Math 551 - Topology I
Homework 1
Fall 2017

1. (a) Let X be any set. Define the discrete metric on X by

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases}$$

   Show that this is indeed a metric.

   (b) Which sets are open in the discrete metric?

   (c) Suppose that X is a discrete metric space and that Y is any metric space. Show that any function $f : X \longrightarrow Y$ is automatically continuous.

   (d) ($\star$) Suppose that Y is a discrete metric space. Show that the only continuous functions $\mathbb{R} \longrightarrow Y$ are the constant functions.

2. Show that $U \subseteq \mathbb{R}^2$ is open in the max metric if and only if it is open in the standard (Euclidean) metric.

3. (a) Show that the wheel metric on $\mathbb{R}^2$ is a metric. Recall that the wheel metric is defined by

$$d(\mathbf{x},\mathbf{y}) = \begin{cases} d_{Euc}(\mathbf{x},\mathbf{y}) & \text{if } \mathbf{x} \text{ and } \mathbf{y} \text{ lie on a common line through } \mathbf{0} \\ d_{Euc}(\mathbf{x},\mathbf{0}) + d_{Euc}(\mathbf{y},\mathbf{0}) & \text{else} \end{cases}$$

   (b) Show that every Euclidean open set in $\mathbb{R}^2$ is also open with respect to the wheel metric, but give an example to show that the converse is not true.

4. Let $f : X \longrightarrow Y$ be a function between metric spaces. Show that f is continuous if and only if

   For every convergent sequence $(x_n) \to x$ in X, the sequence $(f(x_n))$ converges to $f(x)$ in Y.

5. Show that the intersection of two open sets in a metric space is always open.