CLASS NOTES
MATH 751 (FALL 2018)

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CONTENTS

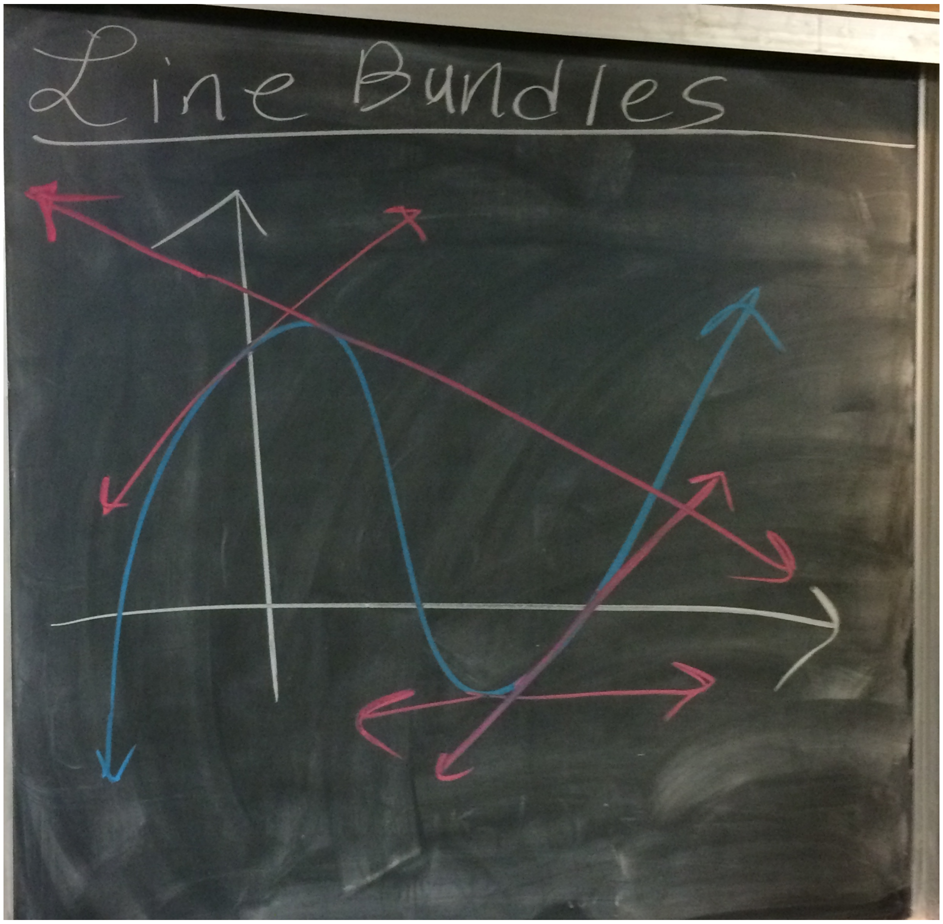

Figure courtesy of Kaelin Cook-Powell

Wed, Aug. 22

Consider the (open) Möbius band $\mathcal{M}$, defined by

$$\mathcal{M} := \Big([0,1] \times (0,1)\Big) / \sim, \qquad (0,t) \sim (1, 1-t)$$

Locally, this "looks like" the product $S^1 \times (0,1)$. But these are not "globally" the same, meaning that $\mathcal{M}$ is not homeomorphic to $S^1 \times (0,1)$.

But actually, we are not so interested in trying to produce an arbitrary homeomorphism. For both of these spaces, projection to the first coordinate produces a map to $[0,1]/\sim \cong S^1$. The fiber of any point in $p : \mathcal{M} \longrightarrow S^1$ is (homeomorphic to) $(0,1)$. So both of these spaces are of the form $E \xrightarrow{p} S^1$, and we might wonder whether these are the same types of spaces "over S^1" (again the answer in this case is no). So we might ask for a homeomorphism f making the following triangle commute:

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "f"] \arrow[dr, "p"'] & & S^1 \times (0,1) \arrow[dl, "p_1"] \\ & S^1 & \end{tikzcd}$$

In language that we will introduce shortly, $\mathcal{M}$ and $S^1 \times (0,1)$ will be examples of "vector bundles" on S^1, and the more precise version of the question above is whether $\mathcal{M}$ and $S^1 \times (0,1)$ are isomorphic as vector bundles over S^1.

