16.1. **Path Connectedness.** Ok, so we have looked at examples and studied this notion of being connected, but if you asked your calculus students to describe what it should mean for a subset of $\mathbb{R}$ to be connected, they probably wouldn’t come up with the “no nontrivial clopen subsets” idea. Instead, they would probably say something about being able to connect-the-dots. In other words, you should be able to draw a line from one point to another while staying in the subset. This leads to the following idea.

**Definition 16.12.** We say that $A \subseteq X$ is **path-connected** if for every pair $a, b$ of points in $A$, there is a continuous function (a path) $\gamma : I \rightarrow A$ with $\gamma(0) = a$ and $\gamma(1) = b$.

This is not unrelated to the earlier notion.

**Proposition 16.13.** If $A \subseteq X$ is path-connected, then it is also connected.

**Proof.** Pick a point $a_0 \in A$. For any other $b \in A$, we have a path $\gamma_b$ in $A$ from $a_0$ to $b$. Then the image $\gamma_b(I)$ is a connected subset of $A$ containing both $a_0$ and $b$. It follows that

$$A = \bigcup_{b \in A} \gamma_b(I)$$

is connected, as it is the overlapping union of connected sets.

For subsets $A \subseteq \mathbb{R}$, we have

$$A \text{ is path-connected} \Rightarrow A \text{ is connected} \iff A \text{ is an interval} \Rightarrow A \text{ is path-connected}.$$ 

So the two notions coincide for subsets of $\mathbb{R}$. But the same is not true in $\mathbb{R}^2$! (The topologist’s sine curve, HW 7).

Path-connectedness has much the same behavior as connectedness.

**Proposition 16.14.**

1. Images of path-connected spaces are path-connected
2. Overlapping unions of path-connected spaces are path-connected
3. Finite products of path-connected spaces are path-connected

However, the topologist’s sine curve shows that closures of path-connected subsets need not be path-connected.

Our proof of connectivity of $\prod_i X_i$ last time used this closure property for connected sets, so the earlier argument does not adapt easily to path-connectedness. But it turns out to be easier to prove.

**Theorem 16.15.** Assume $X_i \neq \emptyset$ for all $i \in I$, where is $I$ is arbitrary. Then $\prod_i X_i$ is path-connected if and only if each $X_i$ is path-connected.

**Proof.** The interesting direction is $(\Leftarrow)$. Thus assume that each $X_i$ is path-connected. Let $(x_i)$ and $(y_i)$ be points in the product $\prod_i X_i$. Then for each $i \in I$ there is a path $\gamma_i$ in $X_i$ with $\gamma_i(0) = x_i$ and $\gamma_i(1) = y_i$. By the universal property of the product, we get a continuous path

$$\gamma = (\gamma_i) : [0, 1] \to \prod_i X_i$$

with $\gamma(0) = (x_i)$ and $\gamma(1) = (y_i)$. 

\textbf{\[40\]}
16.2. Components. The overlapping union property for (path-)connectedness allows us to make the following definition.

**Definition 16.16.** Let \( x \in X \). We define the **connected component** (or simply component) of \( x \) in \( X \) to be

\[
C_x = \bigcup_{C \text{ connected}} C.
\]

Similarly, the **path-component** of \( X \) is defined to be

\[
PC_x = \bigcup_{P \text{ connected}} P.
\]

The overlapping union property guarantees that \( C_x \) is connected and that \( PC_x \) is path-connected. Since path-connected sets are connected, it follows that for any \( x \), we have \( PC_x \subseteq C_x \). An immediate consequence of the above definition(s) is that any (path-)connected subset of \( X \) is contained in some (path-)component.

**Example 16.17.** Consider \( \mathbb{Q} \), equipped with the subspace topology from \( \mathbb{R} \). Then the only connected subsets are the singletons, so \( C_x = \{x\} \). Such a space is said to be **totally disconnected**.

Note that for any space \( X \), each component \( C_x \) is closed as \( \overline{C_x} \) is a connected subset containing \( x \), which implies \( \overline{C_x} \subseteq C_x \). If \( X \) has finitely many components, then each component is the complement of the finite union of the remaining components, so each component is also open, and \( X \) decomposes as a disjoint union

\[
X \cong C_1 \sqcup C_2 \sqcup \cdots \sqcup C_n
\]

of its components. But this does not happen in general, as the previous example shows.

The situation is worse for path-components: they need not be open or closed, as the topologist’s sine curve shows.

