### 22.1. Path-homotopy.

## Mon, Dec. 4

Recall that a path in a space $X$ is simply a continuous map $\gamma: I \longrightarrow X$. It will turn out to be fruitful to study homotopy-classes of paths in a space $X$. But this is not very interesting if we don't impose additional restrictions: every path is null! A contracting homotopy for the path $\gamma$ is given by

$$
H(s, t)=\gamma(s(1-t)) .
$$



We need to modify our notion of homotopy to get an interesting relation for paths.
Definition 22.9. Let $\gamma_{1}$ and $\gamma_{2}$ be paths in $X$ with the same initial and end points. A pathhomotopy between $\gamma_{1}$ and $\gamma_{2}$ is simply a homotopy $h$ such that at each time $t$, the resulting path $h(-, t)$ also has the same initial and end points as $\gamma_{1}$ and $\gamma_{2}$.

Another way to think about this is that a path homotopy is a map from the square $I \times I$ that is constant on the left vertical edge and also on the right vertical edge.


Example 22.10. The two paths $\gamma_{1}(s)=e^{i \pi s}$ and $\gamma_{2}(s)=e^{-i \pi s}$ are path-homotopic in $\mathbb{R}^{2}$. A homotopy is given by $h(s, t)=(1-t) \gamma_{1}(s)+t \gamma_{2}(s)$. This is the straight-line homotopy. For example, when we restrict to $s=1 / 2$, the homotopy gives the vertical diameter of the circle.

On the other hand, we could also consider these as paths in $\mathbb{R}^{2}-\{(0,0)\}$ or as paths in $S^{1}$. We will see later that these are not path-homotopic in either of these spaces.

Proposition 22.11. Given two points $a$ and $b$ in $X$, path-homotopy defines an equivalence relation on the set of paths from a to $b$.

A path in $X$ that begins and ends at the same point is called a loop in $X$. We call the starting/end point the basepoint of the loop (and often of $X$ as well). By the above proposition, path-homotopy defines an equivalence relation on the set of loops in $X$ with basepoint $x_{0}$. The set of equivalence classes is denoted $\pi_{1}\left(X, x_{0}\right)$ and is called the fundamental group of $X$ (with basepoint $x_{0}$ ). Of course, so far we have no reason to call this a group, we only know this as a set.

Example 22.12. Use of straight-line homotopies show that $\pi_{1}\left(\mathbb{R}^{n}, x\right)=\left\{c_{x}\right\}$ for any $n$ and $x$. More generally, $\pi_{1}(X, x)=\left\{c_{x}\right\}$ for any convex subset of $\mathbb{R}^{n}$. This holds even more generally for any star-shaped region in $\mathbb{R}^{n}$. A subset $X \subset \mathbb{R}^{n}$ is said to be star-shaped around $x$ if for any $y \in X$, the straight-line segment $\overline{x y}$ is contained in $X$.

Here is a slightly different perspective on loops. Since a loop is a map $\gamma: I \longrightarrow X$ that is constant on the subspace $\partial I=\{0,1\} \subseteq I$, there is an induced map from the quotient space $\bar{\gamma}: I / \partial I \longrightarrow X$. Recall that $I / \partial I$ is homeomorphic to the circle $S^{1}$. So a loop in $X$ is the same as a map $\bar{\gamma}: S^{1} \longrightarrow X$.

A based map between two spaces with chosen basepoints is simply a map that takes the basepoint of one space to the basepoint of the other. By a based homotopy, we mean a homotopy through based maps (so the homotopy is constant on the basepoint). Based homotopy defines an equivalence relation on the set of based maps, and the set of based homotopy classes is denoted

$$
\left[\left(X, x_{0}\right),\left(Y, y_{0}\right)\right]_{*} .
$$

It is customary to take $(1,0)$ as the basepoint for $S^{1}$, and path-homotopy classes of loops in $X$, based at $x_{0}$, correspond to based homotopy classes of maps $S^{1} \longrightarrow X$. So

$$
\pi_{1}\left(X, x_{0}\right) \cong\left[\left(S_{63}^{1},(1,0)\right),\left(X, x_{0}\right)\right]_{*}
$$

Where does the group structure on homotopy classes of loops come from? Well, you can concatenate paths, by traveling first along one and then along the other.

