Mon, Nov. 27

Last time, we were discussing CW complexes, and we considered two different CW structures on $S^n$. We continue with more examples.

2. (Torus) In general, a product of two CW complexes becomes a CW complex. We will describe this in the case $S^1 \times S^1$, where $S^1$ is built using a single 0-cell and single 1-cell.

Start with a single 0-cell, and attach two 1-cells. This gives $S^1 \vee S^1$. Now attach a single 2-cell to the 1-skeleton via the attaching map $\psi$ defined as follows. Let us refer to the two circles in $S^1 \vee S^1$ as $\ell$ and $r$. We then specify $\psi : S^1 \to S^1 \vee S^1$ by $\ell r \ell^{-1} r^{-1}$. What we mean is to trace out $\ell$ on the first quarter of the domain, to trace out $r$ on the second quarter, to run $\ell$ in reverse on the third quarter, and finally to run $r$ in reverse on the final quarter.

We claim that the resulting CW complex $X$ is the torus. Since the attaching map $\psi : S^1 \to S^1 \vee S^1$ is surjective, so is $\psi : D^2 \to X$. Even better, it is a quotient map.

On the other hand, we also have a quotient map $I^2 \to T^2$, and using the homeomorphism $I^2 \approx D^2$ from before, we can see that the quotient relation in the two cases agrees. We say that this homeomorphism $T^2 \approx X$ puts a cell structure on the torus. There is a single 0-cell (a vertex), two 1-cells (the two circles in $S^1 \vee S^1$), and a single 2-cell.

3. $\mathbb{R}P^n$. Let’s start with $\mathbb{R}P^2$. Recall that one model for this space was as the quotient of $D^2$, where we imposed the relation $x \sim -x$ on the boundary. If we restrict our attention to the boundary $S^1$, then the resulting quotient is $\mathbb{R}P^1$, which is again a circle. The quotient map $q : S^1 \to S^1$ is the map that winds twice around the circle. In complex coordinates, this would be $z \mapsto z^2$. The above says that we can represent $\mathbb{R}P^2$ as the pushout

$$
\begin{array}{ccc}
S^1 & \longrightarrow & D^2 \\
\downarrow q & & \downarrow \psi \\
S^1 & \longrightarrow & \mathbb{R}P^2
\end{array}
$$

If we build the 1-skeleton $S^1$ using a single 0-cell and a single 1-cell, then $\mathbb{R}P^2$ has a single cell in dimensions $\leq 2$.

More generally, we can define $\mathbb{R}P^n$ as a quotient of $D^n$ by the relation $x \sim -x$ on the boundary $S^{n-1}$. This quotient space of the boundary was our original definition of $\mathbb{R}P^{n-1}$. It follows that we can describe $\mathbb{R}P^n$ as the pushout

$$
\begin{array}{ccc}
S^{n-1} & \longrightarrow & D^n \\
\downarrow q & & \downarrow \phi \\
\mathbb{R}P^{n-1} & \longrightarrow & \mathbb{R}P^n
\end{array}
$$

Thus $\mathbb{R}P^n$ can be built as a CW complex with a single cell in each dimension $\leq n$.

4. $\mathbb{C}P^n$. Recall that $\mathbb{C}P^1 \approx S^2$. We can think of this as having a single 0-cell and a single 2-cell. We defined $\mathbb{C}P^2$ as the quotient of $S^3$ by an action of $S^1$ (thought of as $U(1)$). Let $\eta : S^3 \to \mathbb{C}P^1$ be the quotient map. What space do we get by attaching a 4-cell to $\mathbb{C}P^1$ by the map $\eta$? Well, the map $\eta$ is a quotient, so the pushout $\mathbb{C}P^1 \cup_\eta D^4$ is a quotient of $D^4$ by the $S^1$-action on the boundary.

Wed, Nov. 29

Now include $D^4$ into $S^5 \subseteq \mathbb{C}^3$ via the map

$$
\varphi(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4, \sqrt{1 - \sum x_i^2}, 0).
$$
(This would be a hemi-equator.) We have the diagonal $U(1)$ action on $S^5$. But since any nonzero complex number can be rotated onto the positive $x$-axis, the image of $\varphi$ meets every $S^1$-orbit in $S^5$, and this inclusion induces a homeomorphism on orbit spaces

$$D^4/U(1) \cong S^5/U(1) = \mathbb{CP}^2.$$\\

We have shown that $\mathbb{CP}^2$ has a cell structure with a single 0-cell, 2-cell, and 4-cell.\\

This story of course generalizes to show that any $\mathbb{CP}^n$ can be built as a CW complex having a cell in each even dimension.

