quad M\{v\},
\end{array}
$$

where $\partial_1 = 0$. It follows that $H_1^\Delta(S^1;M) = M$ and $H_0^\Delta(S^1;M) = M$.

A more interesting example is

Example 16.3. $X = \mathbb{RP}^2$, $M = k$ is a field. The chain complex with coefficients in $k$ is

$$
\begin{array}{ccc}
C_2^\Delta(\mathbb{RP}^2) & \xrightarrow{\partial_2} & C_1^\Delta(\mathbb{RP}^2) \xrightarrow{\partial_1} C_0^\Delta(\mathbb{RP}^2) \\
k\{z_1, z_2\} & \xrightarrow{(1 \ 1 \ 1 \ 1)} & k\{y_1, y_2, y_3\} \xrightarrow{(0 \ -1 \ -1 \ 0)} k\{x_1, x_2\}.
\end{array}
$$

The Smith Normal Form that we previously found over $\mathbb{Z}$ gives a reduced echelon form over $k$. The echelon form for $\partial_1$ is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, but the Smith Normal Form $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ for $\partial_2$ gives a reduced echelon form of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ if $\text{char}(k) \neq 2$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ if $\text{char}(k) = 2$. Thus we read off the homology groups

$$
H_0^\Delta(\mathbb{RP}^2;F_2) \cong F_2, \quad H_1^\Delta(\mathbb{RP}^2;F_2) \cong F_2, \quad H_2^\Delta(\mathbb{RP}^2;F_2) \cong F_2
$$

and

$$
H_0^\Delta(\mathbb{RP}^2;k) \cong k, \quad H_1^\Delta(\mathbb{RP}^2;k) = 0, \quad H_2^\Delta(\mathbb{RP}^2;k) = 0
$$

if $\text{char}(k) \neq 2$.

For a given space $X$, the assignment $M \mapsto H_*(X;M)$ is functorial in $M$, meaning that any homomorphism $\phi : M \rightarrow N$ induces a homomorphism $\phi_* : H_*(X;M) \rightarrow H_*(X;N)$ by simply applying $\phi$ to the coefficients in any $n$-chain in $X$. Even better, the homomorphisms $\phi_*$ are natural in $X$. But there is an even stronger connection between the $H_*(X;M)$ as $M$ varies.

Recall that a short exact sequence is a chain complex

$$
0 \rightarrow K \xrightarrow{i} M \xrightarrow{q} Q \rightarrow 0
$$

that is exact (has no homology). Exactness at the three spots means

1. $\ker(i) = 0$, so that $i$ is injective
2. $\ker(q) = \im(i)$, and
3. $\im(q) = C$, so that $q$ is surjective.

A standard example is

$$
0 \rightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 0.
$$

The question is what does this short exact sequence of coefficients buy for us at the level of homology?
Let’s first consider what happens at the level of chain complexes. The first observation is that we get a short exact sequence of chain complexes

\[
\cdots \to C_{n+1}(X) \xrightarrow{p} C_n(X) \xrightarrow{\partial} C_{n+1}(X)/p \to 0
\]

\[
\cdots \to C_{n-1}(X) \xrightarrow{p} C_{n}(X) \xrightarrow{\partial} C_{n-1}(X)/p \to 0
\]

This means that each row is a short exact sequence and that moreover all squares in the above diagram commute. (Note that the fact that each row is exact relies on the fact that each group \( C_n(X) \) is free abelian.)

The Long Exact Sequence from a short exact sequence in coefficients

**Proposition 17.1.** A short exact sequence \( 0 \to A \xrightarrow{i} B \xrightarrow{q} C \to 0 \) of chain complexes induces a long exact sequence in homology

\[
\cdots \to H_{n+1}(C) \xrightarrow{\delta} H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{q_*} H_n(C) \xrightarrow{\delta} H_{n-1}(A) \to \cdots
\]

**Proof.** We start with the construction of the “connecting homomorphism \( \delta \)”. Thus let \( c \in C_n \) be a cycle. Choose a lift \( b \in B_n \), meaning that \( q(b) = c \). Then we have \( q(\partial_n(b)) = \partial_n(q(b)) = \partial_n(c) = 0 \). Since the rows are exact, we have \( \partial_n(b) = i(a) \) for some unique \( a \in A_{n-1} \), and we define

\[
\delta(c) := a.
\]

It remains to see how \( a \) depends on the choice of \( b \). Thus let \( d \in \ker(q) \), so that \( q(b + d) = c \). By exactness, we have \( d = i(e) \) for some \( e \in A_n \). Then

\[
i(a + \partial_n(e)) = \partial_n(b) + i(\partial_n(e)) = \partial_n(b) + \partial_n(i(e)) = \partial_n(b) + \partial_n(d) = \partial_n(b + d),
\]

so that \( \delta(c) = a + \partial_n(e) \sim a \). In other words, \( a \) specifies a well-defined homology class.

Since we want \( \delta \) to be well-defined not only on cycles but also on homology, we need to show that if \( c \) is a boundary, then \( \delta(c) \sim 0 \). Thus suppose \( c = \partial(c') \). We can then choose \( b' \) such that \( q(b') = c' \). It follows that \( \partial(b') \) would be a suitable choice for \( b \). But then \( \partial(b) = \partial(\partial(b')) = 0 \), so that \( \delta(c) = 0 \).
Since $H_2$ is a short exact sequence $0 \to Z \xrightarrow{p} Z \to Z/pZ \to 0$ gives rise to a short exact sequence of chain complexes

$$0 \to C^\Delta_+(X) \xrightarrow{p} C^\Delta_+(X) \xrightarrow{q_+} C^\Delta_+(X)/p \to 0$$

and therefore to a long exact sequence

$$\ldots \to H^\Delta_{n+1}(X; Z/pZ) \xrightarrow{\delta} H^\Delta_n(X; Z) \xrightarrow{p} H^\Delta_n(X; Z) \xrightarrow{q_+} H^\Delta_n(X; Z/pZ) \xrightarrow{\delta} H^\Delta_{n-1}(X; Z) \to \ldots$$

Taking $X = \R P^2$, this long exact sequence takes the form

$$0 \to H^0_2(\R P^2; Z) \xrightarrow{p} H^0_2(\R P^2; Z) \xrightarrow{q_+} H^0_2(\R P^2; Z/pZ) \xrightarrow{\delta} H^0_1(\R P^2; Z) \xrightarrow{p} H^0_1(\R P^2; Z) \xrightarrow{q_+} H^0_1(\R P^2; Z/pZ) \to 0.$$  

If $p$ is odd, this sequence becomes

$$0 \to 0 \xrightarrow{p} 0 \xrightarrow{q_+} H^0_2(\R P^2; Z/pZ) \xrightarrow{\delta} Z/2Z \xrightarrow{p} Z/2Z \to 0.$$  

Since $Z/2Z \xrightarrow{p} Z/2Z$ is an isomorphism, we conclude that $H^0_2(\R P^2; Z/pZ) = 0$ and $H^1_1(\R P^2; Z/pZ) = 0$. We also get that $H^0_1(\R P^2; Z/pZ) \cong Z/pZ$.

On the other hand, for $p = 2$, we get the sequence

$$0 \to 0 \xrightarrow{p} 0 \xrightarrow{q_+} H^0_2(\R P^2; Z/2Z) \xrightarrow{\delta} Z/2Z \xrightarrow{p} Z/2Z \to 0.$$  

Since $Z/2Z \xrightarrow{p} Z/2Z$ is zero, we get $H^0_2(\R P^2; Z/2Z) \cong Z/2Z$ and $Z/2Z \cong H^0_1(\R P^2; Z/2Z)$. We also get $H^0_1(\R P^2; Z/2Z) \cong Z/2Z$ as before.

The general result is

**Theorem 18.2.** Suppose that $0 \to K \to M \to Q \to 0$ is a short exact sequence of abelian groups. Then there is a long exact sequence

$$\ldots \to H_n(X; K) \to H_n(X; M) \to H_n(X; Q) \to H_{n-1}(X; K) \to \ldots$$

18. Wed, Oct. 5

**Exactness at C:** We first show that $\delta \circ q_+ = 0$. Thus let $b \in B_n$ be a cycle. We wish to show that $\delta(q_+(b)) = 0$. But the first step in constructing $\delta(q(b))$ is to choose a lift for $q(b)$, which we can of course take to be $b$. Then $\delta(b) = 0$, so that $a = 0$ as well.

Now suppose that $c \in C_n$ is a cycle that lives in the kernel of $\delta$. This means that $a = \partial(e)$ for some $e$. But then $b - i(e)$ is a cycle, and $q(b - i(e)) = c$, so $c$ is in the image of $q_+$.  

**Exactness at A:** First, we show that $i_+ \circ \delta = 0$. Let $c \in C_n$ be a cycle. Then if $\delta(c) = a$, then by construction, we have $i(a) = \partial(b) \sim 0$, so that $i_+ \circ \delta = 0$.

Finally, suppose that $a \in A_n$ is a cycle that lives in $\ker i_+$. Then $i(a) = \partial(b)$ for some $b$, but then $a = \delta(q(b))$.  

**Example 18.1.** The short exact sequence $0 \to Z \xrightarrow{p} Z \to Z/pZ \to 0$ gives rise to a short exact sequence of chain complexes

$$0 \to C^\Delta_+(X) \xrightarrow{p} C^\Delta_+(X) \xrightarrow{q_+} C^\Delta_+(X)/p \to 0$$

and therefore to a long exact sequence

$$\ldots \to H^\Delta_{n+1}(X; Z/pZ) \xrightarrow{\delta} H^\Delta_n(X; Z) \xrightarrow{p} H^\Delta_n(X; Z) \xrightarrow{q_+} H^\Delta_n(X; Z/pZ) \xrightarrow{\delta} H^\Delta_{n-1}(X; Z) \to \ldots$$

Taking $X = \R P^2$, this long exact sequence takes the form

$$0 \to H^0_2(\R P^2; Z) \xrightarrow{p} H^0_2(\R P^2; Z) \xrightarrow{q_+} H^0_2(\R P^2; Z/pZ) \xrightarrow{\delta} H^0_1(\R P^2; Z) \xrightarrow{p} H^0_1(\R P^2; Z) \xrightarrow{q_+} H^0_1(\R P^2; Z/pZ) \to 0.$$  

If $p$ is odd, this sequence becomes

$$0 \to 0 \xrightarrow{p} 0 \xrightarrow{q_+} H^0_2(\R P^2; Z/pZ) \xrightarrow{\delta} Z/2Z \xrightarrow{p} Z/2Z \to 0.$$  

Since $Z/2Z \xrightarrow{p} Z/2Z$ is an isomorphism, we conclude that $H^0_2(\R P^2; Z/pZ) = 0$ and $H^1_1(\R P^2; Z/pZ) = 0$. We also get that $H^0_1(\R P^2; Z/pZ) \cong Z/pZ$.

On the other hand, for $p = 2$, we get the sequence

$$0 \to 0 \xrightarrow{p} 0 \xrightarrow{q_+} H^0_2(\R P^2; Z/2Z) \xrightarrow{\delta} Z/2Z \xrightarrow{p} Z/2Z \to 0.$$  

Since $Z/2Z \xrightarrow{p} Z/2Z$ is zero, we get $H^0_2(\R P^2; Z/2Z) \cong Z/2Z$ and $Z/2Z \cong H^1_1(\R P^2; Z/2Z)$. We also get $H^0_1(\R P^2; Z/2Z) \cong Z/2Z$ as before.
The Long Exact Sequence for a subspace

Let \( A \subseteq X \) be a subspace. Define the group of relative \( n \)-chains by
\[
C_n(X, A) := C_n(X) / C_n(A).
\]
More generally, for any choice of coefficients \( M \) we define
\[
C_n(X, A; M) := C_n(X; M) / C_n(A; M).
\]

**Definition 18.3.** Given \( A \subseteq X \) and an abelian group \( M \), we define the relative homology groups to be
\[
H_n(X, A; M) := H_n(C_*(X, A) \otimes M).
\]

Given our discussion from above, we easily derive

**Proposition 18.4.** For any subspace \( A \subseteq X \) and abelian group \( M \), there is a long exact sequence
\[
\ldots H_n(A; M) \xrightarrow{i_*} H_n(X; M) \rightarrow H_n(X, A; M) \xrightarrow{\delta} H_{n-1}(A; M) \rightarrow \ldots
\]

**Proof.** We have a short exact sequence of chain complexes
\[
0 \rightarrow C_*(A; M) \rightarrow C_*(X; M) \rightarrow C_*(X, A; M) \rightarrow 0.
\]
The result is now a direct application of Prop 17.1. \( \Box \)

**Example 19.1.** If \( (X, x_0) \) is a based space, then we get a long exact sequence
\[
\ldots H_n(x_0) \xrightarrow{i_*} H_n(X) \rightarrow H_n(X, x_0) \xrightarrow{\delta} H_{n-1}(x_0) \rightarrow .
\]
Moreover, the map \( p : X \rightarrow x_0 \) induces a splitting \( p_* : H_n(X) \rightarrow H_n(x_0) \) to \( i_* \). It follows that each connecting homomorphism \( \delta \) is zero, so that the long exact sequence breaks up into a bunch of short exact sequences
\[
0 \rightarrow H_n(x_0) \rightarrow H_n(X) \rightarrow H_n(X, x_0) \rightarrow 0.
\]
Since reduced homology was defined to be the cokernel of \( i_* \), we conclude that
\[
\tilde{H}_n(X) \cong H_n(X, x_0).
\]

However, in general the long exact sequence is of limited use unless we can compute the relative groups. One of the main tools for computing relative homology is the Excision Theorem.

**Definition 19.2.** An excisive triad is a triple \((X; A, B)\), where \( A, B \subseteq X \) and \( X = \text{Int}(A) \cup \text{Int}(B) \).

**Theorem 19.3** (Excision). Let \((X; A, B)\) be an excisive triad. Then the inclusion \((A, A \cap B) \hookrightarrow (X, B)\) induces an isomorphism
\[
H_n(A, A \cap B; M) \cong H_n(X, B; M)
\]
for any coefficient group \( M \).

**Example 19.4.** We use the Excision Theorem to compute \( H_k(S^n) \). We write \( S^n \) as a union
\[
S^n = S^n_+ \cup S^n_-,
\]
where \( S^n_+ \) and \( S^n_- \) are the upper and lower hemispheres (extended by a collar around the equator, so that the equator lies in the interior of each). The intersection \( S^n_+ \cap S^n_- \) is a thickened version of the equator, but we simply write \( S^{n-1} \), since these are homotopy equivalent. Now the long exact sequence for the pair \((S^n, S^n_-)\) takes the form
\[
\rightarrow H_k(S^n_-) \rightarrow H_k(S^n) \rightarrow H_k(S^n, S^n_-) \xrightarrow{\delta} H_{k-1}(S^n_-) \rightarrow .
\]
Since the hemisphere $S^n$ is contractible, the outer two groups are zero if $k \geq 2$. Thus
\[ H_k(S^n) \cong H_k(S^n, S^n_-) \quad \text{if } k \geq 2. \]
In the case $k = 1$, this part of the sequence is
\[ 0 = H_1(S^n) \to H_1(S^n) \to H_1(S^n, S^n_-) \xrightarrow{\delta} H_0(S^n_-) \to H_0(S^n). \]
The rightmost map is an isomorphism $\mathbb{Z} \cong \mathbb{Z}$, so that $\delta = 0$. We conclude that $H_1(S^n) \cong H_1(S^n, S^n_-)$.
Now excision gives $H_k(S^n, S^n_-) \cong H_k(S^n_+, S^{n-1}_+)$, and the long exact sequence for the pair $(S^n_+, S^{n-1}_+)$ is
\[ \cdots \to H_k(S^n) \to H_k(S^n_+, S^{n-1}_+) \xrightarrow{\delta} H_{k-1}(S^{n-1}_+) \to H_{k-1}(S^n_+) \to \cdots. \]
Again, the hemisphere $S^n_+$ is contractible, so the outer two groups are zero if $k \geq 2$. We have shown that
\[ H_k(S^n) \cong H_k(S^n_+, S^{n-1}_+) \cong H_k(S^n_+, S^{n-1}) \cong H_{k-1}(S^{n-1}) \quad \text{if } k \geq 2. \]
If $k = 1$, this becomes
\[ 0 = H_1(S^n_+) \to H_1(S^n_+, S^{n-1}_+) \xrightarrow{\delta} H_0(S^{n-1}_+) \to H_0(S^n_+). \]
If $n \geq 2$, then the right map is an isomorphism $\mathbb{Z} \cong \mathbb{Z}$, so that $H_1(S^n) \cong H_1(S^n_+, S^{n-1}_+) = 0$. The other possible case is $n = 1$, in which case the right map is the fold map $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$, so that $H_1(S^1) \cong H_1(S^1_+, S^0)$ is identified with the kernel of the fold map, which is isomorphic to $\mathbb{Z}$.
Combining the above results, if $k > n$, then
\[ H_k(S^n) \cong H_{k-1}(S^{n-1}) \cong \ldots \cong H_{k-n+1}(S^1) = 0. \]
If $k = n$, we have
\[ H_n(S^n) \cong H_{n-1}(S^{n-1}) \cong \ldots \cong H_1(S^1) \cong \mathbb{Z}. \]
If $k < n$, we have
\[ H_k(S^n) \cong H_{k-1}(S^{n-1}) \cong \ldots \cong H_1(S^{n-k}) \cong 0. \]
In summary, if $k, n \geq 1$, then
\[ H_k(S^n) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n. \end{cases} \]
If we switch to reduced homology, the statement holds and extends to include the $n = 0$ case.

The next example we will discuss is $\mathbb{R}P^2$. Recall that one model of $\mathbb{R}P^2$ is as a quotient of $D^2$ by the relation $z \sim -z$ on the boundary circle. Another way to express this is as the pushout
\[ \begin{array}{ccc}
S^1 & \xrightarrow{i} & D^2 \\
\downarrow \alpha & & \downarrow \\
S^1 & \to & \mathbb{R}P^2.
\end{array} \]
This is an example of what is known as a CW complex. In general, you start out with the 0-skeleton $X_0$, which is just a (discrete) set. You then form the 1-skeleton by attaching 1-cells via a pushout
\[ \begin{array}{ccc}
\coprod_0 \partial D^1 & \xrightarrow{i} & \coprod_0 D^1 \\
\downarrow & & \downarrow \\
\coprod_0 \partial D^1 & \to & \coprod_0 D^1
\end{array} \]
You then attach 2-cells similarly via a pushout:

\[
\begin{array}{c}
\mathbb{I}_a \partial D^2 \\ \downarrow \quad \downarrow \\
\mathbb{I}_a \varphi_a \\ X_1 \\
\end{array} \quad \rightarrow \quad \begin{array}{c}
\mathbb{I}_a D^2 \\ \downarrow \quad \downarrow \\
X_2 \\
\end{array}
\]

We will come back to this idea of a CW complex when discussing cellular homology.
Last time, we introduced the idea of a CW complex. Here are some examples:

(1) $S^1$. There are many models. Two basic ones are (a) take a single 0-cell and a single 1-cell, and (b) start with two 0-cells and attach two 1-cells.