Vector bundles have gotten a lot of attention for a number of reasons: (1) they are a fundamental part of the structure of "smooth" manifolds (2) they are the basic ingredient to $\mathbb{K}$-theory, which was one of the first "generalized" cohomology theories to be studied.

This will be a course on Vector Bundles. Among the topics we aim to study include **Topics:**

- real and complex vector bundles,
- constructions (Whitney sum, tensor product, etc.)
- reduction of structure group
- universal bundles and the classification of bundles
- characteristic classes
- $\mathbb{K}$-theory

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1. Vector bundles

1.1. Definition and examples.

Definition 1.1. A (real) **vector bundle** of rank n over B is a space E and map $p : E \longrightarrow B$ equipped with (real) vector space structure on each fiber $p^{-1}(b)$ such that

- (Local triviality) For each $b \in B$ there is a neighborhood $b \in U$ and a homeomorphism
$$\varphi_U : p^{-1}(U) \cong U \times \mathbb{R}^n$$
which restricts to a vector space isomorphism $\varphi_U : p^{-1}(c) \cong \{c\} \times \mathbb{R}^n$ for all $c \in U$.

Example 1.2. (Trivial bundle) For any base B, the **trivial** rank n bundle over B is $B \times \mathbb{R}^n$, where p is just the projection. The point is that any bundle looks locally like a trivial bundle, though perhaps not globally so. We will sometimes write this bundle as $\underline{n}_B$.

Example 1.3. Let $B = \mathbb{R}^2 - \{\mathbf{0}\}$. Let $E \subset \mathbb{R}^2 \times \mathbb{R}^2$ consist of pairs $(\mathbf{x}, \mathbf{y})$, where $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{x} \perp \mathbf{y}$. Then E, equipped with the projection to the first $\mathbb{R}^2$, becomes a rank 1 bundle over B. We give $p^{-1}(\mathbf{x}) \subset \{\mathbf{x}\} \times \mathbb{R}^2$ the vector space structure inherited from $\mathbb{R}^2$. We may define a local trivialization on the subspace $U_1 = \{\mathbf{x} \mid x_1 > 0\} \subseteq B$ by
$$\varphi_{U_1}(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, y_2) \in U_1 \times \mathbb{R}.$$
Note that the assignment $(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}, y_2)$ is linear in $\mathbf{y}$. And it is injective since $\mathbf{x} \in U_1$, so that the line spanned by $\mathbf{y}$ is not the x-axis. In particular, the projection $\mathbf{y} \mapsto y_2$ is an isomorphism.

$\boxed{\textbf{Fri, Aug. 24}}$

In the previous example, if we recall that $\mathbb{R}^2 - \{\mathbf{0}\} \cong S^1 \times (0, \infty)$, we can observe that the radius is not interacting with the bundle in an essential way, and restricting to S^1 gives the following example

Example 1.4. Let $B = S^1$ and $E \subset \mathbb{R}^2 \times \mathbb{R}^2$ consist of pairs $(\mathbf{x}, \mathbf{y})$, where $\mathbf{x} \in S^1$ and $\mathbf{x} \perp \mathbf{y}$. The projection makes this into a rank one bundle over S^1 just as above.

Rank one bundles are called **line bundles**. More generally, we have

Example 1.5. Let $B = S^n$ and $E \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ consist of pairs $(\mathbf{x}, \mathbf{y})$, where $\mathbf{x} \in S^n$ and $\mathbf{x} \perp \mathbf{y}$. For fixed $\mathbf{x}$, the $\mathbf{y}$ span an n-dimensional subspace of $\mathbb{R}^{n+1}$. This determines a rank n bundle on S^n.

Example 1.6. Taking again $B = S^1$, another choice is to take $E \subset \mathbb{R}^2 \times \mathbb{R}^2$ to consist of pairs $(\mathbf{x}, \mathbf{y})$, where $\mathbf{x} \in S^1$ and $\mathbf{y}$ is any scalar multiple of $\mathbf{x}$. We again take $p : E \longrightarrow B$ to be the projection $p(\mathbf{x}, \mathbf{y}) = \mathbf{x}$. We define a local trivialization on the subspace $U_1 = \{\mathbf{x} \mid x_1 > 0\} \subseteq B$ by

$$\varphi_{U_1}(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, y_1) \in U_1 \times \mathbb{R}.$$

Again the assignment $(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}, y_1)$ is linear in $\mathbf{y}$. And it is injective since $\mathbf{x} \in U_1$, so that the line spanned by $\mathbf{y}$ is not the y-axis. In particular, the projection $\mathbf{y} \mapsto y_1$ is an isomorphism.