Definition 22.13. Let $\gamma$ and $\lambda$ be paths in $X$. We say the two paths are composable in $X$ if $\gamma(1)=\lambda(0)$. When this is the case, we define the concatenation of $\gamma$ and $\lambda$ to be the path

$$
\gamma \cdot \lambda(s)=\left\{\begin{array}{cc}
\gamma(2 s) & s \in[0,1 / 2] \\
\lambda(2 s-1) & s \in[1 / 2,1] .
\end{array}\right.
$$

This formula looks familiar, right? This was the one used in Proposition 22.4 to glue two homotopies together. This is no accident: a path is precisely a homotopy between two constant maps!

Remark 22.14. Beware that $\gamma \cdot \lambda$ means do $\gamma$ first (in double time), and then $\lambda$ (in double time). This is the opposite convention of what we use for function composition.

Concatenation will provide the group structure on $\pi_{1}(X)$.
Proposition 22.15. The above operation only depends on path-homotopy classes. That is, if $\gamma \simeq_{p} \gamma^{\prime}$ and $\lambda \simeq_{p} \lambda^{\prime}$, then $\gamma \cdot \lambda \simeq_{p} \gamma^{\prime} \cdot \lambda^{\prime}$.

Proof. Let $L: \gamma \simeq_{p} \gamma^{\prime}$ and $R: \lambda \simeq_{p} \lambda^{\prime}$ be path-homotopies.
We define a new path homotopy by

$$
H(s, t)=\left\{\begin{array}{cc}
L(2 s, t) & s \in[0,1 / 2] \\
R(2 s-1, t) & s \in[1 / 2,1] .
\end{array}\right.
$$



This tells us that the concatenation operation is well-defined on path-homotopy classes. We will next check that it gives a well-behaved algebraic operation.
22.2. The fundamental group. For any point $x \in X$, we denote by $c_{x}$ the constant path at $x$ in $X$.

Proposition 22.16. Let $\gamma$ (from $x$ to $y$ ), $\lambda$, and $\mu$ be composable paths in $X$. Concatenation of path-homotopy classes satisfies the following properties.
(1) (unit law) $\left[c_{x}\right] \cdot[\gamma]=[\gamma]=[\gamma] \cdot\left[c_{y}\right]$
(2) (associativity) $([\gamma] \cdot[\lambda]) \cdot[\mu]=[\gamma] \cdot([\lambda] \cdot[\mu])$
(3) (inverses) Define $\bar{\gamma}(s)=\gamma(1-s)$. Then $[\gamma] \cdot[\bar{\gamma}]=\left[c_{x}\right]$ and $[\bar{\gamma}] \cdot[\gamma]=\left[c_{y}\right]$.

Proof. (1) Define

$$
h(s, t)=\left\{\begin{array}{cc}
x & 2 s \in[0,1-t] \\
\gamma\left(\frac{2 s-1+t}{1+t}\right) & 2 s \in[1-t, 2] .
\end{array}\right.
$$


(2) Define

$$
h(s, t)=\left\{\begin{array}{cc}
\gamma\left(\frac{4 s}{1+t}\right) & s \in\left[0, \frac{1+t}{4}\right] \\
\lambda(4 s-1-t) & s \in\left[\frac{1+t}{4}, \frac{2+t}{4}\right] \\
\mu\left(\frac{4 s-2-t}{2-t}\right) & s \in\left[\frac{2+t}{4}, \frac{1]}{} .\right.
\end{array}\right.
$$


(3) Define

$$
h(s, t)=\left\{\begin{array}{cc}
\gamma(2 s) & 2 s \in[0,1-t] \\
\gamma(1-t) & 2 s \in[1-t, 1+t] \\
\gamma(2(1-s)) & 2 s \in[1+t, 2] .
\end{array}\right.
$$



Actually, for parts (1) and (2) there is a slicker approach, (this is in Hatcher). A reparametrization of a path $\gamma$ is a composition $\gamma \circ \varphi$, where $\varphi: I \longrightarrow I$ is any map satisfying $\varphi(0)=0$ and $\varphi(1)=1$. It is clear that any such $\varphi$ is homotopic to the identity map of $I$ (just use a straight-line homotopy). For (1), we can write $c_{x} \cdot \gamma$ as a reparametrization of $\gamma$. Thus $c_{x} \cdot \gamma=\gamma \circ \varphi \simeq_{p} \gamma$. A similar argument also works for (2).