(2) $S^2$. The simplest model is to take a single 0-cell and a single 2-cell. Another option is to take any CW structure on $S^1$, and then attach a pair of 2-cells, which will become the northern and southern hemispheres of $S^2$.

(3) $\mathbb{RP}^2$. Recall that one model for this space was as the quotient of $D^2$, where we imposed the relation $x \sim -x$ on the boundary. If we restrict our attention to the boundary $S^1$, then the resulting quotient is $\mathbb{RP}^1$, which is again a circle. The quotient map $q : S^1 \to S^1$ is the map that winds twice around the circle. In complex coordinates, this would be $z \mapsto z^2$.

The above says that we can represent $\mathbb{RP}^2$ as the pushout

$$
\begin{array}{ccc}
S^1 & \xrightarrow{i} & D^2 \\
\downarrow{q} & & \downarrow \\
S^1 & \to & \mathbb{RP}^2
\end{array}
$$

If we build the 1-skeleton $S^1$ using a single 0-cell and a single 1-cell, then $\mathbb{RP}^2$ has a single cell in dimensions $\leq 2$.

(4) $T^2$, the torus. We can start with a single 0-cell and a pair $a$ and $b$ of 1-cells. This yields a 1-skeleton which is $S^1 \vee S^1$. We then attach a single 2-cell using the attaching map

$$
S^1 \to S^1 \vee S^1
$$

specified as the element $aba^{-1}b^{-1}$ of $\pi_1(S^1 \vee S^1)$.

(5) $K$, the Klein bottle. Just like the torus, we start with a 1-skeleton of $S^1 \vee S^1$, but now we attach the 2-cell using the attaching map $aba^{-1}b$. My making the change of coordinates $d = a^{-1}b$, we can alternatively describe the attaching map in the form $aadd$.

(6) $M_g$, the orientable surface of genus $g$. This can be described as the connect sum of $g$ copies of $T^2$. This has a CW structure with a single 0-cell and $2g$ 1-cells, labeled $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$. Thus the 1-skeleton is a wedge of $2g$ circles. There is a single 2-cell, attached via the product of commutators

$$
[a_1, b_1] \cdot [a_2, b_2] \cdot \cdots \cdot [a_g, b_g].
$$

(7) $N_g$, the nonorientable surface of genus $g$. This is the connect sum of $g$ copies of $\mathbb{RP}^2$. This can be given a CW structure with a single 0-cell and $g$ 1-cells labelled $\{c_1, \ldots, c_g\}$, so that the 1-skeleton is a wedge of $g$ circles. There is a single 2-cell, attached via the product

$$
c_1^2 \cdots c_g^2.
$$

(8) $\mathbb{RP}^n$. We described $\mathbb{RP}^2$ above.

More generally, we can define $\mathbb{RP}^n$ as a quotient of $D^n$ by the relation $x \sim -x$ on the boundary $S^{n-1}$. This quotient space of the boundary was our original definition of $\mathbb{RP}^{n-1}$.

It follows that we can describe $\mathbb{RP}^n$ as the pushout

$$
\begin{array}{ccc}
S^{n-1} & \xrightarrow{i} & D^n \\
\downarrow{q} & & \downarrow \\
\mathbb{RP}^{n-1} & \to & \mathbb{RP}^n
\end{array}
$$

Thus $\mathbb{RP}^n$ can be built as a CW complex with a single cell in each dimension $\leq n$. 

31
21. Wed, Oct 12

(9) $\mathbb{C}P^n$. Recall that $\mathbb{C}P^1 \cong S^2$. We can think of this as having a single 0-cell and a single 2-cell. We defined $\mathbb{C}P^2$ as the quotient of $S^3$ by an action of $S^1$ (thought of as $U(1)$). Let $\eta : S^3 \to \mathbb{C}P^1$ be the quotient map. What space do we get by attaching a 4-cell to $\mathbb{C}P^1$ by the map $\eta$? Well, the map $\eta$ is a quotient, so the pushout $\mathbb{C}P^1 \cup_\eta D^4$ is a quotient of $D^4$ by the $S^1$-action on the boundary.

Now include $D^4$ into $S^5 \subseteq \mathbb{C}^3$ via the map

$$\varphi(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4, \sqrt{1 - \sum x_i^2}, 0).$$

(This would be a hemi-equator.) We have the diagonal $U(1)$ action on $S^5$. But since any nonzero complex number can be rotated onto the positive $x$-axis, the image of $\varphi$ meets every $S^1$-orbit in $S^5$, and this inclusion induces a homeomorphism on orbit spaces

$$D^4/U(1) \cong S^5/U(1) = \mathbb{C}P^2.$$

We have shown that $\mathbb{C}P^2$ has a cell structure with a single 0-cell, 2-cell, and 4-cell.

This story of course generalizes to show that any $\mathbb{C}P^n$ can be built as a CW complex having a cell in each even dimension.

**Example 21.1.** $X = \mathbb{R}P^2$. Recall that we can build $\mathbb{R}P^2$ as a CW complex in which we start with a single 1-cell and attach a 2-cell via the attaching map $S^1 \xrightarrow{\delta} S^1$.

Let $x$ be a point in the interior of the attached 2-cell. Then $\mathbb{R}P^2 - \{x\}$ deformation retracts onto the 1-skeleton $S^1$. Write $U = \mathbb{R}P^2 - \{x\}$, and let $V$ be the interior of the 2-cell. Then $U \cap V = V - \{x\} \cong S^1$. The long exact sequence takes the form

$$\to H_2(U) \to H_2(\mathbb{R}P^2) \to H_2(\mathbb{R}P^2, U) \xrightarrow{\delta} H_1(U) \to H_1(\mathbb{R}P^2) \to H_1(\mathbb{R}P^2, U) \xrightarrow{\delta} H_0(U) \to H_0(\mathbb{R}P^2).$$

Since $U \cong S^1$, we know that $H_k(U) = 0$ for $k \geq 2$, so that $H_k(\mathbb{R}P^2) \cong H_k(\mathbb{R}P^2, U)$ for all $k \geq 2$. We have previously identified $H_0(X)$ with $\mathbb{Z}\{\pi_0(X)\}$, so the last map is an isomorphism $\mathbb{Z} \cong \mathbb{Z}$. It follows that the last $\delta$ must be zero, so we can replace our sequence with

$$0 \to H_2(\mathbb{R}P^2) \to H_2(\mathbb{R}P^2, U) \xrightarrow{\delta} \mathbb{Z} \to H_1(\mathbb{R}P^2) \to H_1(\mathbb{R}P^2, U) \to 0.$$ 

We use excision to calculate these relative groups. Excision identifies the above relative groups with the relative groups for $(V, V \cap U) \simeq (D^2, S^1)$. These groups sit in a long exact sequence

$$H_2(D^2) \to H_2(D^2, S^1) \xrightarrow{\delta} H_1(S^1) \to H_1(D^2) \to H_1(D^2, S^1) \xrightarrow{\delta} H_0(S^1) \to H_0(D^2).$$

Since $H_k(D^2)$ and $H_k(S^1)$ both vanish for $k \geq 2$, it follows that the relative groups vanish for $k \geq 3$. By the above, this shows that $H_k(\mathbb{R}P^2) = 0$ for $k \geq 3$. Next, we identify the above sequence with

$$0 \to H_2(D^2, S^1) \xrightarrow{\delta} \mathbb{Z} \to 0 \to H_1(D^2, S^1) \xrightarrow{\delta} \mathbb{Z} \cong \mathbb{Z}.$$

It follows that $H_2(D^2, S^1) \cong \mathbb{Z}$ and $H_1(D^2, S^1) \cong 0$. Plugging this back in above gives the exact sequence

$$0 \to H_2(\mathbb{R}P^2) \to \mathbb{Z} \xrightarrow{\delta} \mathbb{Z} \to H_1(\mathbb{R}P^2) \to 0 \to 0.$$ 

Now we cheat, and assume $H_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$. We will see later that this follows from the Hurewicz theorem. This implies that $\delta$ must be multiplication by 2 and so $H_2(\mathbb{R}P^2) = 0$. 


I skipped the following discussion of the proof of Theorem 19.3.

In order to prove the excision theorem, we introduce a new chain complex: let $C_n^{A,B}(X)$ be the free abelian group on (singular) $n$-simplices of $X$ whose image lies entirely in either $A$ or $B$. This condition is preserved by the differential of $C_*(X)$, so that $C_n^{A,B}(X) \subseteq C_*(X)$ is a sub-chain complex.

**Proposition 22.1.** The inclusion $C_n^{A,B}(X) \hookrightarrow C_*(X)$ is a chain homotopy equivalence.

**Proof.** We only give a brief indication. For a full (and lengthy) proof, see Prop 2.21 of Hatcher.

We need to define a homotopy inverse $f : C_*(X) \to C_n^{A,B}(X)$. The idea is to use barycentric subdivision. The subdivision of an $n$-simplex expresses it as the union of smaller $n$-simplices. By the Lebesgue Number Lemma, repeated barycentric subdivision will eventually decompose any $n$-simplex into a collection of $n$-simplices, each of which is either contained in $A$ or in $B$. This subdivision allows you to define a chain map $f$. You then show that subdivision of simplices is chain-homotopic to the identity. ■

**Proof of Theorem 19.3.** The chain homotopy equivalence $C_n^{A,B}(X) \simeq C_*(X)$ carries $C_*(B)$ into itself, so that we get a chain homotopy equivalence

$$C_n^{A,B}(X)/C_*(B) \simeq C_*(X)/C_*(B).$$

But the inclusion $C_*(A) \hookrightarrow C_n^{A,B}(X)$ induces an isomorphism

$$C_*(A)/C_*(A \cap B) \cong C_n^{A,B}(X)/C_*(B),$$

since both quotients can be identified with the free abelian group on $n$-simplices in $A$ that are not entirely contained in $B$. These chain homotopy equivalences are carried over after tensoring with $M$, which gives the theorem. ■

Recall that, given a map $f : A \to X$, the **mapping cone** $C(f)$ on $f$ is defined to be

$$C(f) := X \cup_A C(A).$$

**Proposition 22.2.** In general, we have $H_*(X,A) \cong \tilde{H}_n(C(f))$, so that the long exact sequence may be written

$$\ldots \to H_n(A;M) \xrightarrow{i_*} H_n(X;M) \to \tilde{H}_n(C(f);M) \xrightarrow{\delta} H_{n-1}(A;M) \to \ldots$$

**Proof.** We write $c$ for the cone point in $C(A) \subseteq C(f)$. Since $C(A) \simeq *$, we have $\tilde{H}_n(C(f)) \cong H_n(C(f),C(A))$. Excision then gives

$$H_n(C(f),C(A)) \cong H_n(C(f) - \{c\},C(A) - \{c\}).$$

But we can deformation retract $C(f) - \{c\}$ onto $X$ and similarly $C(A) - \{c\}$ onto $A$, so that the latter relative homology group can be identified with $H_n(X,A)$. ■

In many “nice” situations, the cofiber $C(f)$ is homotopy equivalent to the quotient $X/A$. For example, if $A \subseteq X$ is a subcomplex of a CW complex, then this follows from [Hatcher, Prop. 0.17] applied to the pair $(C(f),C(A))$.

Hatcher introduces a weaker notion, called “good pairs”. The precise definition of a good pair $(X,A)$ is that $A$ is closed (and nonempty) and that there is a neighborhood $A \subseteq U$ of $A$ in $X$, such that $U$ deformation retracts onto $A$. The point is that this is enough [Hatcher, Prop 2.22] to conclude that $H_*(X/A) \cong H_*(C(f)) \cong H_*(X,A)$.

In the case that $A = x_0$ is a basepoint, we say that $X$ is “well-based”.
**Proposition 22.3** (Suspension isomorphism). If \( X \) is a based space, then
\[
\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(SX),
\]
where \( SX = CX \cup_X CX \) is the (unreduced) suspension and we take one of the cone points as the basepoint.

**Proof.** Consider the pair \((CX, X)\). The quotient \( C(X)/X \) is the (unreduced) suspension \( S(X) \), and \((CX, X)\) is a “good pair”. The long exact therefore takes the form
\[
\ldots \rightarrow H_{n+1}(CX) \rightarrow H_{n+1}(CX, X) \cong \widetilde{H}_{n+1}(SX) \delta \rightarrow H_n(X) \rightarrow H_n(CX) \rightarrow \ldots.
\]
Since the outer two groups are zero for \( n \geq 1 \), we conclude that the connecting homomorphism is an isomorphism. This gives what we wanted if \( n \geq 1 \) since \( H_n(X) \cong \widetilde{H}_n(X) \) for \( n \geq 1 \).

In the case \( n = 0 \), \( H_0(CX) \cong \mathbb{Z} \), and the connecting homomorphism identifies \( \widetilde{H}_1(SX) \) with the kernel of \( H_0(X) \rightarrow H_0(CX) \), which is precisely the group \( \widetilde{H}_0(X) \).  

The unreduced suspension has no canonical basepoint, so the above result is usually stated instead in terms of the reduced suspension.

**Proposition 22.4** (Suspension isomorphism). If \( X \) is a well-based space, then
\[
\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(\Sigma X),
\]
where \( \Sigma X = S^1 \wedge X \) is the (reduced) suspension.

The reduced suspension is \( \Sigma X = SX/(I \times \{x_0\}) \). If \( X \) is well-based, then \((SX, I \times \{x_0\})\) is a good pair, so that the reduced homology of the two versions of suspension are the same.

23. WED, OCT. 7

**Proposition 23.1** (Wedge isomorphism). If \( \{X_a\}_{a \in A} \) are based spaces, with “good” basepoints, then the inclusions \( X_a \hookrightarrow \bigvee_a X_a \) induce an isomorphism
\[
\bigoplus_a \widetilde{H}_n(X_a) \cong \widetilde{H}_n(\bigvee_a X_a).
\]

**Proof.** We apply proposition 22.2 with \( X = \coprod_a X_a \) and \( A = \coprod a \ast \). We have a long exact sequence
\[
\rightarrow H_n(A) \rightarrow H_n(X) \cong \bigoplus_a H_n(X_a) \rightarrow \widetilde{H}_n(\bigvee_a X_a) \delta \rightarrow H_{n-1}(A) \rightarrow \ldots.
\]
The outer two groups are zero if \( n \geq 2 \), so that the middle map becomes an isomorphism. The same conclusion holds when \( n = 1 \) since \( H_0(A) \rightarrow H_0(X) \) is injective, so that the connecting homomorphism must be zero. For \( n = 0 \), we get a short exact sequence
\[
0 \rightarrow H_0(A) \cong \bigoplus_a \mathbb{Z} \rightarrow H_0(X) \cong \bigoplus_a H_0(X_a) \rightarrow \widetilde{H}_0(\bigvee_a X_a) \rightarrow 0,
\]
which gives the desired conclusion.  

\[\square\]
24. Mon, Oct. 17

The identification of Simplicial and Singular homology

If $X$ is a $\Delta$-complex, we can consider the chain complexes $C^\Delta_*(X)$ and $C_*(X)$. In fact, there is a natural map $\eta : C^\Delta_*(X) \to C_*(X)$, which considers a simplex of $X$ as a singular simplex. This works just as well in the relative case, and we will prove

**Theorem 24.1.** Let $X$ be a $\Delta$-complex and $A \subseteq X$ a sub-$\Delta$-complex. Then the chain map $\eta$ induces an isomorphism

$$H^\Delta_*(X, A) \cong H_*(X, A).$$

**Proof.** We only give the proof in the case that $X$ is finite-dimensional and $A = \emptyset$. See [Hatcher, Theorem 22.27] for the general case.

For each $k \geq 0$, denote by $X^k$ the $k$-skeleton of $X$, which is the union of all simplices of dimension $k$ or less. We will argue by induction on $k$ that $\eta : H^\Delta_*(X^k) \to H_*(X^k)$ is an isomorphism. In the base case $k = 0$, this is clear since $X^0$ is discrete and we know that both versions of homology agree on discrete spaces.

For the induction step, the inclusion $X^{k-1} \hookrightarrow X^k$ is a $\Delta$-map, and we have a map of long exact sequences

$$\cdots \longrightarrow H^\Delta_{n+1}(X^k, X^{k-1}) \xrightarrow{\delta} H^\Delta_n(X^{k-1}) \longrightarrow H^\Delta_n(X^k) \longrightarrow H^\Delta_n(X^k, X^{k-1}) \longrightarrow H^\Delta_{n-1}(X^{k-1}) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_{n+1}(X^k, X^{k-1}) \xrightarrow{\delta} H_n(X^{k-1}) \longrightarrow H_n(X^k) \longrightarrow H_n(X^k, X^{k-1}) \longrightarrow H_{n-1}(X^{k-1}) \longrightarrow \cdots$$

We first argue that the vertical maps at the relative groups are isomorphisms. By definition, the simplicial relative homology groups are the homology groups of the chain complex $C^\Delta_*(X^k)/C^\Delta_*(X^{k-1})$. But this quotient group is trivial in every degree except for $k$, in which case we have a free abelian group on the set of $k$-simplices of $X^k$. So this chain complex has zero differential, and the relative homology groups are again just $\mathbb{Z}(\Delta^k(X^k))$ concentrated in degree $k$.

For the relative singular groups, we have

$$H_n(X^k, X^{k-1}) \cong \tilde{H}_n(X^k/X^{k-1}) \cong \tilde{H}_n(\bigvee_{\Delta_k(X)} S_k) \cong \bigoplus_{\Delta_k(X)} \tilde{H}_n(S_k) \cong \left\{ \begin{array}{cl} \mathbb{Z}\{\Delta_k(X)\} & k = n \\ 0 & k \neq n. \end{array} \right.$$  

So the relative groups agree, and the map $\eta$ sends generators to generators, so the vertical maps at the relative groups are isomorphisms.