Example 1.7. Let $B = S^n$ and $E \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ consist of pairs $(\mathbf{x}, \mathbf{y})$, where $\mathbf{x} \in S^n$ and $\mathbf{y}$ is parallel to $\mathbf{x}$. For fixed $\mathbf{x}$, the $\mathbf{y}$ span a line in $\mathbb{R}^{n+1}$. This determines a line bundle on S^n.

Since $\mathbb{RP}^1 \cong S^1$, Example 1.6 generalizes to give a line bundle on $\mathbb{RP}^n$.

Example 1.8. (Canonical line bundle) Take $B = \mathbb{RP}^n$ for $n \geq 1$. Let $E \subseteq \mathbb{RP}^n \times \mathbb{R}^{n+1}$ consist of pairs $(\ell, \mathbf{y})$, where $\mathbf{y} \in \ell$. The map $p : E \longrightarrow B$ is projection. Let $U_1 \subset \mathbb{RP}^n$ consist of lines ℓ that do not lie in the hyperplane $x_1 = 0$. Then the map

$$\varphi_{U_1} : p^{-1}(U_1) \longrightarrow U_1 \times \mathbb{R}$$

defined by $\varphi_{U_1}(\ell, \mathbf{y}) = (\ell, y_1)$ is a local trivialization. This bundle is often written γ_1^n. In algebraic geometry, it is written $\mathcal{O}(-1)$.

1.2. **Tangent bundles & normal bundles.** Example 1.5 is an instance of a more general, important family of examples.

Let $M \subseteq \mathbb{R}^N$ be a manifold and let $p \in M$. A **smooth curve** through p in M is a map

$$\gamma : (-\epsilon, \epsilon) \longrightarrow M$$

with $\gamma(0) = p$ which is infinitely differentiable, as a map $(-\epsilon, \epsilon) \longrightarrow \mathbb{R}^N$. The **velocity vector** of such a curve is $\gamma'(0) \in \mathbb{R}^N$.

Definition 1.9. Let $M \subseteq \mathbb{R}^N$ be a smooth manifold and $p \in M$. Then a **tangent vector** to M at p is a vector in $\mathbb{R}^N$ of the form $\gamma'(0)$ for some γ. The **tangent space** of M at p is the set T_pM of tangent vectors to M at p.

Without restrictions on M, there is no reason for these tangent spaces to behave well. But if we assume that M is a **smooth manifold** of dimension n, then each T_pM turns out to be an n-dimensional subspace of $\mathbb{R}^N$.

Example 1.10. Consider $M = S^1 \subseteq \mathbb{R}^2$. A tangent vector at $(1, 0)$ is the velocity vector of some smooth curve through $(1, 0)$. For example, we have $\gamma(t) = e^{it} = (\cos t, \sin t)$. Then $\gamma'(t) = (-\sin t, \cos t)$, so that $\gamma'(0) = (0, 1)$ is a tangent vector.

$\boxed{\textbf{Mon, Aug. 27}}$

This definition is not ideal for some purposes. For instance, it is not clear from this definition that it does not depend on the embedding into $\mathbb{R}^N$. Another way to think about tangent vectors is via their action on functions.

Suppose $f : M \longrightarrow \mathbb{R}$ is a differentiable (or smooth) function. If $\mathbf{v} = \gamma'(0)$ is a tangent vector at $p \in M$, we define

$$\partial_{\mathbf{v}}(f) := (f \circ \gamma)'(0) \in \mathbb{R}.$$

Now if we consider a product of functions, we have

$$\begin{aligned}
\partial_{\mathbf{v}}(g \cdot f) &= ((g \cdot f) \circ \gamma)'(0) \\
&= (g\gamma \cdot f\gamma)'(0) \\
&= (g\gamma)'(0)(f\gamma)(0) + (g\gamma)(0)(f\gamma)'(0) \\
&= \partial_{\mathbf{v}}(g)f(p) + g(p)\partial_{\mathbf{v}}(f)
\end{aligned}$$

Writing $\mathcal{C}^\infty(M, \mathbb{R})$ for the set of smooth functions $M \longrightarrow R$, a function $\partial_{\mathbf{v}} : \mathcal{C}^\infty(M, \mathbb{R}) \longrightarrow \mathbb{R}$ satisfying this formula is known as a **derivation at** p on $\mathcal{C}^\infty(M, \mathbb{R})$, and it turns out that every derivation arises from a tangent vector. Thus some authors choose to define tangent vectors to be derivations on $\mathcal{C}^\infty$.

