Math 551 - Topology I
Homework 9
Fall 2017

1. (a) Find an example of a bijective continuous map $f : X \longrightarrow Y$, where X is locally compact but Y is not.

   (b) Show that if $f : X \longrightarrow Y$ is a continuous, open surjection and X is locally compact, then Y must be locally compact.

2. Let $X = (\mathbb{R} \times \mathbb{Z})/\sim$, where $\sim$ is the equivalence relation generated by $(x, n) \sim (x, k)$ for all $n, k \in \mathbb{Z}$ and $x \neq 0$. Show that X is locally compact but does not have a basis of precompact open sets.

3. Suppose that Y is locally compact Hausdorff. Let $K \subseteq U \subset Y$ with K compact and U open. Show that there is a precompact open set V with

$$K \subseteq V \subseteq \overline{V} \subseteq U.$$

4. (Stereographic Projection) Let $N = (0, \dots, 0, 1) \in S^n$ be the North Pole. Define a homeomorphism $S^n \setminus \{N\} \cong \mathbb{R}^n$ as follows. For each $x \neq N \in S^n$, consider the ray starting at N and passing through x. This meets the equatorial hyperplane (defined by $x_{n+1} = 0$) in a point, which we call $p(x)$.

   (a) Determine a formula for p and show that it gives a homeomorphism.

   (b) Conclude that the one-point compactification of $\mathbb{R}^n$ is S^n.

5. A map $f : X \longrightarrow Y$ is said to be **proper** if, for any compact subset $K \subseteq Y$, the preimage $f^{-1}(K) \subseteq X$ is compact.

   (a) Show that if X is compact and Y is Hausdorff, then any continuous $f : X \longrightarrow Y$ is automatically proper.

   (b) Let X and Y be locally compact and Hausdorff. Show that a continuous map $f : X \longrightarrow Y$ is proper if and only if it extends to a continuous map $\widehat{f} : \widehat{X} \longrightarrow \widehat{Y}$ with $\widehat{f}(\infty_X) = \infty_Y$.