Now for the induction step assume the vertical maps at $X^{k-1}$ are isomorphisms. The theorem follows from the following important result from homological algebra:

**Lemma 24.2 (5-lemma).** If both rows in

$$\begin{array}{ccccccc}
A_1 & \xrightarrow{g_1} & A_2 & \xrightarrow{g_2} & A_3 & \xrightarrow{g_3} & A_4 & \xrightarrow{g_4} & A_5 \\
| & \equiv & \downarrow & \equiv & \downarrow & \equiv & \downarrow & \equiv & \\
B_1 & \xrightarrow{h_1} & B_2 & \xrightarrow{h_2} & B_3 & \xrightarrow{h_3} & B_4 & \xrightarrow{h_4} & B_5
\end{array}$$

are exact and all $f_i$ except $f_3$ are isomorphisms, then $f_3$ is also an isomorphism.

**Proof.** We give the proof of injectivity. The proof of surjectivity is left as an exercise.

Suppose $x \in A_3$ and $f_3(x) = 0$. We wish to show that $x = 0$. Now $f_4(g_3(x)) = h_3(f_3(x)) = 0$. Since $f_4$ is injective, we know that $g_3(x) = 0$. Thus $x = g_2(w)$, some $w \in A_2$. Now $h_2(f_2(w)) = 0$. 

35
\[ f_3(g_2(w)) = f_3(x) = 0. \text{ It follows that } f_2(w) = h_1(y), \text{ some } y \in B_1. \text{ Since } f_1 \text{ is surjective, there is some } z \in A_1 \text{ with } f_1(z) = y. \]

\[
\begin{array}{ccccccccc}
  z & \xrightarrow{g_1} & w & \xrightarrow{g_2} & x & \xrightarrow{g_3} & 0 & \xrightarrow{g_4} & A_3 \\
  y & \xrightarrow{h_1} & f_2(w) & \xrightarrow{h_2} & 0 & \xrightarrow{h_3} & 0 & \xrightarrow{h_4} & B_3 \\
\end{array}
\]

Now \( f_2(g_1(z)) = h_1(f_1(z)) = h_1(y) = f_2(w) \). Since \( f_2 \) is injective, it follows that \( g_1(z) = w \). But then \( x = g_2(w) = g_2(g_1(z)) = 0 \).

**The Mayer-Vietoris sequence**

It is sometimes convenient to combine the long exact sequence and excision into a different form. On your homework, you are asked to deduce the following Mayer-Vietoris exact sequence simply from the axioms. We give a chain-level argument here.

Let \((X; A, B)\) be an excisive triad and recall that the group \(C_n^{A,B}(X)\) defined in Prop. 22.1 is chain-homotopy equivalent to \(X\).

We have a surjection \(\varphi : C_n(A) \oplus C_n(B) \to C_n^{A,B}(X)\) given by \(\varphi(x, y) = x + y\). The kernel consists of pairs of the form \((x, -x)\). But then \(x\) is a chain in both \(A\) and \(B\), so it is a chain in \(A \cap B\). We conclude that we have a short exact sequence of chain complexes

\[ 0 \to C_*(A \cap B) \xrightarrow{\kappa} C_*(A) \oplus C_*(B) \xrightarrow{\varphi} C_+^{A,B}(X) \to 0, \]

where \(\kappa(x) = (x, -x)\). Again, use of Prop 17.1 gives rise to the Mayer-Vietoris long exact sequence

\[ \ldots \to H_n(A \cap B) \xrightarrow{(j_A, -j_B)} H_n(A) \oplus H_n(B) \xrightarrow{i_A + i_B} H_n(X) \xrightarrow{\delta} H_{n-1}(A \cap B) \to \ldots, \]

where \(j_A : A \cap B \to A\), \(j_B : A \cap B \to B\), \(i_A : A \to X\), and \(i_B : B \to X\) are the various inclusions.

25. **Fri., Oct. 21**

**The Eilenberg-Steenrod Axioms**

By the category of pairs of CW complexes, we mean the category in which the objects are a pair \((X, A)\), where \(X\) is CW and \(A\) is a subcomplex, and a morphism \(f : (X, A) \to (Y, B)\) is a map \(f : X \to Y\) such that \(f(A) \subseteq B\).

**Definition 25.1.** A homology theory on CW complexes is a sequence of functors \(h_n(X, A)\) on pairs of CW complexes and natural transformations \(\delta : h_n(X, A) \to h_{n-1}(A, \emptyset)\) satisfying the following axioms:

1. (Homotopy) If \(f \simeq g\), then \(f_* = g_*\).
2. (Long exact sequence) Writing \(h_n(X) := h_n(X, \emptyset)\), the sequence

\[ \ldots h_n(A) \to h_n(X) \to h_n(X, A) \xrightarrow{\delta} h_{n-1}(A) \to \ldots \]

is exact.
3. (Excision) If \(X\) is the union of subcomplexes \(A\) and \(B\), then the inclusion \((A, A \cap B) \hookrightarrow (X, B)\) induces an isomorphism

\[ h_n(A, A \cap B) \cong h_n(X, B) \]
(4) (Additivity) If \((X, A)\) is the disjoint union of pairs \((X_i, A_i)\), then the inclusions \((X_i, A_i) \hookrightarrow (X, A)\) induce an isomorphism
\[
\bigoplus_i h_n(X_i, A_i) \cong h_n(X, A).
\]

An ordinary homology theory is one that also satisfies the additional axiom

(5) (Dimension) \(h_n(pt) = 0\) if \(n \neq 0\).

It turns out that if \(h\) is an ordinary homology theory and \(G := h_0(pt, \emptyset)\), then \(h_n(X, A) \cong H_n(X, A; G)\). In other words, singular homology is essentially the only ordinary homology theory. There are many “extraordinary” homology theories (\(K\)-theory, bordism, stable homotopy . . . ) but we will not study these in this course.

**Euler characteristic**

The Euler characteristic \(\chi\) started from the simple formula
\[
\chi(X) = V - E + F,
\]
in the case of a 2-dimensional simplicial complex, where \(V\), \(E\), and \(F\) stand for the number of vertices, edges, and faces, respectively. An arbitrary simplicial (or \(\Delta\)-) complex can have simplices of arbitrary dimension, and we can more generally define
\[
\chi(X) := \sum_{i=0}^{\infty} (-1)^i \text{(number of } i\text{-simplices)}.
\]

If we want to define the Euler characteristic to be a topological invariant, meaning that any two homeomorphic simplicial complexes should have the same Euler characteristic, then you can already see why the alternating sum is a good idea: subdividing a simplex does not change the above formula.

\[
\chi(\bullet) = \chi(\bullet \bullet) = 1 \quad \text{and} \quad \chi\left(\begin{array}{c}
\bullet
\end{array}\right) = \chi\left(\begin{array}{c}
\bullet
\end{array} \left\{\begin{array}{c}
\bullet
\end{array}\right\}
\right) = 1
\]

We can also define an algebraic version. Recall that the rank of a finitely generated abelian group is the rank of the free part. In other words, if \(A \cong \mathbb{Z}^r \oplus \text{torsion}\), then \(\text{rank}(A) := r\). This is also the same as the dimension of the \(\mathbb{Q}\)-vector space \(A \otimes_{\mathbb{Z}} \mathbb{Q}\).

We also say that a chain complex \(C_\ast\) of abelian groups is finite if each group \(C_n\) is finitely generated and furthermore if only finitely many groups \(C_n\) are nonzero.

**Definition 25.2.** If \(C_\ast\) is a finite chain complex, we define \(\chi(C_\ast) := \sum_{i=0}^{\infty} (-1)^i \text{rank}(C_i)\).

Our goal will be to show

**Proposition 25.3.** Let \(C_\ast\) be a finite chain complex. Then
\[
\chi(C_\ast) = \chi(H_\ast(C_\ast)).
\]

For this discussion, it will be convenient to use the language of tensor products.

**Definition 25.4.** Given abelian groups \(A\) and \(B\), their **tensor product** is defined to be
\[
A \otimes B := \mathbb{Z}\{a \otimes b \mid (a, b) \in A \oplus B\} / \sim,
\]
where the relation is generated by
\[
a_1 \otimes b + a_2 \otimes b \sim (a_1 + a_2) \otimes b, \quad \text{and} \quad a \otimes b_1 + a \otimes b_2 \sim a \otimes (b_1 + b_2).
\]
Example 25.5. $Z \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z}$. The point is that

$$k \otimes \ell \sim k \cdot (1 \otimes \ell) \sim k\ell(1 \otimes 1),$$

so that the group is cyclic, and furthermore

$$n \cdot (1 \otimes 1) \sim 1 \otimes n = 1 \otimes 0 \sim 0(1 \otimes 1) = 0.$$

More generally, $Z \otimes A \cong A$ for any $A$. 
Example 26.1. \( \mathbb{Q} \otimes \mathbb{Z}/n\mathbb{Z} \cong 0 \). The point is that for any rational number \( \frac{a}{b} \), we have
\[
\frac{a}{b} \otimes k = \frac{an}{bn} \otimes k = \frac{a}{bn} \otimes kn = \frac{a}{bn} \otimes 0 = 0.
\]

Even more useful than the construction of the tensor product given last time is the universal property:

**Proposition 26.2.** The homomorphism \( u : A \oplus B \longrightarrow A \otimes B \) defined by \( u(a, b) = a \otimes b \) is the universal example of a bilinear map out of \( A \oplus B \). That is, if \( f : A \oplus B \longrightarrow C \) is also bilinear, then there is a unique homomorphism \( \overline{f} : A \otimes B \longrightarrow C \) making the diagram commute.

\[
\begin{array}{ccc}
A \oplus B & \xrightarrow{f} & C \\
\downarrow{u} & & \\
A \otimes B & & \\
\end{array}
\]

Beware that \( u : A \oplus B \longrightarrow A \otimes B \) is not surjective in general. For instance \( \mathbb{Z}^2 \otimes \mathbb{Z}^3 \cong \mathbb{Z}^6 \).

We can also make sense of tensor product of vector spaces \( V \otimes W \) in a similar way. This has a similar universal property in terms of bilinear maps. One of the helpful things to know is that if \( \{v_1, \ldots, v_k\} \) is a basis for \( V \) and \( \{w_1, \ldots, w_n\} \) is a basis for \( W \), then the set \( \{v_1 \otimes w_j\} \) gives a basis for \( V \otimes W \). In particular,
\[
\dim(V \times W) = \dim(V) \cdot \dim(W).
\]

Another important property of the tensor product is its relation to Hom groups.

**Proposition 26.3.** Given abelian groups \( A, B, \) and \( C \), there is an isomorphism
\[
\text{Hom}(A \otimes B, C) \cong \text{Hom}(A, \text{Hom}(B, C))
\]
that is natural in \( A, B, \) and \( C \).

This is an example of an ‘adjunction’, and is completely analogous to the homeomorphism
\[
\text{Map}(X \times Y, Z) \cong \text{Map}(X, \text{Map}(Y, Z))
\]
in the world of topological spaces.

**Proposition 27.1.** Given abelian groups \( A_1, A_2, \) and \( B \), there is a natural isomorphism
\[
(A_1 \oplus A_2) \otimes B \cong (A_1 \otimes B) \oplus (A_2 \otimes B).
\]

**Proof.** For any abelian group \( C \), we have
\[
\text{Hom} \left( (A_1 \oplus A_2) \otimes B, C \right) \cong \text{Hom} \left( A_1 \oplus A_2, \text{Hom}(B, C) \right)
\cong \text{Hom}(A_1, \text{Hom}(B, C)) \oplus \text{Hom}(A_2, \text{Hom}(B, C))
\cong \text{Hom}(A_1 \otimes B, C) \oplus \text{Hom}(A_2 \otimes B, C)
\cong \text{Hom} \left( (A_1 \otimes B) \oplus (A_2 \otimes B), C \right).
\]

So we have shown that, for the two groups \( G_1 \) and \( G_2 \) that we want to compare, the functors \( \text{Hom}(G_1, -) \) and \( \text{Hom}(G_2, -) \) are naturally isomorphic. The proposition now follows from the following lemma.
Lemma 27.2. (Yoneda) Let $A$, $B$, and $C$ be abelian groups. Suppose given an isomorphism 

$$\eta_C : \text{Hom}(A, C) \cong \text{Hom}(B, C)$$

that is natural in $C$. Then $A \cong B$.

Proof. We define $f : A \to B$ by $f = \eta_B^{-1}(\text{id}_B)$ and similarly $g : B \to A$ by $g = \eta_A(\text{id}_A)$. You can use the naturality diagram to show $f \circ g = \text{id}$ and $g \circ f = \text{id}$. \[\square\]

Prop. 25.3 will follow from Lemma 27.3.

Tensoring with $\mathbb{Q}$ preserves (short) exact sequences. In other words, if

$$0 \to A \xrightarrow{i} B \xrightarrow{q} C \xrightarrow{r} 0$$

is exact, then so is

$$0 \to \mathbb{Q} \otimes A \to \mathbb{Q} \otimes B \to \mathbb{Q} \otimes C \to 0.$$

Abelian groups with this property are called flat.

Proof. You are asked to show on your homework that for any abelian $D$, the sequence

$$D \otimes A \to D \otimes B \to D \otimes C \to 0$$

is always exact. So it suffices to show that $\mathbb{Q} \otimes A \to \mathbb{Q} \otimes B$ is injective. We will write $\varphi$ for this map of $\mathbb{Q}$-vector spaces.

Let $x = \sum_i r_i \otimes a_i \in \mathbb{Q} \otimes A$ such that $\varphi(x) = 0$ in $\mathbb{Q} \otimes B$. We can clear denominators of the $r_i$ by multiplying by some sufficiently large integer $n$. Thus $nx$ is in the image of $A \to \mathbb{Q} \otimes A$, $a \mapsto 1 \otimes a$. So we can write $nx = 1 \otimes a$ for some $a \in A$. Now

$$1 \otimes i(a) = \varphi(nx) = n\varphi(x) = 0$$

in $\mathbb{Q} \otimes B$, so $i(a)$ must be a torsion class in $B$. Since $i : A \to B$ was injective, it follows that $a$ was torsion in $A$. But then $nx = 1 \otimes a = 0$ in $\mathbb{Q} \otimes A$. It follows that $x = \frac{1}{n} \cdot nx = 0$ as well. \[\square\]

Corollary 27.4. If

$$0 \to A \to B \to C \to 0$$

is short exact, then $\text{rank}(B) = \text{rank}(A) + \text{rank}(C)$.

Proof. This follows from the lemma, given that $\text{rank}(A) = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes A)$. \[\square\]

28. Fri, Oct. 28

Proof of Proposition 25.3. Let $Z_i := \ker(\partial_i) \subseteq C_i$ be the subgroup of cycles and $B_i = \text{im}(\partial_{i+1}) \subseteq Z_i \subseteq C_i$ be the boundaries. The key is to note that we have short exact sequences

$$0 \to Z_i \to C_i \to B_{i-1} \to 0.$$

and

$$0 \to B_i \to Z_i \to H_i \to 0.$$

By the corollary, these tell us that

$$\text{rank}(C_i) = \text{rank}(Z_i) + \text{rank}(B_{i-1})$$

and

$$\text{rank}(Z_i) = \text{rank}(B_i) + \text{rank}(H_i).$$

So

$$\sum_i (-1)^i \text{rank}(C_i) = \sum_i (-1)^i (\text{rank}(B_i) + \text{rank}(H_i) + \text{rank}(B_{i-1})).$$

This is a telescoping sum, and we end up with $\chi(H_*)$. \[\square\]
So this tells us that the Euler characteristic only depends on the homology of the space, not on the particular cellular model. This also allows us to define the Euler characteristic for any space (with “finite” homology), not only for simplicial complexes.

**Definition 28.1.** Let \( X \) be a space such that \( H_*(X) \) is a finite chain complex. We then define

\[
\chi(X) := \chi(H_*(X)).
\]

By Proposition 25.3, this agrees with the previous notion for simplicial complexes.

**Example 28.2.**

1. \( X = S^2 \). We built the sphere as a \( \Delta \)-complex by gluing together two 2-simplices. The leads to the Euler characteristic computation

\[
\chi(S^2) = 3 - 3 + 2 = 2.
\]

On the other hand, the computation via homology is

\[
\chi(S^2) = \chi(H_*(S^2)) = 1 - 0 + 1 = 2.
\]

2. \( X = T^2 \). The torus was similarly built by gluing two 2-simplices. We have, on the one hand

\[
\chi(T^2) = 1 - 3 + 2 = 0
\]

and on the other

\[
\chi(H_*(T^2)) = 1 - 2 + 1 = 0.
\]

3. \( X = \mathbb{RP}^2 \). The projective plane was built from two simplices as in the picture to the right. So

\[
\chi(\mathbb{RP}^2) = 2 - 3 + 2 = 1
\]

and

\[
\chi(\mathbb{RP}^2) = \text{rank}(\mathbb{Z}) - \text{rank}(\mathbb{Z}/2\mathbb{Z}) = 1.
\]

**Degree**

The next topic is yet another variant of homology, this one defined for CW complexes. It will be convenient to first discuss the notion of “degree” of a map of spheres.

**Definition 28.3.** For \( n > 0 \), let \( f : S^n \rightarrow S^n \) be any map. This induces a map

\[
\mathbb{Z} \cong \tilde{H}_n(S^n) \xrightarrow{f_*} \tilde{H}_n(S^n) \cong \mathbb{Z}
\]

which is necessarily of the form \( i \mapsto k \cdot i \) for some \( k \in \mathbb{Z} \). This integer \( k \) is called the **degree** of the map \( f \).

Note that there are two possible choices of isomorphism \( \tilde{H}_n(S^n) \cong \mathbb{Z} \), corresponding to the two generators for the infinite cyclic group. But as long as we use the same choice in both the domain and codomain of \( f_* \), this makes the notion of degree well-defined. Here are some properties of the degree of a map of spheres.

**Proposition 28.4.**

1. \( \deg(f) \) only depends on the homotopy class of \( f \)

2. The degree defines a homomorphism \( \deg : \pi_n(S^n) \rightarrow \tilde{H}_n(S^n) \cong \mathbb{Z} \).

**Proof.** (1) This follows from homotopy-invariance of homology
(2) Recall that the sum $f + g$ of two elements of the homotopy group is defined to be the composite

\[ S^n \xrightarrow{p} S^n \vee S^n \xrightarrow{f \vee g} S^n, \]

where $p$ is a pinch map. Applying homology gives

\[ \tilde{H}_n(S^n) \xrightarrow{p_*} \tilde{H}_n(S^n \vee S^n) \cong \tilde{H}_n(S^n) \oplus \tilde{H}_n(S^n) \xrightarrow{f_* \oplus g_*} \tilde{H}_n(S^n). \]

The isomorphism $\tilde{H}_n(S^n \vee S^n) \cong \tilde{H}_n(S^n) \oplus \tilde{H}_n(S^n)$ is induced by the two collapse maps $c_i : S^n \vee S^n \longrightarrow S^n$. These compose with the pinch map $p$ to give maps (based-)homotopic to the identity, so that the above sequence is isomorphic to

\[ \tilde{H}_n(S^n) \xRightarrow{\Delta} \tilde{H}_n(S^n) \oplus \tilde{H}_n(S^n) \xrightarrow{f_* \oplus g_*} \tilde{H}_n(S^n), \]

which simplifies to the sum $f_* + g_*$. \[\square\]
Proposition 28.4 (continued...)