Definition 1.11. Let $f : M \longrightarrow N$ be a differentiable (or smooth) map, and let $p \in M$. Then define $df : T_pM \longrightarrow T_{f(p)}N$ by

$$\mathbf{v} = \gamma'(0) \mapsto (f \circ \gamma)'(0).$$

With this definition in hand, we have another interpretation of $\mathbf{v}(f)$. Namely,

$$\partial_{\mathbf{v}}(f) = df(\mathbf{v}) \in T_{f(p)}\mathbb{R} \cong \mathbb{R}.$$

Proposition 1.12. *The map $df : T_pM \longrightarrow T_{f(p)}N$ is a linear map.*

Even better, we have

Proposition 1.13. *The assignment $(M, p) \mapsto T_pM$ defines a functor from the category pointed manifolds and smooth maps to* $\mathbf{Vect}_{\mathbb{R}}$*, where the functor takes a morphism f to df.*

Definition 1.14. Let $M \subseteq \mathbb{R}^N$ be a smooth manifold of dimension d. We define the **tangent bundle** of M to be the subspace

$$TM \subset M \times \mathbb{R}^N$$

consisting of pairs $x, \mathbf{v}$, where $\mathbf{v} \in T_xM$. This is a vector bundle of rank d

We have assumed an embedding just for convenience; it is possible to define TM without reference to an embedding. In particular, TM does not depend on the embedding of M into some Euclidean space. This is in contrast to the following definition.

Definition 1.15. Let $M \subseteq \mathbb{R}^N$ be a smooth manifold. We define the **normal bundle** of M in $\mathbb{R}^N$ to be the subspace

$$NM \subset M \times \mathbb{R}^N$$

consisting of pairs $x, \mathbf{v}$, where $\mathbf{v} \perp T_xM$. This is a vector bundle of rank $N - d$.

The normal bundle is often denoted by ν. This bundle most certainly depends on the embedding.

In fact, we have already seen tangent and normal bundles. Example 1.5 was the tangent bundle of S^n, and Example 1.7 was the normal bundle of the embedding $S^n \hookrightarrow \mathbb{R}^{n+1}$.

Wed, Aug. 29

1.3. **Maps of bundles & sections.** We know what vector bundles are. How do we map between them?

Definition 1.16. Let $p : E_1 \longrightarrow B_1$ and $q : E_2 \longrightarrow B_2$ be vector bundles. A **bundle map** is a pair (F, f) making the diagram

$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d, "p"'] & E_2 \arrow[d, "q"] \\ B_1 \arrow[r, "f"'] & B_2 \end{tikzcd}$$

commute, and such that the restriction $F : p^{-1}(b) \longrightarrow q^{-1}(f(b))$ is linear.

We sometimes say that this is a bundle map "over f". Of particular importance is the following special case.

Proposition 1.17. *Let $p : E \longrightarrow B$ be a vector bundle. A bundle map $\mathbf{1}_B \longrightarrow E$ over (the identity map of) B corresponds to a map $s : B \longrightarrow E$ such that $p \circ s = \mathrm{id}_B$.*

Proof. Suppose that $F : B \times \mathbb{R} \longrightarrow E$ is a bundle map. This gives, for each $b \in B$, a linear map $\{b\} \times \mathbb{R} \longrightarrow p^{-1}(b)$. But a linear map out of $\mathbb{R}$ is completely determined by its value at 1. In other words, the map F is determined by its restriction to $B \times \{1\}$. We define s to be this restriction $s = F_{|B\times\{1\}} : B \longrightarrow E$.

Conversely, given s, we define F by $F(b, r) = r \cdot s(b)$, where we are using the vector space structure on $p^{-1}(b)$. To check continuity, it suffices to check it locally. This can be done using local triviality of the bundle. ■

This kind of map s is useful, so we give it a name.

Definition 1.18. A **section** of a bundle $p : E \longrightarrow B$ is a map $s : B \longrightarrow E$ such that $p \circ s = \mathrm{id}_B$.

A section of the tangent bundle on M is called a **vector field** on M. Any vector bundle has the **zero section**, which simply picks out 0 in each fiber. Often we are interested in the existence of other sections.