16.3. Locally (Path-)Connected.

**Definition 16.18.** Let \( X \) be a space. We say that \( X \) is **locally connected** if any neighborhood \( U \) of any point \( x \) contains a connected neighborhood \( \overline{B} \subseteq U \). Similarly \( X \) is **locally path-connected** if any neighborhood \( U \) of any point \( x \) contains a path-connected neighborhood \( \overline{B} \subseteq U \).

The locally path-connected turns out to show up more often, so we focus on that.

**Proposition 16.19.** Let \( X \) be a space. The following are equivalent.

1. \( X \) is locally path-connected
2. \( X \) has a basis consisting of path-connected open sets
3. for every open set \( U \subseteq X \), the path-components of \( U \) are open in \( X \)
4. for every open set \( U \subseteq X \), every component of \( U \) is path-connected and open in \( X \).

**Proof.** We leave the implications \( (1) \iff (2) \iff (3) \) as an exercise. We argue for \( (1) \iff (4) \).

Assume \( X \) is locally path-connected, and let \( C \) be a component of an open subset \( U \subseteq X \). Let \( P \subseteq C \) be a nonempty path-component. Then \( P \) is open in \( X \). But all of the other path-components of \( C \) are also open, so their union, which is the complement of \( P \), must be open. It follows that \( P \) is closed. Since \( C \) is connected, we must have \( P = C \).

On the other hand, suppose that \( (4) \) holds. Let \( U \) be a neighborhood of \( x \). Then the component \( C_x \) of \( x \) in \( U \) is the desired neighborhood \( V \).
In particular, this says that the components and path-components agree if \( X \) is locally path-connected.

Just as path-connected implies connected, locally path-connected implies locally-connected. But, unfortunately, there are no other implications between the four properties.

**Example 16.20.** The topologist’s sine curve is connected, but not path-connected or locally connected or locally path-connected. Thus it is possible to be connected but not locally so.

**Example 16.21.** For any space \( X \), the cone on \( X \) is defined to be \( CX = X \times [0, 1]/X \times \{1\} \). The cone on any space is always path-connected. In particular, the cone on the topologist’s sine curve is connected and-path connected but not locally connected or locally path-connected.

**Example 16.22.** A disjoint union of two topologist’s sine curves gives an example that is not connected in any of the four ways.

**Example 16.23.** Note that if \( X \) is locally path-connected, then connectedness is equivalent to path-connectedness. A connected example would be \( \mathbb{R} \) or a one-point space. A disconnected example would be \( (0, 1) \cup (2, 3) \) or a two-point (discrete) space.

Finally, we have spaces that are locally connected but not locally path-connected.

**Example 16.24.** The cocountable topology on \( \mathbb{R} \) is connected and locally connected but not path-connected or locally path-connected.

**Example 16.25.** The cone on the cocountable topology will give a connected, path-connected, locally connected space that is not locally path-connected.

**Example 16.26.** Two copies of \( \mathbb{R}_{\text{coconutable}} \) give a space that is locally connected but not connected in the other three ways.

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**17. Compactness**

The next topic is one of the major ones in the course: compactness. As we will see, this is the analogue of a “closed and bounded subset” in a general space. The definition relies on the idea of coverings.

**Definition 17.1.** An open cover of \( X \) is a collection \( \mathcal{U} \) of open subsets that cover \( X \). In other words, \( \bigcup_{U \in \mathcal{U}} U = X \). Given two covers \( \mathcal{U} \) and \( \mathcal{V} \) of \( X \), we say that \( \mathcal{V} \) is a subcover if \( \mathcal{V} \subseteq \mathcal{U} \).

**Definition 17.2.** A space \( X \) is said to be compact if every open cover has a finite subcover (i.e. a cover involving finitely many open sets).

**Example 17.3.** Clearly any finite topological space is compact, no matter the topology.

**Example 17.4.** An infinite set with the discrete topology is not compact, as the collection of singletons gives an open cover with no finite subcover.

**Example 17.5.** \( \mathbb{R} \) is not compact, as the open cover \( \mathcal{U} = \{(n, n) \mid n \in \mathbb{N}\} \) has no finite subcover.

**Example 17.6.** Similarly \( [0, \infty) \) is not compact, as the open cover \( \mathcal{U} = \{(0, n)\} \) has no finite subcover. Recall that \( [0, \infty) \cong [a, b] \).

**Theorem 17.7.** Let \( a < b \). Then \( [a, b] \) is a compact subset of \( \mathbb{R} \).
Proof. Let $\mathcal{U}$ be an open cover. Then some element of the cover must contain $a$. Pick such an element and call it $U_0$.