(3) \( \text{deg}(\text{id}) = 1. \)
(4) \( \text{deg}(g \circ f) = \text{deg}(g) \cdot \text{deg}(f) \)

Proof. 
(3) Since \( \tilde{\mathbb{H}}_n \) is a functor, we know that \( \tilde{\mathbb{H}}_n(\text{id}_{S^n}) = \text{id}_{\tilde{\mathbb{H}}_n(S^n)} \), so that the multiplier is just 1.
(4) This again comes from the fact that \( \tilde{\mathbb{H}}_n \) is a functor! We know that \( (g \circ f)^* = g_* \circ f_* \), so that \( \text{deg}(g \circ f) \cdot 1 = (g \circ f)_*(1) = g_*(f_*(1)) = g_*(\text{deg}(f) \cdot 1) = \text{deg}(f) \cdot g_*(1) = \text{deg}(f) \cdot \text{deg}(g) \cdot 1. \)

Proposition 29.1. \( \pi_n(S^n) \cong \mathbb{Z} \oplus ? \) for \( n \geq 1. \)

Proof. We have a homomorphism \( \text{deg} : \pi_n(S^n) \to \mathbb{Z} \). There are two possibilities: either it is the zero homomorphism, or it is surjective. Since \( \text{deg}(\text{id}) = 1 \), it must be surjective. But then we have a splitting \( s : \mathbb{Z} \to \pi_n(S^n) \) defined by \( s(n) = n \cdot \text{id}_{S^n} \). As we have discussed, the splitting induces a direct sum decomposition.

In fact, the ? is trivial, so that \( \pi_n(S^n) \cong \mathbb{Z} \) for all \( n \geq 1. \)

Cellular homology

We now introduce our third version of homology, this one defined for CW complexes. The idea is to define the cellular chain complex by

\[ C_n^{\text{cell}}(X) := \mathbb{Z}\{n\text{-cells of } X \}. \]

For the differential \( \partial_n^{\text{cell}} : C_n^{\text{cell}}(X) \to C_{n-1}^{\text{cell}}(X) \), let \( e_n^n \) be an \( n \)-cell of \( X \). Then \( e_n^n \) is determined by its attaching map \( \varphi_n : S^{n-1} \to X^{n-1} \). The idea is that \( \partial_n^{\text{cell}}(e_n^n) \) should capture how the attaching map interacts with the various \((n-1)\)-cells. If we write

\[ \partial_n^{\text{cell}}(e_n^n) = \sum_\beta d_{n\beta}[\beta], \]

where \( \beta \) are the \((n-1)\)-cells of \( X \), then we take \( d_{n\beta} \) to be the degree of the map

\[ S^{n-1} \xrightarrow{\varphi_n} X^{n-1} \to X^{n-1}/X^{n-2} \cong \bigvee_\beta S^{n-1} \xrightarrow{p_\beta} S^{n-1}. \]

It remains to show that \( \partial_{n-1}^{\text{cell}} \circ \partial_n^{\text{cell}} = 0 \) and to then define cellular homology as the homology of this cellular chain complex. This can be done, but there is another, slick, approach, using the machinery we have already built up.

Example 29.2. Before we give the precise definition, let’s turn to an example. For \( n \geq 2 \), consider the CW structure on \( S^n \) having a single 0-cell and single \( n \)-cell. Then the cellular chain complex will be just \( \mathbb{Z} \) in degrees 0 and \( n \), with no possible differential. So we immediately read off the homology groups.

For \( n = 1 \), there is a possible \( d_1 : C_1(S^1) \to C_0(S^1) \). But in fact the differential is zero.
30. Wed, Nov. 2

We wanted to define
\[ C_n^{\text{cell}}(X) := \mathbb{Z}\{n\text{-cells of } X\}. \]

Now in a truly perverse act, we can rewrite this as
\[ \mathbb{Z}\{n\text{-cells of } X\} \cong \check{H}_n(\bigvee_{\beta} S^n) \cong \check{H}_n(X^n / X^{n-1}) \cong H_n(X^n, X^{n-1}), \]

and we now instead choose to define
\[ C_n^{\text{cell}}(X) := H_n(X^n, X^{n-1}). \]

The differential is defined as the composite
\[ C_n^{\text{cell}}(X) = H_n(X^n, X^{n-1}) \xrightarrow{\partial} H_{n-1}(X^{n-1}, X^{n-2}) = C_{n-1}^{\text{cell}}(X). \]

But now with this definition, it is simple to check that \( \partial_n^{\text{cell}} \circ \partial_{n+1}^{\text{cell}} = 0 \): this composition is displayed in the diagram
\[
\begin{array}{ccc}
C_{n+1}^{\text{cell}}(X) & \xrightarrow{\partial} & H_{n+1}(X^{n+1}, X^n) \\
\downarrow & & \downarrow \\
H_n(X^n) & \xrightarrow{\partial} & H_n(X^n, X^{n-1}) = C_n^{\text{cell}}(X).
\end{array}
\]

But the two arrows surrounding \( C_n^{\text{cell}}(X) \) are part of the long exact sequence in homology for the pair \( (X^n, X^{n-1}) \) and therefore compose to zero. It follows that we have a chain complex, so that the following definition makes sense.

**Definition 30.1.** Given a CW structure on a space \( X \), we define
\[ H_n^{\text{cell}}(X) := H_n(C_n^{\text{cell}}(X)). \]

We can also introduce coefficients or consider a reduced theory, just as in the other versions of homology.

**Theorem 30.2.** For any CW complex \( X \), we have
\[ H_n^{\text{cell}}(X) \cong H_n(X). \]

Before we prove the theorem, it will be convenient to establish the following.

**Lemma 30.3.**
1. For any \( k < n \), the inclusion \( X^n \hookrightarrow X \) induces an isomorphism \( H_k(X^n) \cong H_k(X) \).
2. For any \( k > n \), we have \( H_k(X^n) = 0 \).

**Proof.** We only prove (i) in the case that \( X \) is finite-dimensional. See p. 138 of Hatcher for the general case. We have an exact sequence
\[ H_{k+1}(X^n, X^{n-1}) \xrightarrow{\partial} H_k(X^{n-1}) \xrightarrow{\partial} H_k(X^n) \xrightarrow{\partial} H_k(X^n, X^{n-1}). \]

These outer two groups are zero if \( k \notin \{n, n-1\} \). So if \( k > n \), we have \( H_k(X^n) \cong H_k(X^{n-1}) \cong \ldots H_k(X^0) = 0 \). Similarly, if \( k < n \), we conclude that \( H_k(X^n) \cong H_k(X^{n+1}) \cong \ldots H_k(X) \). ■
Proof of Theorem 30.2. Consider the following diagram.

\[
\begin{array}{ccccccccc}
0 = H_n(X^{n-1}) & \rightarrow & H_n(X^n) & \rightarrow & H_n(X^n, X^{n-1}) & \rightarrow & H_{n-1}(X^{n-1}, X^{n-2}) & \rightarrow & 0 \\
\delta & \downarrow & j_n & \downarrow & \partial_{n+1} & \downarrow & j_{n-1} & \downarrow & \delta \\
H_{n+1}(X^{n+1}, X^n) & \rightarrow & H_n(X^n) & \rightarrow & H_n(X^n, X^{n-1}) & \rightarrow & H_{n-1}(X^{n-1}, X^{n-2}) & \rightarrow & 0 = H_{n-1}(X^{n-2})
\end{array}
\]

First, we have

\[H_n(X) \cong H_n(X^n) / \text{im}(\delta).\]

Since \(j_n\) is injective, the latter quotient is identified with \(\text{im}(j_n)/\text{im}(\partial_{n+1})\). But since the down-right sequence is exact, we can replace this with \(\ker(\delta)/\text{im}(\partial_{n+1})\). Finally, since \(j_{n-1}\) is injective, the latter is the same as the quotient

\[\ker(\partial_{n+1})/\text{im}(\partial_{n+1}) = H_n^{\text{cell}}(X).\]

\[\blacksquare\]

31. FRI, NOV. 4

Having established this theorem, we will now drop the decoration “cell” on cellular homology. We turn now to examples. In practice, many (connected) examples are given a CW structure with a single 0-cell, so it is useful to have

Proposition 31.1. Suppose that \(X\) is a CW complex with a single 0-cell. Then the differential \(d_1\) is trivial.

Proof. The differential is \(H_1(X^1, \ast) \xrightarrow{\partial} H_0(\ast) \xrightarrow{\sim} H_0(\ast, \emptyset)\). But that connecting homomorphism \(\partial\) is trivial, since the next map in that long exact sequence is the isomorphism \(H_0(\ast) \xrightarrow{\sim} H_0(X^1)\).

\[\blacksquare\]

Example 31.2.

1. \(T^2\) has a CW structure with a single 0 cell, two 1-cells \(a\) and \(b\), and a single 2-cell attached by the map \(S^1 \rightarrow S^1 \vee S^1\) represented by \(aba^{-1}b^{-1}\). It follows that the coefficients in the differential \(\partial_2 : C_2 = \mathbb{Z}\{e\} \rightarrow C_1 = \mathbb{Z}\{a, b\}\) are both \(1 + (-1) = 0\). So the cellular chain complex has no differentials!

2. The Klein bottle \(K\) has a CW structure with a single 0 cell, two 1-cells \(a\) and \(b\), and a single 2-cell attached by the map \(S^1 \rightarrow S^1 \vee S^1\) represented by \(abab^{-1}\). It follows that the differential \(\partial_2 : C_2 = \mathbb{Z}\{e\} \rightarrow C_1 = \mathbb{Z}\{a, b\}\) is \(\partial_2(e) = (2a, 0)\).

3. \(\mathbb{RP}^n\) has a CW structure with a single cell in each dimension. The \(k\)-skeleton is \(\mathbb{RP}^k\), and the attaching map \(q : S^k \rightarrow \mathbb{RP}^k\) for the \((k+1)\)-cell is the defining double cover of \(\mathbb{RP}^k\). To determine the degree of the composition

\[S^k \xrightarrow{q} \mathbb{RP}^k \rightarrow \mathbb{RP}^k/\mathbb{RP}^{k-1} \cong S^k,\]
note that the cover $q$ sends the equator $S^{k-1}$ to $\mathbb{R}P^{k-1}$ and therefore gets collapsed in the next map. It follows that our map factors as

$$S^k \longrightarrow S^k / (S^{k-1}) \cong S^k \vee S^k \longrightarrow S^k.$$  

Thinking now of $\mathbb{R}P^k$ as the quotient of the northern hemisphere of $S^k$, modulo a relation on the equator, we see that the degree of our map on the northern $S^k$ is 1, whereas the degree on the southern $S^k$ is the degree of the antipodal map.

**Lemma 31.3.** Let $a : S^k \longrightarrow S^k$ be the antipodal map. Then $\deg(a) = (-1)^{k+1}$.

**Proof.** The sphere $S^k$ has a standard embedding inside $\mathbb{R}^{k+1}$. The antipodal map is

$$(x_1, \ldots, x_{k+1}) \mapsto (-x_1, \ldots, -x_{k+1})$$

and can therefore be described as the composition of $k+1$ reflections (one in each coordinate). It suffices to show that any reflection has degree $-1$.

If $r$ is a reflection in a hyperplane $H$, then we can think of $H \cap S^k$ as an equator and describe $S^k$ as a CW-complex obtained by attaching two $k$-cells $e_1^k$ and $e_2^k$ along this equator. The difference of the chains $e_1^k - e_2^k$ is a cycle that represents the generator of $\tilde{H}_k(S^k)$. But $r_* (e_1^k - e_2^k) = e_2^k - e_1^k = -(e_1^k - e_2^k)$, so $\deg(r) = -1$.

$$\blacksquare$$

32. MON, NOV. 7

If follows that the differential

$$\partial_{k+1} : C_{k+1}(\mathbb{R}P^n) \cong \mathbb{Z} \longrightarrow C_k(\mathbb{R}P^n) \cong \mathbb{Z}$$

is

$$\partial_{k+1}(e^{k+1}) = (1 + (-1)^{k+1})e^k = \begin{cases} 0 & k \text{ even} \\ 2 & k \text{ odd.} \end{cases}$$

So our cellular chain complex is

$$\ldots \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$  

If $n$ is even, then the first differential is $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$, whereas if $n$ is odd, then the first differential is $\mathbb{Z} \xrightarrow{0} \mathbb{Z}$. We read off

$$H_k(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2\mathbb{Z} & k \text{ odd, } k < n \\ 0 & k \text{ even, } k \leq n \\ \mathbb{Z} & k = n \text{ odd} \\ 0 & k > n. \end{cases}$$

If we want to calculate $H_i(\mathbb{R}P^n; \mathbb{F}_2)$, we first tensor the cellular chain complex with $\mathbb{F}_2$. But then all differentials become zero, and we see that $H_i(\mathbb{R}P^n; \mathbb{F}_2) \cong \mathbb{F}_2$ for $0 \leq i \leq n$.

(4) We can build an infinite-dimensional CW complex $\mathbb{R}P^{\infty}$ as the union of the $\mathbb{R}P^n$’s. The homology of this space is then

$$H_k(\mathbb{R}P^{\infty}) \cong \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2\mathbb{Z} & k \text{ odd} \\ 0 & \text{else.} \end{cases}$$
(5) \( \mathbb{C}P^n \) has a CW structure with a single cell in every even dimension. There is no room for differentials, so we conclude that
\[
H_k(\mathbb{C}P^n) \cong \begin{cases} 
\mathbb{Z} & k \text{ even}, k \leq 2n \\
0 & \text{else}.
\end{cases}
\]

(6) We can build an infinite-dimensional CW complex \( \mathbb{C}P^\infty \) as the union of the \( \mathbb{C}P^n \)'s. The homology of this space is then
\[
H_k(\mathbb{C}P^\infty) \cong \begin{cases} 
\mathbb{Z} & k \text{ even} \\
0 & \text{else}.
\end{cases}
\]

We long ago gave a description of \( H_0(X) \), but we have put off describing \( H_1(X) \). We do this now.

**Theorem 32.1** (Hurewicz). Assume that \( X \) is a connected CW complex. Then
\[
H_1(X) \cong \pi_1(X)_{ab}.
\]

**Proof.** First, note that cells in dimensions 3 or higher affect neither \( \pi_1 \) nor \( H_1 \). In other words, if \( X^2 \) is the 2-skeleton, then \( \pi_1(X^2) \cong \pi_1(X) \) and \( H_1(X^2) \cong H_1(X) \).

By the van Kampen theorem, we know that \( \pi_1(X^1) \to \pi_1(X^2) \) is surjective. Moreover, if we denote by \( \beta_1, \ldots, \beta_k \) the 2-cells of \( X \) (or really, their attaching maps, thought of as elements of \( \pi_1(X^1) \)), then the van Kampen theorem tells us that
\[
\pi_1(X^2) \cong \pi_1(X^1)/(\beta_1, \ldots, \beta_k).
\]

Denote by \( \tilde{X}^1 \) the result of collapsing out a maximal tree in the graph \( X^1 \), and recall that the natural map \( X^1 \to \tilde{X}^1 \) is a homotopy equivalence. The space \( \tilde{X}^1 \) is a wedge of circles \( \tilde{X}^1 \cong \vee S^1 \), each circle corresponding to a generator of \( \pi_1(X^1) \). We now have
\[
\pi_1(X^2) \cong \pi_1(X^1)/(\beta_1, \ldots, \beta_k) \cong F(a_1, \ldots, a_n)/(\beta_1, \ldots, \beta_k).
\]

Let’s now turn to homology. We know that \( H_1(X) \) is computed as a quotient
\[
C_2(X) \to Z_1(X).
\]

**Lemma 32.2.** We have \( Z_1(X) = Z_1(X^1) = H_1(X^1) \cong H_1(\tilde{X}^1) = Z_1(\tilde{X}^1) = C_1(\tilde{X}^1) \).

The homology isomorphism follows from the fact that \( X \to \tilde{X}^1 \) is a homotopy equivalence. The lemma implies that \( H_1(X) \) is the quotient
\[
H_1(X) \cong Z(\alpha_1, \ldots, a_n)/(\beta_1, \ldots, \beta_k).
\]

There is now an obvious surjection
\[
\pi_1(X) \to H_1(X)
\]
induced by the abelianization map \( F(a_1, \ldots, a_n) \to \mathbb{Z}[a_1, \ldots, a_n] \). The following lemma implies that the map \( \pi_1(X) \to H_1(X) \) is also abelianization. \( \blacksquare \)

**Lemma 32.3.** Let \( \phi : F \to G \) be a surjection of groups with kernel \( N \). Then the map \( G = F/N \xrightarrow{\lambda} F_{ab}/N_{ab} \) induces an isomorphism \( G_{ab} \cong F_{ab}/N_{ab} \).

**Proof.** The map out of \( G_{ab} \) comes from the universal property of abelianization:
\[
\begin{array}{ccc}
G = F/N & \xrightarrow{\lambda} & F_{ab}/N_{ab} \\
\downarrow \mu & & \\
G_{ab} & \xrightarrow{\text{universal property}} & F_{ab}/N_{ab}
\end{array}
\]
Since \( \lambda \) is surjective, so is \( \mu \). To see that \( \mu \) is injective, suppose that \( \mu(gN) = 0 \). This means that \( g \in N \cdot [F,F] = [F,F] \cdot N \). But this is the commutator subgroup of \( F/N \), so we are done. \( \blacksquare \)

There is also a statement in higher dimensions, assuming that all lower homotopy groups vanish. We state it without proof.

**Theorem 32.4** (Hurewicz). Assume that \( X \) is a CW complex satisfying \( \pi_k(X) = 0 \) for \( k < n \) (we say that \( X \) is \((n-1)\)-connected), where \( n \geq 2 \). Define

\[
h_n : \pi_n(X) \to H_n(X)\]

by

\[
h_n(\alpha) = \alpha_*(x_n),\]

where \( x_n \in H_n(S^n) \) is the class of the unique \( n \)-cell (in the minimal CW structure on \( S^n \)). Then \( h_n \) is an isomorphism of groups, known as the Hurewicz map.

Using induction and the fundamental group Hurewicz theorem, this implies the following result.

**Corollary 32.5.** Suppose that \( X \) is a CW complex that is \((n-1)\)-connected. Then \( H_k(X) = 0 \) for \( 0 < k < n \) as well.