## Wed, Dec. 6

Ok, now we know that we have a group structure on $\pi_{1}\left(X, x_{0}\right)$ ! Next semester, we will show the following result:
Theorem 22.17. The fundamental group $\pi_{1}\left(S^{1}, 1\right)$ is an infinite cyclic group. In other words, it is isomorphic to $\mathbb{Z}$.

It is easy to write down a group homomorphism $\mathbb{Z} \xrightarrow{\phi} \pi_{1}\left(S^{1}, 1\right)$. We define $\phi(n)$ to be the loop that winds around the circle $n$ times. In other words,

$$
\phi(n)(t)=e^{t \cdot 2 n \pi i}
$$

The content of the theorem is that this homomorphism is bijective.
We can derive a number of very interesting consequences from our knowledge of the fundamental group of $S^{1}$.

First, we discuss how the fundamental group interacts with maps.
Proposition 22.18. Let $\left(X, x_{0}\right) \xrightarrow{f}\left(Y, y_{0}\right) \xrightarrow{g}\left(Z, z_{0}\right)$ be based maps.
(1) The induced map $f_{*}: \pi_{1}\left(X, x_{0}\right) \longrightarrow \pi_{1}\left(Y, y_{0}\right)$ is a group homomorphism.
(2) The composition $g_{*} \circ f_{*}$ agrees with $(g \circ f)_{*}$.
(3) If $\left(Y, y_{0}\right)=\left(X, x_{0}\right)$ and $f \simeq_{p} \mathrm{id}$, then $f_{*}=\mathrm{id}$.

Taken together these statements say that the assignment $\left(X, x_{0}\right) \mapsto \pi_{1}\left(X, x_{0}\right)$ defines a functor from the category of based spaces and basepoint-preserving maps to the category of groups and group homomorphisms.

Theorem 22.19. (Brouwer fixed point theorem) For any map $f: D^{2} \longrightarrow D^{2}$, there exists at least one point $x \in D^{2}$ such that $f(x)=x$. Such an $x$ is called a fixed point of the map $f$.

Proof. Assume for a contradiction that $f$ has no fixed points. Then $x-f(x)$ is not the origin, and for each point $x$ there is a unique $t_{x} \geq 1$ such that $f(x)+t_{x}(x-f(x))$ lies on $S^{1}$. This is where the ray starting at $f(x)$ and passing through $x$ meets the circle. Define $g(x): D^{2} \longrightarrow S^{1}$ by the formula

$$
g(x)=f(x)+t_{x}(x-f(x))
$$

You should convince yourself that $t_{x}$, and therefore $g(x)$, is a continuous function of $x$.
Now the key point is that if $x$ starts in the boundary $S^{1}$ of $D^{2}$, then $t_{x}=1$ and $g(x)=x$. In other words, the composition

$$
S^{1} \xrightarrow{i} D^{2} \xrightarrow{g} S^{1}
$$

is the identity map of $S^{1}$. Consider what happens on the fundamental group. The conclusion would be that the composition

$$
\pi_{1}\left(S^{1}\right)=\mathbb{Z} \xrightarrow{i_{*}} \pi_{1}\left(D^{2}\right)=0 \xrightarrow{g_{*}} \pi_{1}\left(S^{1}\right)=\mathbb{Z}
$$

is the identity map of $\mathbb{Z}$, which is impossible.
Application: Take a cup of coffee and move it around, so that the coffee gets mixed up. When it comes to rest, there is some particle that ends up where it started. (Okay, this is sort of BS since it assumes every particle stays on the surface, but it is a common description of the Brouwer fixed point theorem.)