Proposition 1.19. *A section of $\underline{1}_B$ corresponds precisely to a map $B \longrightarrow \mathbb{R}$. Similarly, a section of $\underline{n}_B$ corresponds to $B \longrightarrow \mathbb{R}^n$.*

So one can think of sections as "generalized functions". Proposition 1.17 moreover gives

Proposition 1.20. *A nowhere-vanishing section of $E \xrightarrow{p} B$ corresponds to an injective map of bundles $\underline{1}_B \hookrightarrow E$.*

It is easy to write down a non-vanishing vector field on S^1 or S^3, but the "Hairy Ball Theorem" states that this is not possible on S^2. Every vector field on S^2 has a "bald spot".

Corollary 1.21. *A line bundle is trivial if and only if it admits a nowhere-vanishing section.*

As an application, we show that the canonical line bundle on $\mathbb{RP}^n$ is nontrivial.

Example 1.22. (From [MS, Theorem 2.1]) Consider the canonical line bundle γ_1^n on $\mathbb{RP}^n$, and let $s : \mathbb{RP}^n \longrightarrow E(\gamma_1^n)$ be any section. We will show that $s(b) = 0$ for some $b \in \mathbb{RP}^n$.

Consider the composition

$$S^n \xrightarrow{q} \mathbb{RP}^n \xrightarrow{s} E(\gamma_1^n).$$

Since we know the total space of γ_1^n, this composition takes the form

$$\mathbf{x} \mapsto (\mathbb{R}\mathbf{x}, t(\mathbf{x})\mathbf{x})$$

for some function t of $\mathbf{x}$. But since this map agrees on antipodal points, we must have $t(-\mathbf{x}) = -t(\mathbf{x})$. In other words, t is an odd function. But S^n is connected, so $t(\mathbf{x}_0) = 0$ for some $\mathbf{x}_0 \in S^n$. It follows that $s(\mathbf{x}_0) = 0$.

Generalizing Corollary 1.21, we have

Proposition 1.23. *Let $E \xrightarrow{p} B$ be a vector bundle of rank d. Then E is trivial if and only if there are d sections $s_1, \dots, s_d$, which are linearly independent at each b.*

Proof. Certainlly if $E \cong \underline{n}_B$, then the maps $b \mapsto (b, \mathbf{e}_i)$ specify n linearly independent sections.

Suppose that E has n linearly independent sections $s_1, \dots, s_n$. Then define $\underline{n}_B \longrightarrow E$ by

$$(b, r_1, \dots, r_n) \mapsto r_1 s_1(b) + \cdots + r_n s_n(b).$$

Since the s_i are linearly independent, this map is injective. Since $\underline{n}_B$ and E are both rank n bundles, the map is an isomorphism. ∎

Wed, Sept. 5

1.4. **Pullbacks.** We will talk about some constructions which allow us to produce new bundles out of old ones. The first is the pullback. We start by reviewing the pullback of spaces.

Definition 1.24. Let $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ be maps. The pullback

$$\begin{tikzcd} X \times_Z Y \arrow[r, "p_Y"] \arrow[d, "p_X"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

is the subspace of $X \times Y$ consisting of pairs (x, y) such that $f(x) = g(y)$.

The pullback is characterized by as the universal object with a pair of maps to X and Y that agree after composing with f and g.

Definition 1.25. Let $p : E \longrightarrow B$ be a vector bundle of rank n and let $f : X \longrightarrow B$ be any map. We define the **pullback bundle** $f^*(E)$, which is a vector bundle over X of rank n, as in the diagram

$$\begin{tikzcd} f^*(E) = X \times_B E \arrow[r, "p_E"] \arrow[d, "p_X"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"] & B. \end{tikzcd}$$

The total space is $X \times_B E$, with projection map to X given by p_X. The map p_E induces a homeomorphism on fibers $p_X^{-1}(x) \cong p^{-1}(f(x))$, and we use this isomorphism to equip $p_X^{-1}(x)$ with a vector space structure. For local triviality, let $x \in X$. Then $f(x) \in B$ has a neighborhood U with a trivialization $\varphi_U : p^{-1}(U) \cong U \times \mathbb{R}^n$. We want a trivialization of $f^*(E)$ over $f^{-1}(U)$. But we can think of the restriction of $f^*(E)$ to $f^{-1}(U)$ as first restricting E to U (which makes it trivial), and then pulling back to $f^{-1}(U)$. Since a trivial bundle pulls back to a trivial bundle, we conclude that $f^*(E)$ must be trivial over $f^{-1}(U)$.