Note that the torus \( T^2 \) shows that Theorem 32.4 fails if we drop the connectivity hypothesis.

33. WED, NOV. 9

**Homology of products**

Our next goal will be to describe \( H_*(X \times Y) \) in terms of \( H_*(X) \) and \( H_*(Y) \). We will work with cellular homology and will therefore assume that \( X \) and \( Y \) are CW complexes. On your homework, you showed that \( X \times Y \) has a CW structure in which the \( n \)-cells correspond to pairs of \( k \)-cells in \( X \) and \( j \)-cells in \( Y \), where \( k + j = n \).

In other words, we have a bijection

\[
\{n\text{-cells in } X \times Y\} \cong \bigsqcup_{k+j=n} \{k\text{-cells in } X\} \times \{j\text{-cells in } Y\}
\]

Applying the free abelian group functor, we get that

\[
C_n(X \times Y) \cong \bigoplus_{k+j=n} C_k(X) \otimes C_j(Y).
\]

We would like to say that we have an isomorphism of chain complexes, but we first need to discuss how to make the right side into a chain complex.

**Definition 33.1.** If \( C_* \) and \( D_* \) are chain complexes, define a chain complex \( C_* \otimes D_* \) by

\[
(C_* \otimes D_*)_n := \bigoplus_{k+j=n} C_k \otimes D_j
\]

and where the differential \( \partial_{C_* \otimes D_*} \) is defined by

\[
\partial_n(x \otimes y) = \partial(x) \otimes y + (-1)^{\deg(x)} x \otimes \partial(y).
\]
We need to check that this is in fact a complex, in the sense that $\partial_{n-1} \circ \partial_n = 0$. We have

\[
\partial_{n-1}(\partial_n(x \otimes y)) = \partial_{n-1}\left(\partial(x) \otimes y + (-1)^{\deg(x)}x \otimes \partial(y)\right) \\
= \partial(\partial(x)) \otimes y + (-1)^{\deg(\partial(x))}\partial(x) \otimes \partial(y) \\
+ (-1)^{\deg(x)}\partial(x) \otimes \partial(y) + (-1)^{2\deg(x)}x \otimes \partial(\partial(y)) \\
= 0 + (-1)^{\deg(x)-1}\partial(x) \otimes \partial(y) + (-1)^{\deg(x)}\partial(x) \otimes \partial(y) + 0 = 0.
\]

So $C_\ast \otimes D_\ast$ is in fact a chain complex.

**Proposition 33.2.** The above isomorphism extends to an isomorphism of chain complexes $C_\ast(X \otimes Y) \cong C_\ast(X) \otimes C_\ast(Y)$.

**Proof.** We know that $e_{n,\beta}^{\ast} \in C_n(X \times Y)$ maps to $e_k^{\ast} \otimes e_j^{\ast} \in C_k(X) \otimes C_j(Y)$, and that the differential on the latter is

\[
\partial(e_k^{\ast} \otimes e_j^{\ast}) = \partial(e_k^{\ast}) \otimes e_j^{\ast} + (-1)^k e_k^{\ast} \otimes \partial(e_j^{\ast}).
\]

So it remains to describe the differential $\partial(e_{n,\beta}^{\ast})$.

By naturality, it suffices to consider the universal case, in which $X = I^0$, $Y = I^1$, and $X \times Y = I^0 \times I^1 \cong I^n$. We give the argument for $k = j = 1$ and $k = 1, j = 2$. For the general case, see Hatcher, section 3.B.

For $k = j = 1$, we want to compute $\partial(e^2)$ in $C_\ast(I^2)$. If we consider this 2-cell as being oriented counterclockwise, then the formula for $\partial(e^2)$ is

\[
\partial(e^2) = -e^1_{0 \times e^1} + e^1_{1 \times e^1} + e^1_{e^1 \times 0} - e^1_{e^1 \times 1}.
\]

And this exactly maps over to $\partial(e^1) \otimes e^1 - e^1 \otimes \partial(e^1) \in C_\ast(I^1) \otimes C_\ast(I^1)$.

For $k = 1$ and $j = 2$, we want to compute $\partial(e^3)$ in $C_\ast(I^3)$, where we are thinking of $I^3$ as $I^1 \times I^2$. Again, we orient each face of $\partial(I^3)$ with a counterclockwise orientation, looking from the outside of the cube. Then the formula for $\partial(e^3)$ is

\[
\partial(e^3) = -e^2_{0 \times e^2} + e^2_{e^2 \times 0} + e^2_{e^2 \times e^1} - e^2_{e^1 \times e^1} - e^2_{e^1 \times 0} + e^2_{e^1 \times e^1}.
\]

Again, this maps over exactly to $\partial(e^1) \otimes e^2 - e^1 \otimes \partial(e^2) \in C_\ast(I^1) \otimes C_\ast(I^2)$. 

34. FRI, NOV. 11

It follows that the homology of $X \times Y$ is the homology of the complex $C_\ast(X) \otimes C_\ast(Y)$, and it remains to compute this latter homology. The answer is much simpler if we use field coefficients.

**Proposition 34.1.** Let $k$ be a field, and let $C_\ast$ and $D_\ast$ be chain complexes of $k$-vector spaces. Then

\[
H_n(C_\ast \otimes_k D_\ast) \cong \bigoplus_{k+j=n} H_k(C_\ast) \otimes_k H_j(D_\ast).
\]

Before turning to the proof, we consider an example.

**Example 34.2.** Consider $X = Y = \mathbb{R}P^2$. We know that $H_k(\mathbb{R}P^2; \mathbb{F}_2)$ is $\mathbb{F}_2$ when $k = 0, 1, 2$ and is zero in other degrees. The corollary gives us that

\[
\dim_{\mathbb{F}_2} H_k(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{F}_2) \cong \begin{cases} 
1 & k = 0, 4 \\
2 & k = 1, 3 \\
3 & k = 2 \\
0 & \text{else}
\end{cases}.
\]
If we try to compute this directly, we use the cellular chain complex for $\mathbb{RP}^2 \times \mathbb{RP}^2$, which takes the form

$$
\begin{array}{cccccccc}
\mathbb{Z}\{e_{2,2}^4\} & \mathbb{Z}\{e_{1,2}^3, e_{2,1}^3\} & \mathbb{Z}\{e_{0,2}^2, e_{1,1}^2, e_{2,0}^2\} & \mathbb{Z}\{e_{0,1}^1, e_{1,0}^1\} & \mathbb{Z}\{e_{0,0}^0\} \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
\mathbb{Z}\{e_{2,2}^2\} & \mathbb{Z}\{e_{1,2}^2, e_{2,1}^2\} & \mathbb{Z}\{e_{0,2}^1, e_{1,1}^1, e_{2,0}^1\} & \mathbb{Z}\{e_{0,1}^0, e_{1,0}^0\} & \\
\end{array}
$$

If we tensor with $\mathbb{F}_2$, then all differentials become zero, and the homology is as given above.

On the other hand, the above example shows that Corollary 34.3 does not hold with $\mathbb{Z}$-coefficients. Recall that the integral homology of $\mathbb{RP}^2$ is $\mathbb{Z}$ in degree zero and $\mathbb{Z}/2\mathbb{Z}$ in degree 1. So if we just take tensor product of the homology, we don’t get anything above degree two. But the above complex has a $\mathbb{Z}/2\mathbb{Z}$ in the homology in degree 3.

**Proof.** There are several advantages to working with vector spaces. For one, every short exact sequence always splits (since every vector space is a free module). This implies that tensoring with a vector space will always preserve short exact sequences as well.

More generally, if $C$ is a vector space and $D_s$ is a chain complex of vector spaces, we will have $H_n(C \otimes D_s) \cong C \otimes H_n(D_s)$ (on the homework, you are asked to show this in the context of free abelian groups). In particular, we can take $C$ to be any of the $C_j$. Now if $C_s$ is a chain complex in which all differentials are zero, we are done.

Now consider a general complex $C_s$, and let $B_s \subseteq Z_s \subseteq C_s$ be the subcomplexes of boundaries and cycles, respectively. Then the complexes $B_s$ and $Z_s$ have no differentials, and moreover we have a short exact sequence of complexes

$$
0 \longrightarrow Z_s \longrightarrow C_s \overset{\partial}{\longrightarrow} B_s \longrightarrow 0.
$$

Again, this will still be exact after tensoring with a complex $D_s$, so that we have

$$
0 \longrightarrow Z_s \otimes D_s \longrightarrow C_s \otimes D_s \overset{\partial \otimes \text{id}}{\longrightarrow} B_s \otimes D_s \longrightarrow 0.
$$

This short exact sequence gives rise to a long exact sequence in homology

$$
\cdots \longrightarrow H_n(Z_s \otimes D_s) \longrightarrow H_n(C_s \otimes D_s) \longrightarrow H_n(B_s \otimes D_s) \longrightarrow H_{n-1}(Z_s \otimes D_s) \longrightarrow \cdots
$$

Tracing through, you can show that the connecting homomorphism $H_n(B_s \otimes D_s) \longrightarrow H_{n-1}(Z_s \otimes D_s)$ is simply induced by the inclusing of subcomplexes $B_s \hookrightarrow Z_s$.

Since $B_s$ and $Z_s$ are both complexes with trivial differentials, we can rewrite the sequence as

$$
\longrightarrow (Z_s \otimes H_s(D_s))_n \longrightarrow H_n(C_s \otimes D_s) \longrightarrow (B_s \otimes H_s(D_s))_n \longrightarrow (Z_s \otimes H_s(D_s))_{n-1} \longrightarrow \cdots
$$

This now splits as a bunch of short exact sequences

$$
0 \longrightarrow B_s \otimes H_s(D_s) \longrightarrow Z_s \otimes H_s(D_s) \longrightarrow H_s(C_s \otimes D_s) \longrightarrow 0.
$$

Again, since tensoring with $H_s(D_s)$ preserves exact sequences, we conclude that $H_s(C_s \otimes D_s) \cong H_s(C_s) \otimes H_s(D_s)$.

**Corollary 34.3.** Let $k$ be a field and $X$ and $Y$ CW complexes. Then

$$
H_n(X \times Y; k) \cong \bigoplus_{k+j=n} H_k(X; k) \otimes_k H_j(Y; k).
$$

**Proof.** This will follow from Proposition 34.1. We have

$$
C_s(X) \otimes_Z C_s(Y) \otimes_Z k \cong C_s(X) \otimes_Z C_s(Y) \otimes_Z k \otimes k \cong (C_s(X) \otimes_Z k) \otimes k (C_s(Y) \otimes_Z k).
$$

Now just apply Proposition 34.1.
Last time, we proved the Kunneth theorem for field coefficients. The example of $\mathbb{RP}^2 \times \mathbb{RP}^2$ shows that the result does not always hold with integer coefficients. As we will see, it holds if the homology groups of $X$ and $Y$ are torsion-free, as in the case of the torus.

**Example 35.1.** $X = T^2 = S^1 \times S^1$. Here we do have an isomorphism

$$H_*(T^2; \mathbb{Z}) \cong H_*(S^1; \mathbb{Z}) \otimes H_*(S^1; \mathbb{Z}).$$

Looking back to the proof of Proposition 34.1, we can try to give the argument with integral chains and see where it breaks down. Since each cellular chain groups $C_*(\tilde{X})$ is free abelian, and since $B_n \subseteq C_n(X)$ is a subgroup, it follows that $B_n$ is also free abelian. This implies that every short exact sequence

$$0 \to Z_n \to C_n(X) \to B_{n-1} \to 0$$

splits, so that tensoring with any group will again produce a short exact sequence. Free abelian groups are flat (i.e., tensoring with them preserves exact sequences) and the complexes $Z_*$ and $B_*$ have zero differentials, so it follows that

$$H_*(Z_* \otimes D_*) \cong Z_* \otimes H_*(D_*) \quad \text{and} \quad H_*(B_* \otimes D_*) \cong B_* \otimes H_*(D_*).$$

The spot where the argument breaks down is that although the connecting homomorphisms in the long exact sequence

$$\lambda \otimes \text{id} \quad (Z_* \otimes H_*(D_*)) \to H_*(C_* \otimes D_*) \to (B_* \otimes H_*(D_*)) \to \ldots$$

are induced by the inclusion $\lambda_{n-1} : B_{n-1} \to Z_{n-1}$, we do not know that these are injective after tensoring with the groups $H_*(D)$. The best we can say is that we have short exact sequences

$$0 \to \text{coker}(\lambda_n \otimes \text{id}) \to H_*(C_* \otimes D_*) \to \ker(\lambda_{n-1} \otimes \text{id}) \to 0.$$ 

But tensoring with any abelian group is right-exact, meaning that it preserves quotients. So $\text{coker}(\lambda_n \otimes \text{id}) \cong \text{coker}(\lambda_n) \otimes H_*(D) \cong H_*(C) \otimes H_*(D)$. So we have a short exact sequence

$$0 \to (H_*(C) \otimes H_*(D)) \to H_*(C_* \otimes D_*) \to \ker(\lambda_{n-1} \otimes \text{id}) \to 0.$$ 

It remains to identify the kernel of $\lambda_{n-1} \otimes \text{id}$.

**Definition 35.2.** Let $A$ be an abelian group. Then a **free resolution** of $A$ is an exact sequence

$$\ldots \to F_2 \to F_1 \to F_0 \to A \to 0$$

in which each group $F_n$ is free abelian.

**Proposition 35.3.** Any abelian group has a free resolution of length 1, meaning that $F_n = 0$ for $n > 1$.

**Proof.** First pick any surjection $F_0 \xrightarrow{\epsilon} A$, where $F_0$ is free abelian. This amounts to choosing a set of generators for $A$. Define $F_1 = \ker(\epsilon)$. Then $F_1$ is a subgroup of a free abelian group and is therefore free abelian. \hfill \square

**Definition 35.4.** Let $F_1 \xrightarrow{\phi} F_0 \xrightarrow{\epsilon} A$ be a free resolution, and let $B$ be an abelian group. Define

$$\text{Tor}(A, B) := \ker(\phi \otimes \text{id}_B : F_1 \otimes B \to F_0 \otimes B).$$

We need to show that this does not depend on the choice of resolution.

**Lemma 35.5.** Any two free resolutions of $A$ are chain-homotopy equivalent.
Proof. Let

\[
\begin{array}{c}
F_1 \xrightarrow{\phi} F_0 \xrightarrow{\epsilon} A \\
| \quad | \\
\downarrow f_1 \quad \downarrow \phi \quad \downarrow \delta \\
G_1 \xrightarrow{\psi} G_0
\end{array}
\]

be free resolutions. Since \(F_0\) and \(G_0\) are free, we can find maps \(f_0\) and \(g_0\) as in the diagram, and this induces factorizations \(f_1\) and \(g_1\). To see, for example, that \(g_* f_* : F_* \rightarrow F_*\) is chain-homotopic to the identity, we need a chain homotopy \(h_0 : F_0 \rightarrow F_1\) with

\[
g_0 f_0(x) - x = \varphi h_0(x) \quad \text{and} \quad g_1 f_1(x) - x = h_0 \varphi(x).
\]

But

\[
\epsilon (g_0 f_0(x) - x) = \epsilon g_0 f_0(x) - \epsilon(x) = \epsilon f_0(x) - \epsilon(x) = 0,
\]

so \(g_0 f_0 - \text{id}\) lands in the kernel of \(\epsilon\), which is \(F_1\). That is, we have a factorization \(F_0 \xrightarrow{h_0} F_1 \xrightarrow{\phi} F_0\) of \(g_0 f_0 - \text{id}\). For the second equation, since \(\varphi\) is injective, it suffices to check it after applying \(\varphi\). But

\[
\varphi (g_1 f_1(x) - x) = \varphi g_1 f_1(x) - \varphi(x) = g_0 \varphi f_1(x) - \varphi(x) = g_0 f_0 \varphi(x) - \varphi(x) = \varphi h_0 \varphi(x),
\]

so we are done.

\[\blacksquare\]

36. Wed, Nov. 16

The ideas in Lemma 35.5 can be used to more generally prove

**Proposition 36.1.** Suppose that \(f_* : C_* \rightarrow D_*\) is a quasi-isomorphism between chain complexes of free abelian groups. Then \(f_*\) is a chain homotopy-equivalence.

We are now ready to prove

**Proposition 36.2.** The group \(\text{Tor}(A, B)\) does not depend on the choice of free resolution of \(A\). Moreover, this group can also be computed by choosing instead a free resolution for \(B\) rather than \(A\).

Proof. By Lemma 35.5, any two resolutions are chain homotopy-equivalent. But chain homotopy-equivalences are preserved by tensoring with \(B\), so it follows that \(\text{Tor}(A, B)\) is independent of the choice of resolution.