Let’s talk about some of the (nice!) topological properties of CW complexes.

21.3. Niceness.

**Theorem 21.12** (Hatcher, Prop A.3). Any CW complex $X$ is normal.

Even better,

**Theorem 21.13** (Lee, Theorem 5.22). Every CW complex is paracompact.

**Proposition 21.14.** Any CW complex $X$ is locally path-connected.

*Proof.* Let $x \in X$ and let $U$ be any open neighborhood of $x$. We want to find a path-connected neighborhood $V$ of $x$ in $U$. Recall that a subset $V \subseteq X$ is open if and only if $V \cap X^n$ is open for all $n$. We will define $V$ by specifying open subsets $V^n \subseteq X^n$ with $V^{n+1} \cap X^n = V^n$ and then setting $V = \bigcup V^n$.

Suppose that $x$ is contained in the (interior of the) cell $e^n_i$. We set $V^k = \emptyset$ for $k < n$. We specify $V_n$ by defining $\Phi_i^{-1}(V^n)$ for each $n$-cell $e^n_i$. If $j \neq i$, we set $\Phi_j^{-1}(V_n) = \emptyset$. We define $\Phi_i^{-1}(V_n)$ to be an open $n$-disc around $\Phi_i^{-1}(x)$ whose closure is contained in $\Phi_i^{-1}(U)$. Now suppose we have defined $V^k$ for some $k \geq n$. Again, we define $V^{k+1}$ by defining each $\Phi_j^{-1}(V^{k+1})$. By assumption, $\overline{\Phi_i^{-1}(V^k)} \subseteq \partial D^{k+1} \subseteq \Phi_i^{-1}(U)$. By the Tube lemma, there is an $\epsilon > 0$ such that (using radial coordinates) $\Phi_j^{-1}(V^k) \times (1 - \epsilon, 1] \subseteq \Phi_j^{-1}(U)$. We define

$$\Phi_j^{-1}(V^{k+1}) = \Phi_j^{-1}(V^k) \times [1, 1 - \epsilon/2),$$

which is path-connected by induction. Note that this forces $\Phi_j^{-1}(V^{k+1})$ to be empty if the image of the attaching map for the cell $e_j^{k+1}$ does not meet $V_k$. Now by construction $V^{k+1}$ is the overlapping union of path-connected sets and therefore path-connected. This also guarantees that $\overline{V^{k+1}} \subseteq U \cap X^{k+1}$, allowing the induction to proceed. ■

**Proposition 21.15** (Hatcher, A.1). Any compact subset $K$ of a CW complex $X$ meets finitely many cells.

**Corollary 21.16.** Any CW complex has the closure-finite property, meaning that the closure of any cell meets finitely many cells.

*Proof.* The closure of $e_i$ is $\Phi_i(D^n_i)$, which is compact. The result follows from the proposition. ■

**Corollary 21.17.**

(i) A CW complex $X$ is compact if and only if it has finitely many cells.

(ii) A CW complex $X$ is locally compact if and only if the collection $\mathcal{E}$ of cells is locally finite.
Part 6. Homotopy and the fundamental group

22. Homotopy

Fri, Dec. 1

We have studied a number of topological properties of spaces, but how would we use these to distinguish $S^2$, $\mathbb{RP}^2$, and $T^2$? These are all compact, connected 2-manifolds. It turns out that the fundamental group will allow us to distinguish these spaces. This is the start of **algebraic** topology. We first introduce the idea of a homotopy.

**Definition 22.1.** Given maps $f$ and $g : X \rightarrow Y$, a **homotopy** $h$ between $f$ and $g$ is a map $h : X \times I \rightarrow Y$ ($I = [0,1]$) such that $f(x) = h(x,0)$ and $g(x) = h(x,1)$. We say $f$ and $g$ are **homotopic** if there exists a homotopy between them (and write $h : f \simeq g$).