Example 1.26. In the world of spaces, if $A \hookrightarrow B$ is an inclusion and $g : Y \longrightarrow B$ is a map, then the pullback $A \times_B Y$ is just the preimage $g^{-1}(A)$. Similarly, in the world of vector bundles, if $\iota : A \hookrightarrow B$ is the inclusion of a subspace, then $\iota^*(E)$ is just the restriction of E to a bundle over the subspace A.

Example 1.27. If X is any space and $c : X \longrightarrow *$ is a point, then we can realize the trivial bundle $\underline{n}_X$ over X as the pullback $\underline{n}_X \cong \iota^*(\mathbb{R}^n)$ of the trivial bundle over a point. This just uses that, in spaces, the pullback of $c : X \longrightarrow *$ and $c : \mathbb{R}^n \longrightarrow *$ is the product $X \times \mathbb{R}^n$.

Example 1.28. Consider the quotient map $q : S^n \longrightarrow \mathbb{RP}^n$. We have a map of bundles

$$\begin{tikzcd} \nu S^n \arrow[r, "Q"] \arrow[d] & E(\gamma_1^n) \arrow[d, "\gamma_1^n"] \\ S^n \arrow[r, "q"] & \mathbb{RP}^n, \end{tikzcd}$$

where νS^n is the normal bundle of S^n in $\mathbb{R}^{n+1}$. The map Q is given by $Q(\mathbf{x}, \mathbf{v}) = (\{\pm \mathbf{x}\}, \mathbf{v})$. This bundle map is an isomorphism on fibers, and Proposition 1.29 below implies that $\nu S^n \cong q^*(\gamma_1^n)$. Note that νS^n is trivial, as shown on Worksheet 1, but γ_1^n is nontrivial (Example 1.22).

Proposition 1.29. *Let (F, f) be a map of bundles from $E_1 \xrightarrow{p_1} B_1$ to $E_2 \xrightarrow{p_2} B_2$ which induces an isomorphism on fibers. Then the map $E_1 \longrightarrow f^*(E_2)$ given by $e \mapsto (p_1(e), F(e))$ is an isomorphism of bundles over B_1.*

We saw in Example 1.28 that the canonical line bundle pulls back to the normal bundle on S^n. Later, we will see that every line bundle is the pullback of some γ_1^n. There is also an analogue of this for bundles of higher rank.

Definition 1.30. Let $k \leq n$. Then **Grassmannian** $Gr_k(\mathbb{R}^n)$ of k-planes in $\mathbb{R}^n$ is the set of k-dimensional subspaces of $\mathbb{R}^n$. The topology comes from thinking of $Gr_k(\mathbb{R}^n)$ as an orbit space. The topological group $O(n)$ acts transitively on $Gr_k(\mathbb{R}^n)$, and the stabilizer of the subspace $\mathbb{R}^k \times \{\mathbf{0}\}$ is the subgroup $O(k) \times O(n-k)$. It follows that we get a bijection

$$Gr_k(\mathbb{R}^n) \cong \frac{O(n)}{O(k) \times O(n-k)}.$$

We define the topology on the Grassmannian using this bijection. In particular, it is compact since $O(n)$ is compact.

There is a canonical k-plane bundle on $Gr_k(\mathbb{R}^n)$, namely the subspace $E_k(\mathbb{R}^n) \subseteq Gr_k(\mathbb{R}^n) \times \mathbb{R}^n$ given by pairs $(\mathbf{V}, \mathbf{v})$, where $\mathbf{v} \in \mathbf{V}$.

Mon, Sept. 10

1.5. **Bundle constructions.** Last time, we introduced the important idea of a pullback. We can use this to define another useful construction.

Proposition 1.31. *Let $E_1 \xrightarrow{p_1} B_1$ and $E_2 \xrightarrow{p_2} B_2$ be vector bundles of rank n_1 and n_2, respectively. Then*

$$E_1 \times E_2 \xrightarrow{p_1 \times p_2} B_1 \times B_2$$

is a vector bundle of rank $n_1 + n_2$.

We leave the proof as an exercise.