Now let \(F_* \xrightarrow{\epsilon} A\) and \(G_* \xrightarrow{\delta} B\) be free resolutions. Note that we can think of \(\epsilon\) and \(\delta\) as quasi-isomorphisms of chain complexes. Then we have a zig-zag of chain maps

\[
F_* \otimes B \xrightarrow{\text{id} \otimes \delta} F_* \otimes G_* \xrightarrow{\epsilon \otimes \text{id}} A \otimes G_*.
\]

By a problem on your homework, these are both quasi-isomorphisms (since \(F_*\) and \(G_*\) are complexes of free abelian groups). By Proposition 36.1, these are both chain homotopy equivalences, so that composing \(\epsilon \otimes \text{id}\) with a homotopy inverse for \(\text{id} \otimes \delta\) gives the desired result.

\[\blacksquare\]

Going back to the reason we introduced \(\text{Tor}\), recall that we saw the group

\[
\ker \left( B_{n-1} \otimes H_j(D) \xrightarrow{\lambda_{n-1} \otimes \text{id}} Z_{n-1} \otimes H_j(D) \right)
\]

showing up in an exact sequence. Since \(\text{coker}(\lambda_{n-1}) \cong H_j(C)\), it follows that the kernel in question is precisely \(\text{Tor}(H_{n-1}(C), H_j(D))\). We have now proved
Theorem 36.3. [Künneth] For CW complexes $X$ and $Y$, there is an exact sequence

$$0 \longrightarrow H_* (X; \mathbb{Z}) \otimes H_* (Y; \mathbb{Z}) \longrightarrow H_* (X \times Y; \mathbb{Z}) \longrightarrow \text{Tor} (H_{*-1} (X), H_* (Y)) \longrightarrow 0.$$  

In fact this sequence is always split, so that there is an isomorphism

$$H_n (X \times Y; \mathbb{Z}) \cong \left( \bigoplus_{i+j=n} H_i (X; \mathbb{Z}) \otimes H_j (Y; \mathbb{Z}) \right) \oplus \left( \bigoplus_{i+j=n} \text{Tor} \left( H_{i-1} (X; \mathbb{Z}), H_j (Y; \mathbb{Z}) \right) \right).$$

Example 36.4. We turn back to $X = \mathbb{R}P^2 \times \mathbb{R}P^2$. Using the Künneth theorem and remembering that $\mathbb{R}P^2$ only has nontrivial homology in degree 0 and 1, we get

$$H_0 (\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}) \cong H_0 (\mathbb{R}P^2; \mathbb{Z}) \otimes_{\mathbb{Z}} H_0 (\mathbb{R}P^2; \mathbb{Z}) \cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z},$$

$$H_1 (\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}) \cong H_1 (\mathbb{R}P^2; \mathbb{Z}) \otimes_{\mathbb{Z}} H_0 (\mathbb{R}P^2; \mathbb{Z}) \oplus H_0 (\mathbb{R}P^2; \mathbb{Z}) \otimes_{\mathbb{Z}} H_1 (\mathbb{R}P^2; \mathbb{Z})$$

$$\cong \mathbb{Z} / 2 \mathbb{Z} \oplus \mathbb{Z} / 2 \mathbb{Z} \oplus \text{Tor} (\mathbb{Z}, \mathbb{Z}),$$

$$H_2 (\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}) \cong H_1 (\mathbb{R}P^2; \mathbb{Z}) \otimes_{\mathbb{Z}} H_1 (\mathbb{R}P^2; \mathbb{Z}) \oplus \text{Tor} (H_0 (\mathbb{R}P^2; \mathbb{Z}), H_1 (\mathbb{R}P^2; \mathbb{Z}))$$

$$\cong \mathbb{Z} / 2 \mathbb{Z} \oplus \text{Tor} (\mathbb{Z}, \mathbb{Z} / 2 \mathbb{Z}) \oplus \text{Tor} (\mathbb{Z} / 2 \mathbb{Z}, \mathbb{Z}),$$

$$H_3 (\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}) \cong \text{Tor} (H_1 (\mathbb{R}P^2; \mathbb{Z}), H_1 (\mathbb{R}P^2; \mathbb{Z}))$$

$$\cong \text{Tor} (\mathbb{Z} / 2 \mathbb{Z}, \mathbb{Z} / 2 \mathbb{Z}).$$

There are three Tor groups to compute. Using the free resolutions $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z} / 2 \mathbb{Z}$, we see that these groups are

$$\text{Tor} (\mathbb{Z}, \mathbb{Z}) = 0 \quad \text{Tor} (\mathbb{Z}, \mathbb{Z} / 2 \mathbb{Z}) = 0, \quad \text{Tor} (\mathbb{Z} / 2 \mathbb{Z}, \mathbb{Z}) = 0, \quad \text{Tor} (\mathbb{Z} / 2 \mathbb{Z}, \mathbb{Z} / 2 \mathbb{Z}) \cong \mathbb{Z} / 2 \mathbb{Z}.$$

It follows that

$$H_0 (\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}) \cong \mathbb{Z}, \quad H_1 (\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}) \cong \mathbb{Z} / 2 \mathbb{Z} \oplus \mathbb{Z} / 2 \mathbb{Z},$$

$$H_2 (\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}) \cong \mathbb{Z} / 2 \mathbb{Z}, \quad H_3 (\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}) \cong \mathbb{Z} / 2 \mathbb{Z}.$$

This is the same answer that comes from the chain complex we wrote down in Example 34.2.
Cohomology

We have now developed quite a bit of machinery, so let’s try to answer the following question:

**Problem:** Show that \( \mathbb{C}P^2 \) is not homotopy equivalent to \( S^2 \vee S^4 \).

The first tool we learned about for distinguishing homotopy types is the fundamental group, but both of these spaces are simply-connected (the 2-skeleton of both spaces is \( S^2 \)). The next tool we learned about was homology, but the homology of both of these spaces is \( \mathbb{Z} \) in dimensions 0, 2, 4 and trivial in other dimensions. So we need something else! Cohomology will allow us to distinguish these spaces.

In defining homology, we always worked with chain complexes. Cohomology starts with cochain complexes.

**Definition 37.1.** A cochain complex \( C^* \) is a sequence \( C^n \) of abelian groups, together with differentials \( \partial^n : C^n \to C^{n+1} \), such that \( \partial^{n+1} \circ \partial^n = 0 \). Given a cochain complex \( C^* \), we define its cohomology groups to be

\[
H^n(C^*) := \ker(\partial^n) / \text{im}(\partial^{n-1}).
\]

There is a canonical way to obtain a cochain complex from a chain complex, simply by dualizing. Namely, if \( C^* \) is a chain complex, we define the dual cochain complex by

\[
C^n := \text{Hom}(C_n, \mathbb{Z}),
\]

with differential given by \( \partial^n = \text{Hom}(\partial_{n+1}, \mathbb{Z}) \). More precisely, if \( f \in \text{Hom}(C_n, \mathbb{Z}) \), then \( \partial^n(f) \in \text{Hom}(C_{n+1}, \mathbb{Z}) \) is defined by

\[
(37.2) \quad \partial^n(f)(x) = -(-1)^n f(\partial_{n+1}(x)),
\]

where the sign arises from the Koszul sign rule. The “extra” negative sign out front appears from the general formula \( \partial(f) = \partial \circ f - (-1)^{\deg(f)} f \circ \partial \).

Since \( \text{Hom}(\cdot, \mathbb{Z}) \) is only left-exact, the cohomology groups are not simply the duals of the homology groups, as we will see in examples below.

**Definition 37.3.** We define the cohomology of a space \( X \) by

\[
H^n(X; \mathbb{Z}) := H^n(\text{Hom}(C_*(X), \mathbb{Z})).
\]

We can define this in any setting in which we defined homology before.

**Example 37.4.**

1. \( \mathbb{X} = S^1 \). If we dualize the cellular chain complex, \( \mathbb{Z} \to \mathbb{Z} \), we get the cochain complex \( \mathbb{Z} \leftarrow \mathbb{Z} \), so that the cohomology groups agree with the homology groups in this case.

2. \( \mathbb{X} = T^2 \). If we dualize the cellular chain complex \( \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \), we get the cochain complex

\[
\mathbb{Z} \leftarrow \mathbb{Z}^2 \leftarrow \mathbb{Z},
\]

so that again the cohomology groups are the same as the homology groups.

3. \( \mathbb{X} = \mathbb{R}P^2 \). If we dualize the cellular chain complex \( \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \), we get the cochain complex

\[
\mathbb{Z} \leftarrow \mathbb{Z}^2 \leftarrow \mathbb{Z},
\]

so that we have

\[
H^n(\mathbb{R}P^2; \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & n = 0 \\
\mathbb{Z}/2\mathbb{Z} & n = 2 \\
0 & \text{else}.
\end{cases}
\]
This finally gives us an answer which differs from homology.

We can also compute the cohomology using coefficients in \( \mathbb{F}_2 \). If we start the (integral) cellular chain complex into \( \mathbb{F}_2 \), we get the cochain complex of \( \mathbb{F}_2 \)-vector spaces

\[
\mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2 \leftarrow \mathbb{F}_2.
\]

The cohomology groups are

\[
H^n(\mathbb{F}_2; \mathbb{F}_2) \cong \begin{cases} 
\mathbb{F}_2 & n = 0 \\
0 & \text{else.}
\end{cases}
\]

These agree with the mod 2 homology groups \( H_n(\mathbb{R}P^2; \mathbb{F}_2) \).

38. MON, NOV. 21

So we see that, sometimes the cohomology groups of a space agree with the homology groups, but not always. Let’s now determine the precise relationship.

We will again work in the general context of a chain complex \( C_* \) of free abelian groups, and we will let \( M \) be an arbitrary abelian group of coefficients. Like in the proof of the K"unneth theorem, we have the short exact sequence of chain complexes

\[
0 \rightarrow Z_* \rightarrow C_* \rightarrow B_{*-1} \rightarrow 0.
\]

Here \( B_{*-1} \) is the chain complex with \( (B_{*-1})_n = B_{n-1} \). Since \( B_{*-1} \) is a complex of free abelian groups, this sequence splits. This means that applying \( \text{Hom}(-, M) \) will produce a (split) short exact sequence of cochain complexes. Taking cohomology then gives a long exact sequence in cohomology

\[
\text{Hom}^n(B_{*-1}, M) \rightarrow \text{Hom}^n(\text{Hom}(C_*), M) \rightarrow \text{Hom}^n(\text{Hom}(Z_*), M) \rightarrow \text{Hom}^n+1(B_{*-1}, M) \rightarrow \ldots
\]

Now the complexes \( B_* \) and \( Z_* \) have trivial differentials, so this remains true after applying \( \text{Hom}(-, M) \). The above long exact sequence then becomes

\[
\text{Hom}^n(B_{*-1}, M) \rightarrow \text{Hom}^n(\text{Hom}(C_*), M) \rightarrow \text{Hom}^n(\text{Hom}(Z_*), M) \rightarrow \text{Hom}^n+1(B_{*-1}, M) \rightarrow \ldots
\]

Note that \( (B_{*-1})_{n+1} = B_{(n+1)-1} = B_n \), so that \( \text{Hom}^{n+1}(B_{*-1}, M) = \text{Hom}^n(B_* , M) \). The connecting homomorphism

\[
\text{Hom}^n(Z_* , M) \rightarrow \text{Hom}^n(B_* , M)
\]

is \( \text{Hom}(\iota, M) \), where \( \iota : B_* \hookrightarrow Z_* \) is the inclusion. It follows that our long exact sequence splits into a bunch of short exact sequences

\[
0 \rightarrow \text{coker}(\text{Hom}(\iota, M))^{n-1} \rightarrow \text{Hom}^n(\text{Hom}(C_*), M) \rightarrow \ker(\text{Hom}(\iota, M))^n \rightarrow 0.
\]

We have a short exact sequence

\[
0 \rightarrow B_* \rightarrow Z_* \rightarrow H_*(C_*) \rightarrow 0.
\]

By HW 11, \( \text{Hom}(-, M) \) is left exact, so that \( \ker(\text{Hom}(\iota, M))^n = \text{Hom}(H_*(C_*), M) \). We have a short exact sequence

\[
0 \rightarrow \text{coker}(\text{Hom}(\iota, M))^{n-1} \rightarrow \text{Hom}^n(\text{Hom}(C_*), M) \rightarrow \text{Hom}(H_*(C_*), M) \rightarrow 0.
\]

Like in the proof of the K"unneth theorem, this sequence splits, and we are left with an “error” term to understand.

**Definition 38.1.** Let \( F_1 \rightarrow F_0 \rightarrow A \) be a free resolution and let \( M \) be an abelian group. We define

\[
\text{Ext}(A, M) := \text{coker} \big( \text{Hom}(F_0, M) \rightarrow \text{Hom}(F_1, M) \big).
\]

**Proposition 38.2.** The group \( \text{Ext}(A, M) \) does not depend on the choice of resolution of \( A \).
This follows from Lemma 35.5.

To summarize, we have

**Theorem 38.3 (Universal Coefficients).** For any chain complex $C_*$ of free abelian groups and any abelian group $M$, we have isomorphisms

$$H^n(\Hom(C_*, M)) \cong \Hom(H_n(C_*), M) \oplus \Ext(H_{n-1}(C_*), M).$$

When applied to the cohomology of a space, this theorem reads as

**Theorem 38.4 (Universal Coefficients).** For any space $X$ and any abelian group $M$, we have isomorphisms

$$H^n(X; M) \cong \Hom(H_n(X; \mathbb{Z}), M) \oplus \Ext(H_{n-1}(X; \mathbb{Z}), M).$$

**Proposition 38.5.** $\Ext(\mathbb{Z}, A) = 0$ and $\Ext(\mathbb{Z}/n\mathbb{Z}, A) \cong A/nA$.

*Proof.* The first statement is immediate since $\mathbb{Z}$ has a free resolution of length 0. The second follows immediately from the free resolution $\mathbb{Z} \xrightarrow{n} \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$. ■

Note that it follows that, unlike Tor, the groups $\Ext(A, M)$ are not symmetric in $A$ and $M$.

**Example 38.6.** Starting from the integral homology of $\mathbb{RP}^2$, which is $H_0 \cong \mathbb{Z}$ and $H_1 \cong \mathbb{Z}/2\mathbb{Z}$, we can deduce the integral cohomology, as well as the mod 2 cohomology. The integral cohomology is as found in example 37.4 because $\Hom(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$ and $\Ext(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. The mod 2 cohomology is found similarly.
There is also a Universal Coefficients Theorem for homology. It reads

**Theorem 39.1** (Universal Coefficients, Homology). For any space $X$ and abelian group $M$, there are isomorphisms

$$H_n(X; M) \cong (H_n(X; \mathbb{Z}) \otimes \mathbb{Z} M) \oplus \text{Tor}(H_{n-1}(X; \mathbb{Z}), M).$$

On your homework, you showed that $\text{Tor}(\mathbb{Z}, A) = 0$ and that $\text{Tor}(\mathbb{Z}/n\mathbb{Z}, A)$ is the $n$-torsion subgroup of $A$.

**Example 39.2.** This gives, for example, the mod 2 homology of $\mathbb{R}P^n$ from the integral homology. On the other hand, the mod 2 homology is easier, and it is often possible to deduce the integral homology from the mod $p$ homology.

For instance, the Kunneth theorem easily gives us that

$$\dim H_n(\mathbb{R}P^2 \times \mathbb{R}P^2; F_2) = \begin{cases} 1 & n = 0, 4 \\ 2 & n = 1, 3 \\ 3 & n = 2 \end{cases}$$

while

$$\dim H_n(\mathbb{R}P^2 \times \mathbb{R}P^2; F_p) = \begin{cases} 1 & n = 0 \\ 0 & n > 0 \end{cases}$$

for $p$ odd. By the Universal Coefficient theorem, the homology of $\mathbb{R}P^2 \times \mathbb{R}P^2$ must all be 2-torsion in positive degrees, since any other summands would be detected in homology with mod $p$ coefficients for some $p$. (The free summands would be detected at every odd prime.)

We already know $H_0(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}) \cong \mathbb{Z}$. Now

$$\mathbb{F}_2 \oplus \mathbb{F}_2 \cong H_1(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{F}_2) \cong H_1(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}) \otimes \mathbb{F}_2 \oplus \text{Tor}(H_0(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}), \mathbb{F}_2)$$

$$= H_1(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}) \otimes \mathbb{F}_2.$$ 

This implies that $H_1(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}) \cong \mathbb{Z}/2^i \oplus \mathbb{Z}/2^j$ for some natural numbers $i$ and $j$. Continuing in this way, we find that $H_2(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}) \cong \mathbb{Z}/2^l$ and $H_3(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}) \cong \mathbb{Z}/2^k$. If we had the homology with coefficients in $\mathbb{Z}/4$ as input rather than just with $\mathbb{Z}/2$, we would see that the integers $i, j, k,$ and $\ell$ are all equal to 1.

**Cohomology as a functor**

We defined the cohomology of a space by dualizing a chain complex $C_\ast(X)$ and then passing to cohomology of the cochain complex. If we start with a chain functor $C_\ast(\ast) : \text{Top} \rightarrow \text{Ch}_{\geq 0}(\mathbb{Z})$, like singular chains, then it follows that the resulting cohomology theory is also a functor on spaces. However, in the process of dualizing, we turn a covariant functor into a contravariant functor, so that we have

**Proposition 39.3.** Singular cohomology defines a contravariant functor

$$H^\ast(\ast; \mathbb{Z}) : \text{Top}^{op} \rightarrow \text{GrAb}.$$ 

Just as for homology, simplicial cohomology is only functorial with respect to $\Delta$-maps. We did not previously discuss functoriality of cellular homology.

**Definition 39.4.** Let $X$ and $Y$ be CW complexes. We say that $f : X \rightarrow Y$ is cellular if, for each $n \geq 0$, we have $f(X^n) \subseteq Y^n$. 

57
In other words, \( f \) should map the \( n \)-skeleton of \( X \) into the \( n \)-skeleton of \( Y \). A composition of two cellular maps is again cellular, and the identity map of any CW complex is cellular. This means that the following definition is valid.

**Definition 39.5.** Let \( \text{CW}_{\text{cell}} \) denote the category whose objects are CW complexes and whose morphisms are cellular maps.

**Proposition 39.6.** Cellular homology and cohomology determine functors

\[
H^*_\text{cell}: \text{CW}_{\text{cell}} \to \text{GrAb}, \quad H^*\text{cell}: (\text{CW}_{\text{cell}})^{op} \to \text{GrAb}.
\]

The point is that you need the assumption that \( f \) is cellular in order to make sense of an induced map \( C^*_\text{cell}(X) \to C^*_\text{cell}(Y) \). The formula for \( f_* \) is given in much the same way as the cellular differential. For an \( n \)-cell \( e^n_\alpha \) of \( X \), then we set

\[
f_*(e^n_\alpha) := \sum_{\beta} n^f_{\alpha, \beta} e^n_\beta,
\]

where \( n^f_{\alpha, \beta} \) is the degree of the map

\[
S^n \to Y^n / Y^{n-1} \to S^n 
\]

The middle map only makes sense if \( f \) is assumed to be cellular.

It is certainly a deficiency in cellular (co)homology that it is only functorial with respect to cellular maps. For example, a famously noncellular map is the diagonal \( X \to X \times X \), for any space \( X \). On the other hand, we can always use the following to replace an arbitrary map by a cellular one.

**Theorem 39.7** (Cellular approximation, Theorem 4.8 of Hatcher). Let \( f: X \to Y \) be a map between CW complexes. Then \( f \) is homotopic to a cellular map \( \tilde{f}: X \to Y \). Furthermore, any two such cellular replacements for \( f \) are cellularly homotopic to each other, meaning that the homotopy \( h: X \times I \to Y \) is cellular.

This means that if we denote by \( \text{Ho}(\text{CW}) \) the category whose objects are CW complexes and whose morphisms are homotopy classes of (arbitrary) maps, then we have the following result.

**Proposition 39.8.** Cellular homology and cohomology determine functors

\[
H^*_\text{cell}: \text{Ho}(\text{CW}) \to \text{GrAb}, \quad H^*\text{cell}: (\text{Ho}(\text{CW}))^{op} \to \text{GrAb}.
\]

There is a similar story for simplicial (co)homology, using

**Theorem 39.9** (Simplicial approximation, Theorem 2C.1 of Hatcher). Let \( f: X \to Y \) be a map between \( \Delta \)-complexes. If \( X \) is a finite complex, then \( f \) is homotopic to a \( \Delta \)-map after applying barycentric subdivision to \( X \) finitely many times.