Consider the set 

$$ E = \{ c \in [a, b] \mid [a, c] \text{ is finitely covered by } \mathcal{U} \}. $$

Certainly $a \in E$ and $E$ is bounded above by $b$. By the Least Upper Bound Axiom, $s = \sup E$ exists. Note that $a \leq s \leq b$, so we must have $s \in U_s$ for some $U_s \in \mathcal{U}$. In general, $U_s$ may not be connected, so let $s \in V \subseteq U_s$ be an open interval. But then for any $c < s$ with $c \in V \subseteq U_s$, we have $c \in E$. This means that

$$ [a, c] \subseteq U_1 \cup \cdots \cup U_k $$

for $U_1, \ldots, U_k \in \mathcal{U}$. But then $[a, s] \subseteq U_1 \cup \cdots \cup U_k \cup U_s$. This shows that $s \in E$. On the other hand, the same argument shows that for any $s < d < b$ with $d \in U_s$, we would similarly have $d \in E$. Since $s = \sup E$, there cannot exist such a $d$. This implies that $s = b$. \hfill $\square$

Like connectedness, compactness is preserved by continuous functions.

**Proposition 17.8.** Let $f : X \to Y$ be continuous, and assume that $X$ is compact. Then $f(X)$ is compact.

**Proof.** Let $\mathcal{V}$ be an open cover of $f(X)$. Then $\mathcal{U} = \{ f^{-1}(V) \mid V \in \mathcal{V} \}$ is an open cover of $X$. Let $\{U_1, \ldots, U_k\}$ be a finite subcover. It follows that the corresponding $\{V_1, \ldots, V_k\}$ is a finite subcover of $\mathcal{V}$. \hfill $\square$

**Example 17.9.** Recall that we have the quotient map $\exp : [0, 1] \to S^1$. It follows that $S^1$ is compact.

**Theorem 17.10 (Extreme Value Theorem).** Let $f : [a, b] \to \mathbb{R}$ be continuous. Then $f$ attains a maximum and a minimum.

**Proof.** Since $f$ is continuous and $[a, b]$ is both connected and compact, the same must be true of its image. But the compact, connected subsets are precisely the closed intervals. \hfill $\square$

The following result is also quite useful.

**Proposition 17.11.** Let $X$ be Hausdorff and let $A \subseteq X$ be a compact subset. Then $A$ is closed in $X$.

**Proof.** Pick any point $x \in X \setminus A$ (if we can’t, then $A = X$ and we are done). For each $a \in A$, we have disjoint neighborhoods $a \in U_a$ and $x \in V_a$. Since the $U_a$ cover $A$, we only need finitely many, say $U_{a_1}, \ldots, U_{a_k}$ to cover $A$. But then the intersection

$$ V = V_{a_1} \cap \cdots \cap V_{a_k} $$

of the corresponding $V_a$’s is disjoint from the union of the $U_a$’s and therefore also from $A$. Since $V$ is a finite intersection of open sets, it is open and thus gives a neighborhood of $x$ in $X \setminus A$. It follows that $A$ is closed. \hfill $\square$

**Exercise 17.12.** If $A \subseteq X$ is closed and $X$ is compact, then $A$ is compact.

Combining these results gives the following long-awaited consequence.

**Corollary 17.13.** Let $f : X \to Y$ be continuous, where $X$ is compact and $Y$ is Hausdorff, then $f$ is a closed map.

In particular, if $f$ is already known to be a continuous bijection, then it is automatically a homeomorphism. For example, this shows that the map $I/\partial I \to S^1$ is a homeomorphism. Similarly, from Example 14.10 we have $D^n/\partial D^n \cong S^{n-1}$. 

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17.1. **Products.** We will next show that finite products of compact spaces are compact, but we first need a lemma.

**Lemma 17.14** (Tube Lemma). Let $X$ be compact and $Y$ be any space. If $W \subseteq X \times Y$ is open and contains $X \times \{y\}$, then there is a neighborhood $V$ of $y$ with $X \times V \subseteq W$.

**Proof.** For each $x \in X$, we can find a basic neighborhood $U_x \times V_x$ of $(x, y)$ in $W$. The $U_x$'s give an open cover of $X$, so we only need finitely many of them, say $U_{x_1}, \ldots, U_{x_n}$. Then we may take $V = V_{x_1} \cap \cdots \cap V_{x_n}$.

\[\Box\]