## Fri, Dec. 8

22.3. Change of basepoint and degree. Let $f: S^{1} \longrightarrow S^{1}$ be any map. Then, as we saw last time, it defines a homomorphism

$$
f_{*}: \pi_{1}\left(S^{1}, 1\right) \longrightarrow \pi_{1}\left(S^{1}, f(1)\right)
$$

If $f$ is not based, then it is a little annoying that the target fundamental group has a different choice of basepoint. There is a way to fix this. First, let $\gamma$ be any path $\gamma: 1 \rightsquigarrow f(1)$.Then if $\alpha$ is any loop based at $f(1)$, we can create a loop based at 1 by the path composition

$$
\Phi_{\gamma}(\alpha)=\gamma \cdot \alpha \cdot \gamma^{-1}
$$

The trouble is that, in general, the map $\Phi_{\gamma}$ does depend on the choice of (path-homotopy class of) $\gamma$. Any other such path is necessarily of the form $\delta=\gamma \cdot \beta$ for some loop $\beta$ based at $f(1)$. Then

$$
\Phi_{\delta}(\alpha)=\gamma \cdot \beta \cdot \alpha \cdot \beta^{-1} \cdot \gamma^{-1}=\Phi_{\gamma}\left(\beta \alpha \beta^{-1}\right) .
$$

In our case, since $\pi_{1}\left(S^{1}, 1\right)$ is abelian, this conjugation disappears, so that the change-of-basepoint map $\Phi_{\gamma}$ does not depend on any choice. Thus, given any (continuous) map $f: S^{1} \longrightarrow S^{1}$, we get a well-defined homomorphism

$$
\mathbb{Z} \cong \pi_{1}\left(S^{1}, 1\right) \xrightarrow{f_{*}} \pi_{1}\left(S^{1}, f(1)\right) \xrightarrow{\Phi_{\gamma}} \pi_{1}\left(S^{1}, 1\right) \cong \mathbb{Z} .
$$

Definition 22.20. We define the degree of $f$ to be $\operatorname{deg}(f)=\Phi_{\gamma}\left(f_{*}(1)\right)$, the image of 1 under this composition.
Proposition 22.21. If $f$ and $g$ are homotopic as maps $S^{1} \longrightarrow S^{1}$, then $\operatorname{deg}(f)=\operatorname{deg}(g)$.
Proof. We suppose WLOG that $f$ is based. Note that if we know that $f$ and $g$ are homotopic through based maps, then the result follows. The map $g$ may not be based. Let $h: f \simeq g$ be a homotopy of maps $S^{1} \longrightarrow S^{1}$ and let $\gamma$ be the path $\gamma(t)=h(1, t)$ from 1 to $g(1)$.

If we define $\tilde{g}:=\Phi_{\gamma}(g)$, then $\operatorname{deg}(\tilde{g})=\operatorname{deg}(g)$ by the definition of the degree. So it suffices to identify $\operatorname{deg}(f)$ with $\operatorname{deg}(\tilde{g})$. But we can build a based homotopy as in the picture to the right.


Theorem 22.22. (Fundamental theorem of algebra) Every nonconstant polynomial with complex coefficients has a solution in $\mathbb{C}$.
Proof. Assume that $p(z) \neq 0$ for all $z \in \mathbb{C}$. We will show that $p$ must be constant. Define a function $f: S^{1} \longrightarrow S^{1}$ by $f(z)=p(z) /\|p(z)\|$. We can define a homotopy by

$$
h(z, t)=\underset{66}{p(z t) /\|p(z t)\| .}
$$

Thus $f$ is homotopic to a constant map, which means that it has "degree" zero.
On the other hand, write $a_{i}$ for the coefficients of the degree $n$ polynomial $p(z)$. For convenience, we assume $p(z)$ is monic. Let $k(z, t)$ be the homotopy between $z^{n}$ and $p(z)$ given by the formula

$$
k(z, t)=\sum_{i=0}^{n} a_{i} z^{i} t^{n-i}=z^{n}+a_{n-1} z^{n-1} t+\cdots+a_{0} t^{n} .
$$

Note that, for $t \neq 0$ this can be rewritten as $k(z, t)=t^{n} p(z / t)$. In particular, this is never 0 by hypothesis. It follows that the map $H: S^{1} \times I \longrightarrow S^{1}$ defined by the formula

$$
H(z, t)=\frac{k(z, t)}{\|k(z, t)\|}
$$

defines a homotopy from $z^{n}$ to $f$. This shows that $f$ has degree $n$.
Combining the two statements gives that $n=0$, so that $p$ is a constant polynomial.
Application: ...everything?