40. Wed, Nov. 30

Last time, we mentioned that cohomology is a **contravariant** functor. To see this, let \( f: X \to Y \) be a (suitable) map, and let \( \alpha \in C^n(Y) \) be a cochain (in whichever variant of cohomology you prefer). Then \( \alpha \) is a homomorphism \( C^n(Y) \xrightarrow{\alpha} \mathbb{Z} \), and it can be precomposed with \( C_n(f) \) to define

\[
C_n(X) \xrightarrow{f^*} C_n(Y) \xrightarrow{\alpha} \mathbb{Z}
\]

58
For instance, suppose that \( \gamma \in \pi_1(Y) \). Since \( \gamma \) is represented by a map \( S^1 \to Y \), it induces a homomorphism \( H^*(Y) \to H^*(S^1) \). Working the other way, a map \( X \to S^1 \) will induce a homomorphism \( Z \cong H^1(S^1) \to H^1(X) \). Such a homomorphism is determined by its value on a generator, and it turns out that this defines a bijection
\[
H^1(X; Z) \leftrightarrow [X, S^1].
\]
Similarly, there are bijections
\[
H^2(X; Z) \leftrightarrow [X, \mathbb{CP}^\infty]
\]
and
\[
H^1(X; \mathbb{F}_2) \leftrightarrow [X, \mathbb{RP}^\infty].
\]
These bijections are all natural in \( X \). If we plug in the spheres \( X = S^n \) as \( n \) varies, these bijections correspond to the fact that the spaces \( S^1, \mathbb{CP}^\infty, \) and \( \mathbb{RP}^\infty \) all have homotopy groups concentrated in a single degree. Such spaces are known as **Eilenberg-Mac Lane spaces**, and it can be shown that for each abelian group \( G \) and \( n \geq 1 \), there is a space \( K(G, n) \) whose only nontrivial homotopy group is \( G \), concentrated in degree \( n \) (and \( G \) can be nonabelian if \( n = 1 \)). In this language, we would say
\[
S^1 \cong K(\mathbb{Z}, 1) \quad \mathbb{CP}^\infty \cong K(\mathbb{Z}, 2), \quad \mathbb{RP}^\infty \cong K(\mathbb{Z}/2, 1).
\]
For most groups and most values of \( n \), we do not have such nice geometric models.

**Cup products**

It turns out that, for any space \( X \) and any commutative ring \( R \) of coefficients, \( H^*(X; R) \) will be a graded ring. To say it is a graded ring means that

1. The unit 1 is in degree 0 and
2. If \( x \) and \( y \) are in degree \( n \) and \( k \), respectively, then \( x \cdot y \) is in degree \( n + k \).

The unit is quite easy to describe: define \( u \in C^0(X; R) = \text{Hom}(C_0(X), R) \) to be the function which takes value 1 on each basis element.

**Lemma 40.1.** \( u \) is a cocycle and therefore determines a cohomology class.

**Proof.** In any of our three versions of homology, the differential \( \delta_1 : C_1(X) \to C_0(X) \) is given by \( \delta_1(e) = e_1 - e_0 \). Since \( u(e_1) = 1 = u(e_0) \), we conclude that \( \delta^0(u)(e) = 0 \) for all \( e \), so that \( \delta^0(u) = 0 \). \( \blacksquare \)

Note that since there is no \( \delta^{-1} \) coming into \( C^0(X; R) \), it follows that \( u \) is a nontrivial cohomology class, and this will play the role of the unit.

**41. Fri, Dec. 2**

We are left with specifying the multiplication
\[
H^n(X; R) \otimes H^k(X; R) \to H^{n+k}(X; R).
\]
There are several ways to do this. One way is to first write down an “external” product
\[
H^n(X; R) \otimes H^k(Y; R) \to H^{n+k}(X \times Y; R).
\]
This is also known as the **cross product**.

Let’s consider first cellular cohomology. Recall that we have an isomorphism \( C_*(X) \otimes C_*(Y) \cong C_*(X \times Y) \). Let \( \varphi \) be the composition
\[
C^*(X; R) \otimes C^*(Y; R) = \text{Hom}(C_*(X), R) \otimes \text{Hom}(C_*(Y), R) \to \text{Hom}(C_*(X) \otimes C_*(Y), R \otimes R)
\]
\[
\cong \text{Hom}(C_*(X \times Y), R \otimes R) \to \text{Hom}(C_*(X \times Y), R),
\]

where the last map is simply induced by the multiplication \( R \otimes R \to R \) in the ring \( R \). Then we define the external product as

\[
H^*(X; R) \otimes H^*(Y; R) \to H^*(C^*(X; R) \otimes C^*(Y; R)) \xrightarrow{H^*(\varphi)} H^*(X \times Y; R).
\]

Finally, the cup product in cellular cohomology is defined as the composition

\[
H^*(X; R) \otimes H^*(X; R) \to H^*(X \times X; R) \xrightarrow{\Delta^*} H^*(X; R).
\]

However, recall that, as we discussed last time, the diagonal \( \Delta : X \to X \times X \) is not a cellular map, so in order to actually compute the cup product, a cellular approximation of the diagonal must be used.

**Proposition 41.1.** The cup product makes \( H^*(X; R) \) into a graded ring.

**Proof.** We must check that the cup product is associative and unital. To show that \( u \) is a left unit, we first note that \( u \) can also be described as \( u = c^*(1) \), where \( c : X \to * \). Note also that

\[
X \xrightarrow{\Delta} X \times X \xrightarrow{c \times \text{id}} * \times X = X
\]

is the identity map of \( X \). Then the commutative diagram

\[
\begin{array}{ccc}
H^0(*; R) \otimes H^n(X; R) & \xrightarrow{c \otimes \text{id}} & H^0(X) \otimes H^n(X; R) \\
\downarrow & & \downarrow \\
H^n(* \times X; R) & \xrightarrow{(c \times \text{id})^*} & H^n(X \times X; R) \\
\downarrow & & \downarrow \\
H^n(X; R) & & H^n(X; R)
\end{array}
\]

shows that \( u \cdot x = x \). A similar argument shows that \( x \cdot u = x \). Associativity similarly follows from the space-level commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\Delta} & X \times X \\
\downarrow & & \downarrow \text{id} \times \Delta \\
X \times X & \xrightarrow{\Delta \times \text{id}} & X \times X \times X.
\end{array}
\]

**Proposition 41.2.** The cup product is natural.

**Example 41.3.** \( X = S^1 \). This is not a very interesting example, since there is no room for a nontrivial product. If \( x \) is a generator in degree 1, then \( x^2 \) must be zero since \( H^2(S^1) = 0 \). It follows that the cohomology ring is

\[
H^*(S^1; \mathbb{Z}) \cong \mathbb{Z}[x]/x^2.
\]

This is often called an exterior algebra.

**Example 41.4.** For a similar reason, we see that

\[
H^*(S^n; \mathbb{Z}) \cong \mathbb{Z}[x_n]/x_n^2
\]

where \( x_n \) has degree \( n \).
Example 41.5. $X = T^2 = S^1 \times S^1$. We know that the cohomology is free abelian on generators $w_0$, $x_1$, $y_1$, and $z_2$, where the subscript indicates the degree of the class. Thus the only question about the ring structure is what are the products $x_1^2$, $y_1^2$, and $x_1 y_1$.

Let $p_i : T^2 \longrightarrow S^1$, for $i = 1, 2$ be the projection maps. These induce ring homomorphisms

$$p_i^* : H^*(S^1) \longrightarrow H^*(T^2).$$

Since the projection is cellular, we can calculate these maps explicitly. We claim that $p_1^*(v_1) = x_1$ and $p_2^*(v_1) = y_1$. To see this, note that we can take $v_1$ to be the dual basis element to the 1-cell of $S^1$, so that $v_1(e_1) = 1$. Similarly, we take $x_1$ to be dual to $e_1^{1,0}$ and $y_1$ to be dual to $e_1^{1,1}$. Then

$$p_i^*(v_1)(ie_1^{1,0} + je_2^{1,0}) = v_1(i(p_1) \cdot e_1^{1,0} + j(p_1) \cdot e_1^{1,1}) = v_1(ie_1 + j0) = i,$$

so that $p_1^*(v_1) = x_1$.

Now since the $p_i$ are ring homomorphisms and $\delta_1^2 = 0$ in $H^*(S^1)$, we conclude that $x_1$ and $y_1$ both square to zero in $H^*(T^2)$. It only remains to determine the product $x_1 \cdot y_1$.

We did not discuss this approach to $x_1 y_1$ in class.

Recall that, by definition, $x_1 \cdot y_1 = \Delta^*(x_1 \cdot y_1)$. Here $x_1 \cdot y_1 \in H^2(T^2 \times T^2)$. In order to calculate the cup product, we must take a cellular approximation of the diagonal on $T^2$. Since $T^2 = S^1 \times S^1$, we can start with a cellular approximation $\tilde{\Delta}_{S^1}$ of the diagonal on $S^1$ and then define our approximation on $T^2$ to be

$$\tilde{\Delta}_{T^2} : T^2 = S^1 \times S^1 \xrightarrow{\tilde{\Delta}_{S^1} \times \tilde{\Delta}_{S^1}} S^1 \times S^1 \times S^1 \times S^1 \xrightarrow{id \times f \times id} S^1 \times S^1 \times S^1 \times S^1 = T^2 \times T^2.$$  

The approximation $\tilde{\Delta}_{S^1}$ can be taken from an approximation on $I$, and we see that the induced map on chains is $e^1 \mapsto e_1^{1,0} + e_1^{1,1}$. Recalling that $f : S^1 \times S^1 \longrightarrow S^1 \times S^1$ induces the map

$$e_1^{1,0} \mapsto e_1^{1,1}, \quad e_1^{1,1} \mapsto e_1^{1,0},$$

on chains, it follows that $\tilde{\Delta}_{T^2}$ induces the map

$$e_2^{2,1,1,1,0} \mapsto e_1^{2,1,1,0,0} - e_0^{2,1,1,0,0} + e_2^{0,1,0,0,1} + e_0^{0,0,1,1}$$

on $C_2$. Now we have

$$(x_1 \cdot y_1)(e_2^{2,1,1,1,0}) := (x_1 \cdot y_1)(e_2^{2,1,1,0,0} - e_2^{0,1,1,0,0} + e_2^{0,1,0,0,1} + e_0^{0,0,1,1}) = x_1(e_1^{0,1,1,0})y_1(e_1^{1,0}) - x_1(e_1^{1,0})y_1(e_1^{0,1}) = 0 \cdot 0 - 1 \cdot 1 = -1.$$  

It follows that $x_1 \cdot y_1 = \pm z_2$ (depending on which generator we choose $z_2$ to be).

Another (easier) way to think about the above example is using the K"unneth theorem. First, as we indicated in the previous example, the projections $p_X$ and $p_Y$ induce ring maps

$$p_X^* : H^*(X) \longrightarrow H^*(X \times Y), \quad p_Y^* : H^*(Y) \longrightarrow H^*(X \times Y).$$

Proposition 41.6. Let $R \xrightarrow{f} T$ and $S \xrightarrow{g} T$ be ring homomorphisms (all rings are assumed to be commutative). Then there is a unique ring homomorphism making the following diagram commute:

$$\begin{tikzcd}
R \arrow{r}{\eta_1} \arrow{d}[swap]{\eta_2} & R \otimes S \arrow{r}{\rho} \arrow{d}{\delta} & T \\
S & T
\end{tikzcd}$$
In other words, \( R \otimes S \) is the coproduct in the category of commutative rings. Here, \( \eta_1(r) = r \otimes 1 \) and \( \eta_2(s) = 1 \otimes s \). The multiplication on \( R \otimes S \) is given on simple tensors by

\[(r_1 \otimes s_1) \cdot (r_2 \otimes s_2) := r_1 r_2 \otimes s_1 s_2\]

and then extended linearly to all of \( R \otimes S \). The unit is \( 1 \otimes 1 \).

**Proof.** Given \( f \) and \( g \), then \( \varphi : R \otimes S \longrightarrow T \) may be defined on simple tensors by the formula

\[\varphi(r \otimes s) = f(r)g(s).\]

This clearly makes the diagram commute, and it is simple to check that this is a ring homomorphism. \( \blacksquare \)

Note that if \( R^* \) and \( S^* \) are graded rings, the same result holds, but signs must be introduced appropriately. For instance, the multiplication on \( R^* \otimes S^* \) is given by

\[(r_1 \otimes s_1) \cdot (r_2 \otimes s_2) := (-1)^{\deg(s_1) \deg(r_2)} r_1 r_2 \otimes s_1 s_2.\]
In order to apply this, we first need to show that cohomology is a commutative ring (in the graded sense).

**Definition 42.1.** A graded ring $A^*$ is said to be (graded-)commutative if

$$x \cdot y = (-1)^{ab} y \cdot x,$$

where $a = \deg(x)$ and $b = \deg(y)$.

**Proposition 42.2.** The cohomology ring is graded commutative.

**Proof.** This follows from a combination of topological and algebraic results. The topological result is that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\Delta} & X \times X \\
\downarrow & & \downarrow \tau \\
X & \xrightarrow{t} & X \times X
\end{array}
$$

commutes, where $t$ is the transposition. The algebraic result is that the square

$$
\begin{array}{ccc}
C_n(X) \otimes C_k(Y) & \xrightarrow{\tau} & C_n+k(X \times Y) \\
\downarrow & & \downarrow t_* \\
C_k(Y) \otimes C_n(X) & \xrightarrow{t} & C_{n+k}(X \times Y)
\end{array}
$$

commutes, where $\tau(x \otimes y) = (-1)^{nk} y \otimes x$. The reason for the sign $(-1)^{nk}$ is as follows. Say $e^n_a$ is an $n$-cell in $X$ and $e^k_\beta$ is a $k$-cell in $Y$. We wish to know what is the coefficient of $e^{n+k}_{\alpha \times \beta} \times \alpha$ in $t^* (e^n_a \times e^k_\beta)$. Recall that this coefficient is the degree of the map

$$S^{n+k} \hookrightarrow \bigvee S^{n+k} \cong (X \times Y)^n/(X \times Y)^{n-1} \xrightarrow{l} (Y \times X)^n/(Y \times X)^{n-1} \cong \bigvee S^{n+k} \rightarrow S^{n+k}.$$ 

But this map is the permutation of coordinates

$$S^{n+k} = S^n \wedge S^k \cong S^k \wedge S^n = S^{n+k},$$ 

which has degree $(-1)^{nk}$ since it can be expressed as $nk$ iterations of a twist $S^1 \wedge S^1 \cong S^1 \wedge S^1$. ■

Applying the previous result to the ring maps $p_X^* \times p_Y^*$ defines a ring homomorphism

$$H^*(X) \otimes H^*(Y) \rightarrow H^*(X \times Y).$$

A cohomological version of the Künneth theorem is

**Theorem 42.3** (Theorem 3.16 of Hatcher). Suppose that the groups $H^k(Y; \mathbb{Z})$ are finitely generated free abelian groups for all $k$. Then the cross product

$$H^*(X; \mathbb{Z}) \otimes_Z H^*(Y; \mathbb{Z}) \rightarrow H^*(X \times Y; \mathbb{Z})$$

is an isomorphism of rings.

Of course, by symmetry the hypothesis on $H^*(Y; \mathbb{Z})$ could equally well be placed on $H^*(X; \mathbb{Z})$ instead.

**Example 42.4.** Turning back to $X = T^2$, this result tells us that

$$H^*(T^2; \mathbb{Z}) \cong (\mathbb{Z}[x_1]/x_1^2) \otimes \mathbb{Z} (\mathbb{Z}[y_1]/y_1^2) \cong \mathbb{Z}[x_1, y_1]/(x_1^2, y_1^2).$$

In particular, $x_1 y_1 \neq 0$ in this ring.
We must check the formula

\[ d^n_i(\sigma) := \sigma \circ d^n \circ d^{n-1} \circ \cdots \circ d^{n-i+1} = \sigma_{|\{0\ldots n-i\}|} \]

be the “left” \( i \)-dimensional face and similarly

\[ d^j_i(\sigma) := \sigma \circ d^0 \circ \cdots \circ d^0 = \sigma_{|\{n-1\ldots n-i\}|} \]

be the “right” \( i \)-dimensional face. Then given \( \alpha \in \text{H}^n(X; R) \) and \( \beta \in \text{H}^k(X; R) \), we define \( \alpha \cup \beta \) on an \((n + k)\)-simplex \( \sigma \) by

\[ (\alpha \cup \beta)(\sigma) := (-1)^n \alpha(d^n_0(\sigma)) \cdot \beta(d^k_0(\sigma)). \]

**Proposition 42.5.** The above cup product defines a chain map

\[ C^*(X; R) \otimes C^*(X; R) \longrightarrow C^*(X; R), \]

where \( C^*(X; R) \) means either singular or simplicial cochains.