**Example 22.2.** Let $f = \text{id} : \mathbb{R} \rightarrow \mathbb{R}$ and take $g : \mathbb{R} \rightarrow \mathbb{R}$ to be the constant map $g(x) = 0$. Then a homotopy $h : f \simeq g$ is given by

$$h(x,t) = x(1-t).$$

Check that $h(x,0) = f(x)$ and $h(x,1) = g(x)$. Since $f$ is homotopic to a constant map, we say that $f$ is **null-homotopic** (and $h$ is a **null-homotopy**).

**Example 22.3.** Consider $f = \text{id} : S^1 \rightarrow S^1$ and the map $g : S^1 \rightarrow S^1$ defined by $g(\cos(\theta), \sin(\theta)) = (\cos(2\theta), \sin(2\theta))$. Thinking of $S^1$ as the complex numbers of unit norm, the map $g$ can alternatively be described as $g(z) = z^2$. Then the maps $f$ and $g$ are not homotopic. Furthermore, neither is null-homotopic. (Though we won’t be able to show this until next semester.)

**Proposition 22.4.** The property of being homotopic defines an equivalence relation on the set of maps $X \rightarrow Y$.

**Proof.** (Reflexive): Need to show $f \simeq f$. Use the constant homotopy defined by $h(x,t) = f(x)$ for all $t$.

(Symmetric): If $h : f \simeq g$, we need a homotopy from $g$ to $f$. Define $H(x,t) = h(x,1-t)$ (reverse time).

(Transitive): If $h_1 : f_1 \simeq f_2$ and $h_2 : f_2 \simeq f_3$, we define a new homotopy $h$ from $f_1$ to $f_3$ by the formula

$$h(x,t) = \begin{cases} h_1(x,2t) & 0 \leq t \leq 1/2 \\ h_2(x,2t-1) & 1/2 \leq t \leq 2. \end{cases}$$

We write $[X,Y]$ for the set of homotopy classes of maps $X \rightarrow Y$.

**Proposition 22.5.** (Interaction of composition and homotopy) Suppose given maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $X \xrightarrow{f'} Y \xrightarrow{g'} Z$. If $f \simeq f'$ and $g \simeq g'$ then $g \circ f \simeq g' \circ f'$.

**Proof.** We will show that $g \circ f \simeq g' \circ f$. The required homotopy is given by

$$H(x,t) = h'(f(x),t).$$

It is easily verified that $H(x,0) = g \circ f(x)$ and $H(x,1) = g' \circ f(x)$. Why is the map $H : X \times I \rightarrow Z$ continuous? It is the composition of the continuous maps

$$X \times I \xrightarrow{f \times \text{id}} Y \times I \xrightarrow{h'} Z.$$

That the map $f \times \text{id}$ is continuous can be easily verified using the universal property. ■
**Definition 22.6.** A map $f : X \rightarrow Y$ is a **homotopy equivalence** if there is a map $g : Y \rightarrow X$ such that both composites $f \circ g$ and $g \circ f$ are homotopic to the identity maps. We say that spaces $X$ and $Y$ are **homotopy equivalent** if there exists some homotopy equivalence between them, and we write $X \simeq Y$.

**Remark 22.7.** It is clear that any homeomorphism is a homotopy equivalence, since then both composites are equal to the identity maps.

The following example shows that the converse is not true.

**Example 22.8.** The (unique) map $f : \mathbb{R} \rightarrow \ast$, where $\ast$ is the one-point space, is a homotopy equivalence. Pick *any* map $g : \ast \rightarrow \mathbb{R}$ (for example, the inclusion of the origin). Then $f \circ g = \text{id}$. The other composition $g \circ f : \mathbb{R} \rightarrow \mathbb{R}$ is constant, but we have already seen last time that the identity map of $\mathbb{R}$ is null-homotopic. So $\mathbb{R} \simeq \ast$. The same argument works equally well to show that $\mathbb{R}^n \simeq \ast$ for any $n$. Even more generally, if $X$ is a convex subset of $\mathbb{R}^n$, then $X \simeq \ast$.

Here's some more terminology: any space that is homotopy-equivalent to the one-point space is said to be **contractible**. As we have just seen in the example above, this is equivalent to the statement that the identity map is null-homotopic.

More generally, we can show that any two maps $f, g : X \rightarrow \mathbb{R}^n$ are homotopic. The **straight-line homotopy** between $f$ and $g$ is given by

$$h(x, t) = (1 - t)f(x) + tg(t).$$

We will see next semester that the spaces $S^2$, $\mathbb{R}P^2$, and $T^2$ are not homotopy-equivalent (and therefore not homeomorphic).