Math 551 - Topology I
Homework 7
Fall 2017

1. Let G and H be groups and let X be a set.

   (a) Show that an action of $G \times H$ on X corresponds to a commuting pair of G and H-actions on X. (Commuting means that $g \cdot (h \cdot x) = h \cdot (g \cdot x)$.)

   (b) Show that an action of $G \times H$ on X gives rise to an action of G on X/H.

   (c) Suppose that X is a set with a $G \times H$-action. Show that $X/(G \times H) \cong (X/G)/H$.

2. Consider the $O(n)$-action described in class on the set of k-dimensional subspaces of $\mathbb{R}^n$. Show that this restricts to a transitive action of the subgroup $SO(n)$. What is the stabilizer of the point $E_k = \mathrm{Span}\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$?

3. Let $k \leq n$ be natural numbers. A **k-frame** in $\mathbb{R}^n$ is a collection of k linearly independent vectors. Denote by $\mathrm{Fr}_k(\mathbb{R}^n)$ the set of k-frames in $\mathbb{R}^n$.

   (a) Show that $Gl_n(\mathbb{R})$ acts transitively on $\mathrm{Fr}_k(\mathbb{R}^n)$.

   (b) What is the stabilizer of $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$?

4. Let $k \leq n$ be natural numbers. An **orthonormal k-frame** in $\mathbb{R}^n$ is an ordered orthonormal set of k (necessarily linearly independent) vectors. Denote by $\mathrm{OFr}_k(\mathbb{R}^n)$ the set of orthonormal k-frames in $\mathbb{R}^n$.

   (a) Show that $O(n)$ acts transitively on $\mathrm{OFr}_k(\mathbb{R}^n)$.

   (b) What is the stabilizer of $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$?

   (c) (**Optional**) Show that the function sending an orthonormal frame to the subspace it spans defines a topological quotient map $\mathrm{OFr}_k(\mathbb{R}^n) \longrightarrow \mathrm{Gr}_k(\mathbb{R}^n)$.

5. (Topologist's sine curve) Let $\Gamma \subseteq \mathbb{R}^2$ be the graph of $\sin(1/x)$ for $0 < x \leq 1/\pi$. Show that the closure $\overline{\Gamma}$ is connected but not path-connected.