**Proof.** We must check the formula

\[ \partial(\alpha \cup \beta) = \partial(\alpha) \beta + (-1)^n a \partial(\beta) \]

if \( \alpha \in C^n(X; R) \) and \( \beta \in C^k(X; R) \). Recall from (37.2) that \( \partial(\alpha) = (-1)^{n+1} \alpha \circ \partial \). For simplicity, we consider the case \( n = 2 \) and \( k = 1 \). Then

\[ \partial^3(\alpha \cup \beta)(\sigma) = (\alpha \cup \beta)(\partial_4(\sigma)) = (\alpha \cup \beta)(\sigma \circ d^0 - \sigma \circ d^1 + \sigma \circ d^2 - \sigma \circ d^3 + \sigma \circ d^4) \]

\[ = a(\alpha \cup \beta)(\sigma_{|\{1,2,3,4\}}) - \alpha(\sigma_{|\{0,1,2,3\}}) + \sigma_{|\{0,1,2,3\}} - \sigma_{|\{0,1,2,4\}} + \sigma_{|\{0,1,3,4\}} \]

\[ = a(\sigma_{|\{1,2,3,4\}}) \beta(\sigma_{|\{3,4\}}) - a(\sigma_{|\{0,1,2,3\}}) \beta(\sigma_{|\{3,4\}}) + a(\sigma_{|\{0,1,2,4\}}) \beta(\sigma_{|\{3,4\}}) \]

\[ - a(\sigma_{|\{0,1,3,4\}}) \beta(\sigma_{|\{2,3\}}) + a(\sigma_{|\{0,2,3,4\}}) \beta(\sigma_{|\{2,3\}}). \]

On the other hand,

\[ \partial^2(\alpha \beta)(\sigma) = -\partial^2(\alpha)(\sigma_{|\{0,1,2,3\}}) \beta(\sigma_{|\{3,4\}}) = a(\partial_3(\sigma_{|\{0,1,2,3\}})) \beta(\sigma_{|\{3,4\}}) \]

\[ = a(\sigma_{|\{1,2,3,4\}}) \beta(\sigma_{|\{3,4\}}) - a(\sigma_{|\{0,1,2,3\}}) \beta(\sigma_{|\{3,4\}}) + a(\sigma_{|\{0,1,2,4\}}) \beta(\sigma_{|\{3,4\}}) - a(\sigma_{|\{0,1,3,4\}}) \beta(\sigma_{|\{2,3\}}) \]

and

\[ \partial d^1(\beta)(\sigma) = a(\sigma_{|\{0,1,2\}}) \partial_1(\beta)(\sigma_{|\{3,4\}}) = a(\sigma_{|\{0,1,2\}}) \beta(\partial_2(\sigma_{|\{3,4\}})) \]

\[ = a(\sigma_{|\{0,1,2\}}) \beta(\sigma_{|\{3,4\}}) - a(\sigma_{|\{0,1,3\}}) \beta(\sigma_{|\{2,4\}}) + a(\sigma_{|\{0,2,3\}}) \beta(\sigma_{|\{2,4\}}) \]

\[ \square \]

**Example 43.1.** \( X = \mathbb{R}P^2 \). Recall that the projective plane was built from two simplices as in the picture to the right. Taking coefficients in \( \mathbb{F}_2 \), this gives the chain complex

\[
\begin{array}{ccc}
C^2_0(\mathbb{R}P^2) \otimes \mathbb{F}_2 & \overset{d_2}{\longrightarrow} & C^1_0(\mathbb{R}P^2) \otimes \mathbb{F}_2 \\
\mathbb{F}_2 \{z_1, z_2\} & \| & \mathbb{F}_2 \{y_1, y_2, y_3\} \\
\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \| & \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\
\end{array}
\]
and therefore the cochain complex

\[
\begin{array}{c}
\mathbb{F}_2 \{ z_1^+, z_2^+ \} \xleftarrow{(1 \ 1 \ 1)} \mathbb{F}_2 \{ y_1^+, y_2^+, y_3^+ \} \xleftarrow{(0 \ 0 \ 1)} \mathbb{F}_2 \{ x_1^+, x_2^+ \}.
\end{array}
\]

Representatives for the nonzero cohomology classes are

\[ a_0 = [x_1^+ + x_2^+], \quad a_1 = [y_1^+ + y_2^+], \quad a_2 = [z_1^+] = [z_2^+]. \]

We want to establish that \( a_1^2 = a_2 \), or, equivalently, that \( a_1^2 \neq 0 \). We have

\[ a_1^2(x_1) := a_1(y_1)a_1(y_3) = 0 \]

and

\[ a_1^2(x_2) := a_1(y_1)a_1(y_2) = 1. \]

It follows that \( a_1^2 = a_2 \).

More generally,

\[ H^*(\mathbb{R}P^n; \mathbb{F}_2) \cong \mathbb{F}_2[x_1]/x_1^{n+1}, \quad H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[x_1]. \]

For complex projective space, we know that the cohomology (integrally) is concentrated in even degrees, and there the answer is

\[ H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[x_2]/x_2^{n+1}, \quad H^*(\mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[x_2]. \]

**Example 43.2.** We can use the cohomology ring structure to show that \( \mathbb{C}P^2 \) is not homotopy equivalent to \( S^2 \vee S^4 \). We know they have the same cohomology groups, but a homotopy equivalence also induces an isomorphism of cohomology rings, so it suffices to show that

\[ H^*(S^2 \vee S^4; \mathbb{Z}) \not\cong H^*(\mathbb{C}P^2; \mathbb{Z}) \cong \mathbb{Z}[z_2]/z_2^3. \]

The long exact sequence for the pair \((S^2 \vee S^4, S^2)\) shows that the restriction \( H^2(S^2 \vee S^4; \mathbb{Z}) \cong H^2(S^2; \mathbb{Z}) \) is an isomorphism.

Write \( H^2(S^2 \vee S^4; \mathbb{Z}) \cong \mathbb{Z}\{y_2\} \) and \( H^4(S^2 \vee S^4; \mathbb{Z}) \cong \mathbb{Z}\{y_4\} \). Note that we have a retraction \( S^2 \rightarrow S^2 \vee S^4 \rightarrow S^2 \). By functoriality, we also get a retraction on cohomology,

\[
\begin{array}{ccc}
H^*(S^2; \mathbb{Z}) & \xrightarrow{p^*} & H^*(S^2 \vee S^4; \mathbb{Z}) \\
\cong & & \cong \\
\mathbb{Z}[x_2]/x_2^2 & \xrightarrow{\pi^*} & \mathbb{Z}\{y_2, y_4\} \\
\cong & & \cong \\
\mathbb{Z}\{y_2\} & \xrightarrow{\text{id}} & \mathbb{Z}[y_2, y_4] \\
\end{array}
\]

Since \( \pi^* \) is an isomorphism on \( H^2 \), it follows that the same is true for \( p^* \). In particular \( p^*(x_2) = \pm y_2 \). It follows that

\[ y_2^2 = (p^*(x_2))^2 = p^*(x_2^2) = 0. \]

It follows that \( H^*(S^2 \vee S^4; \mathbb{Z}) \not\cong H^*(\mathbb{C}P^2; \mathbb{Z}) \).

Note that this shows that the attaching map \( S^3 \xrightarrow{\eta} S^2 \) for the 4-cell in \( \mathbb{C}P^2 \) is not null-homotopic. If \( \eta \) were null-homotopic, this would give a homotopy equivalence \( \mathbb{C}P^2 \simeq S^2 \vee S^4 \).
The ideas in the previous example show more generally that the inclusions of the wedge summands induce a ring isomorphism
\[ \tilde{H}^* (X \vee Y) \cong \tilde{H}^*(X) \times \tilde{H}^*(Y). \]

44. WED, DEC. 2

Orientations

When we restrict our attention to manifolds, we can say quite a bit more about cohomology. We start by recalling

Definition 44.1. A (topological) \( n \)-manifold \( M \) is a Hausdorff, second-countable space such that each point has a neighborhood homeomorphic to an open subset of \( \mathbb{R}^n \).

Last semester, you discussed orientability for surfaces (2-manifolds), and we can now give a general, rigorous treatment. The two main properties we would want of an orientation are

1. an orientation should be determined by a coherent family of “local” orientations around each point \( x \in M \)
2. an orientation of \( \mathbb{R}^n \) should be preserved by a rotation but reversed by a reflection.

Since a manifold is locally like \( \mathbb{R}^n \), we should first define an orientation of \( \mathbb{R}^n \). There are many ways to do this, but it will be convenient for us to give a definition in terms of homology. With that in mind, we note that for any \( x \in \mathbb{R}^n \), the relative homology group \( H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}; \mathbb{Z}) \) is isomorphic to \( \mathbb{Z} \). We then define an orientation of \( \mathbb{R}^n \) at \( x \) to be a choice of generator of \( H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}; \mathbb{Z}) \cong \mathbb{Z} \). Note that since rotations have degree 1 and reflections have degree \(-1\), our definition satisfies condition (2).

Since a manifold \( M \) is locally like \( \mathbb{R}^n \), this allows us to define local orientations on any \( M \). The key is that excision shows that
\[ H_n(M, M - \{x\}; \mathbb{Z}) \cong H_n(U, U - \{x\}; \mathbb{Z}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}; \mathbb{Z}) \cong \mathbb{Z}. \]

In fact, it will be convenient for us to consider a general commutative ring \( R \) as the coefficient group.

Definition 44.2. Let \( R \) be a commutative ring and \( M \) an \( n \)-manifold. Then, for any \( x \in M \), a local \( R \)-orientation at \( x \) is a choice \( \mu_x \) of \((R\text{-module})\) generator of \( H_n(M, M - \{x\}; R) \).

This gives us the local definition. Now we want to say that \( M \) is \( R \)-orientable if there is a compatible family of orientations.

Definition 44.3. An \( R \)-orientation of \( M \) is an open cover \( U = \{U\} \) of \( M \) together with a homology class \( \mu_U \in H_n(M, M - U; R) \) for each \( U \in U \) such that for each \( x \in U \), \( \mu_U \) restricts to a \((R\text{-module})\) generator under \( H_n(M, M - U; R) \rightarrow H_n(M, M - \{x\}; R) \cong R \). We also require that if \( U \cap V \neq \emptyset \) for \( U, V \in U \), then \( \mu_U \) and \( \mu_V \) determine the same element of \( H_n(M, M - (U \cap V); R) \). We say that \( M \) is \( R \)-orientable if there exists an \( R \)-orientation.

An equivalent definition is to say that an \( R \)-orientation is a collection \( \mu_x \) of local orientations such that each point \( x \) has a neighborhood \( U \) and class \( \mu_U \) restricting to each \( \mu_y \) for all \( y \in U \).

The two choices of \( R \) of primary interest are \( R = \mathbb{Z} \) and \( R = \mathbb{F}_2 \). In the case \( R = \mathbb{Z} \), we simply say “orientable” without referencing the coefficients.

Note that a \( \mathbb{Z} \)-orientation \( \mu \) of \( M \) determines an \( R \)-orientation of \( M \) for any \( R \), using the ring homomorphism \( \mathbb{Z} \rightarrow R, 1 \mapsto 1 \). However, this is not an if and only if.

Proposition 44.4. Any manifold has a (unique) \( \mathbb{F}_2 \)-orientation.
Proof. The point is that orientability is about being able to make consistent choices of generators. But there is always a canonical choice of generator of a 1-dimensional $\mathbb{F}_2$-vector space: the (unique) nonzero element.

Recall that a closed manifold is one that is compact and without boundary.

**Theorem 44.5.** Let $M$ be a connected, closed $n$-manifold. Then either

1. $M$ is orientable and $H_n(M; \mathbb{Z}) \to H_n(M, M - \{x\}; \mathbb{Z}) \cong \mathbb{Z}$ is an isomorphism for all $x \in M$
   OR
2. $M$ is nonorientable and $H_n(M; \mathbb{Z}) = 0$.

45. Fri, Dec. 9

Working with $\mathbb{F}_2$-coefficients, it turns out that $H_n(M; \mathbb{F}_2) \cong \mathbb{F}_2$ for any $M$, corresponding to the fact that every manifold is $\mathbb{F}_2$-orientable. See [Hatcher, Theorem 3.26] for the statement over an arbitrary coefficient ring. In the orientable case, a generator of $H_n(M; \mathbb{Z})$ is called a fundamental class or orientation class for $M$. Note that there are two such classes (the two choices of generator).

The key step in the proof is to show that for connected noncompact $n$-manifolds $N$, we have $\tilde{H}_n(N; \mathbb{Z}) = 0$. Applying this in the case $N = M - \{x\}$, we get that

$$H_n(M; \mathbb{Z}) \to H_n(M, M - \{x\}; \mathbb{Z}) \cong \mathbb{Z}$$

is injective. This already shows that $H_n(M; \mathbb{Z})$ must be either $\mathbb{Z}$ or 0.

**Example 45.1.** In Example 31.2, we computed that $H_n(\mathbb{R}P^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$.

It follows that $\mathbb{R}P^n$ is orientable if and only if $n$ is odd.

**Poincaré Duality**

Our last main topic for the course is duality, given as the following result.

**Theorem 45.2** (Poincaré Duality). Let $M$ be a closed, orientable $n$-manifold. Then there is an isomorphism

$$D : H^k(M; \mathbb{Z}) \cong H_{n-k}(M; \mathbb{Z})$$

for all $k$.

Under this isomorphism, the unit $1 \in H^0(M)$ corresponds to the fundamental class $\mu \in H_n(M)$. The map $D$ can be described in terms of the cap product.

**Definition 45.3.** The cap product in the singular/simplicial theories is a map

$$H^p(X; \mathbb{Z}) \otimes H_q(X; \mathbb{Z}) \to H_{p-q}(X; \mathbb{Z}).$$

On the level of cochains, the formula is

$$\alpha \cap \sigma = \alpha(\sigma|_{v_0,\ldots,v_p})\sigma_{v_p,\ldots,v_q}.$$ 

There is an important relation of the cap product to the cup product, which comes immediately from the definitions:

**Proposition 45.4.** For $\alpha \in H^p(X)$, $\beta \in H^q(X)$, and $\sigma \in H_{p+q}(X)$, we have

$$\langle \alpha \cup \beta, \sigma \rangle = (-1)^{pq} \langle \beta \cup \alpha, \sigma \rangle = \langle \alpha, \beta \cap \sigma \rangle \in \mathbb{Z}.$$
Now that we have defined the cap product, we can define the map $D$ of Theorem 45.2. We assume that $M$ is closed and orientable, so that according to Theorem 44.5 it has a fundamental class $\mu_M \in H_n(M; \mathbb{Z})$. Then we define

$$D(\alpha) := \alpha \cap \mu_M \in H_{n-k}(M; \mathbb{Z}).$$

**Corollary 45.5.** A closed, odd-dimensional manifold $M$ has Euler characteristic $\chi(M) = 0$.

**Proof.** Since any manifold is $\mathbb{F}_2$-orientable, we apply the Poincaré Duality theorem with $\mathbb{F}_2$-coefficients. Recall that $\chi(M)$ can be calculated as

$$\chi(M) = \sum_i (-1)^i \text{rank}(H_i(M; \mathbb{Z})) = -\sum_i (-1)^i \text{rank}(C_i(M))$$

by Proposition 25.3. But since the groups $C_i(M)$ are free abelian, the latter sum agrees with $\sum_i (-1)^i \text{dim}_{\mathbb{F}_2}(C_i(M) \otimes \mathbb{F}_2)$. By an argument similar to that given in the proof of Proposition 25.3, this agrees with $\sum_i (-1)^i \text{dim}_{\mathbb{F}_2} H_i(M; \mathbb{F}_2)$.

But now by combining duality and universal coefficients, we have

$$\text{dim}_{\mathbb{F}_2} H_i(M; \mathbb{F}_2) = \text{dim}_{\mathbb{F}_2} H^{n-i}(M; \mathbb{F}_2) = \text{dim}_{\mathbb{F}_2} H_{n-i}(M; \mathbb{F}_2).$$

Since $n$ is odd it follows that $\text{dim}_{\mathbb{F}_2} H_i(M; \mathbb{F}_2)$ will always cancel $\text{dim}_{\mathbb{F}_2} H_{n-i}(M; \mathbb{F}_2)$ in the formula for $\chi(M)$. \qed

If $M$ is closed and $R$-orientable, then consider the mapping

$$H^k(M; R) \otimes H^{n-k}(M; R) \longrightarrow R$$

defined by $(\alpha, \beta) \mapsto \langle \alpha \cup \beta, \mu_M \rangle$. This defines a bilinear pairing on the cohomology groups. Recall that, a bilinear pairing $A \otimes_R B \longrightarrow R$ is called nonsingular if the adjoint maps $A \longrightarrow \text{Hom}_R(B, R)$ and $B \longrightarrow \text{Hom}_R(A, R)$ are isomorphisms. The following result is a consequence of the Poincaré duality theorem.

**Proposition 45.6.** Taking $R = \mathbb{F}$ a field, the above pairing is nonsingular (again assuming that $M$ is closed and $\mathbb{F}$-orientable).

**Proof.** Let $\alpha \neq 0 \in H^k(M; \mathbb{F})$. We need to know that there is a $\beta \in H^{n-k}(M; \mathbb{F})$ such that $\langle \alpha \cup \beta, \mu_M \rangle \neq 0$.

But recall that

$$\langle \alpha \cup \beta, \mu_M \rangle = \langle \alpha \cap \beta \cap \mu_M \rangle = \langle \alpha, D(\beta) \rangle.$$

Since $\alpha \neq 0$ and the evaluation pairing $H^k(M; \mathbb{F}) \otimes_{\mathbb{F}} H_k(M; \mathbb{F}) \longrightarrow \mathbb{F}$ is nonsingular by the homework, there must be some homology class $\gamma \in H_k(M; \mathbb{F})$ such that $\langle \alpha, \gamma \rangle \neq 0$. But since the duality map is an isomorphism, we can write $\gamma = D(\beta)$ for some $\beta$, which gives the result. \qed

The same result holds for $R = \mathbb{Z}$ if we quotient the homology and cohomology by their torsion subgroups.

**Example 45.7.** $M = \mathbb{RP}^n$. We have already determined the cup product structure on $H^*(\mathbb{RP}^n; \mathbb{F}_2)$, but this was not so easy. We can instead obtain the cup product structure immediately from the preceding results (recall that every manifold is $\mathbb{F}_2$-orientable). In the case of $\mathbb{RP}^2$, the previous result says that the cup product

$$H^1(\mathbb{RP}^2; \mathbb{F}_2) \otimes H^1(\mathbb{RP}^1; \mathbb{F}_2) \longrightarrow H^2(\mathbb{RP}^2; \mathbb{F}_2)$$

cannot be zero, which was the only nontrivial step in determining the cohomology ring.

In the case of $\mathbb{RP}^3$, we learn that

$$H^1(\mathbb{RP}^3; \mathbb{F}_2) \otimes H^2(\mathbb{RP}^3; \mathbb{F}_2) \longrightarrow H^3(\mathbb{RP}^3; \mathbb{F}_2)$$
is nonzero. The only remaining question is whether $x_2 = x_1^2$. But we can determine this by restricting along the inclusion $\mathbb{R}P^2 \hookrightarrow \mathbb{R}P^3$. An induction proof now easily shows that

$$H^*(\mathbb{R}P^n; \mathbb{F}_2) \cong \mathbb{F}_2[x_1]/(x_1^{n+1}).$$

By restricting to finite skeleta, it now follows that

$$H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[x_1].$$

**Example 45.8.** $M = \mathbb{C}P^n$. Since $\mathbb{C}P^n$ is simply-connected, it is $\mathbb{Z}$-orientable, so that Poincaré Duality applies. Also, we know that all homology and cohomology is torsion-free. The preceding result then tells us that

$$H^2(\mathbb{C}P^2; \mathbb{Z}) \otimes H^2(\mathbb{C}P^2; \mathbb{Z}) \to H^4(\mathbb{C}P^2; \mathbb{Z})$$

is nonzero and further that there exists $i \in \mathbb{Z}$ such that $z_2 \cup iz_2$ is a generator for $H^4$. Certainly $i$ must be $\pm 1$, so that $z_2^2$ is a generator. Now a similar argument as above shows that $z_2^i$ is a generator in $H^{2k}(\mathbb{C}P^n; \mathbb{Z})$ whenever $k \leq n$. We get

$$H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[x_2]/(x_2^{n+1})$$

and

$$H^*(\mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[x_2].$$