Definition 1.32. Let $E \xrightarrow{p} B$ and $E' \xrightarrow{q} B$ be vector bundles of rank n and k over B, and let $\Delta_B : B \longrightarrow B \times B$ be the diagonal map. We define their **Whitney sum** (or simply direct sum) to be the vector bundle

$$E \oplus E' := \Delta_B^*(E \times E').$$

over B, of rank $n + k$.

Example 1.33. For any base space B, we have $\underline{n}_B \cong \underline{1}_B^{\oplus n}$.

Example 1.34. Recall the tangent bundle and normal bundle of S^n. A tangent vector at $\mathbf{x}$ is one that is orthogonal to $\mathbf{x}$, whereas a normal vector is one that is parallel. It follows that we have an isomorphism

$$TS^n \oplus \nu_{S^n} \cong \underline{n+1}_{S^n}.$$

Note that since the normal bundle for the sphere is trivial this can be rewritten as

$$TS^n \oplus \underline{1}_{S^n} \cong \underline{n+1}_{S^n}.$$

Beware that, however tempting it may be, we cannot simply cancel a $\underline{1}_{S^n}$ from each side, since the tangent bundle for S^n is not always trivial. But we do say that TS^n is **stably trivial**, because it becomes trivial after adding on a trivial bundle.

Definition 1.35. Let $M \subseteq \mathbb{R}^N$ be a d-dimensional manifold. We define the **normal bundle** of M in $\mathbb{R}^N$ as follows. The embedding $M \hookrightarrow \mathbb{R}^N$ gives an inclusion $TM \hookrightarrow \underline{N}_M = M \times \mathbb{R}^N$. For any $x \in M$, define the ν_x to be the orthogonal complement (in $\mathbb{R}^N$) of T_xM.

Proposition 1.36. *The ν_x's assemble, as x varies in M, to define a vector bundle of rank $N-d$ over M. Furthermore, we have an isomorphism*

$$TM \oplus \nu(M \hookrightarrow \mathbb{R}^N) \cong \underline{N}_M.$$

Proof. TM and ν are contained in the trivial bundle $M \times \mathbb{R}^N$, and we do have $T_xM \oplus \nu_x = \mathbb{R}^N$ for each x. It remains to find local trivializations for ν, so that it really does define a bundle. Let U be a neighborhood of x on which TM is trivial. Then we have linearly independent sections $s_1, \dots, s_d$ of TM on U. By using the Gram-Schmidt process, we may suppose that the sections are orthonormal (i.e. their values in any fiber gives an orthonormal basis for that fiber).

Now we claim that, after further restricting our neighborhood, we can extend the collection $\{s_i\}$ to a collection of N linearly independent sections of the trivial bundle $\underline{N}_M$. The key is that, locally, the inclusion $M \hookrightarrow \mathbb{R}^N$ is homeomorphic (actually diffeomorphic) to the inclusion $\mathbb{R}^d \times \{\mathbf{0}\} \hookrightarrow \mathbb{R}^N$. In the standard inclusion $\mathbb{R}^d \times \{\mathbf{0}\} \hookrightarrow \mathbb{R}^N$, we can find sections $s'_{d+1}, \dots, s'_N$ that are independent of the tangent space of $\mathbb{R}^d \times \{\mathbf{0}\}$, and under the homeomorphism, they give us the desired (local) sections of $\underline{N}_M$. Again, by applying Gram-Schmidt, we can force these to be orthonormal. But then the $s'_{d+1}, \dots, s'_N$ give $N-d$ linearly independent (local) sections for ν, which give the local trivialization. ∎

Wed, Sept. 12

Today, we will discuss a few more constructions for vector bundles. The idea is to transport constructions from the vector space world (such as direct sum) over to the setting of vector bundles. In order for this to work out, the vector space construction must be appropriately continuous.

We start by observing that the category $\mathbf{Vect}_{\mathbb{R}}$ of finite-dimensional real vector spaces is **enriched** over topological spaces. This means that,

(1) for vector spaces $\mathbf{V}$ and $\mathbf{W}$, we can equip the set $\mathbf{Vect}_{\mathbb{R}}(\mathbf{V}, \mathbf{W})$ of linear maps with the subspace topology of $\mathrm{Map}(\mathbf{V}, \mathbf{W})$.

(2) the composition map

$$\mathbf{Vect}_R(\mathbf{V}, \mathbf{W}) \times \mathbf{Vect}_{\mathbb{R}}(\mathbf{U}, \mathbf{V}) \longrightarrow \mathbf{Vect}_{\mathbb{R}}(\mathbf{U}, \mathbf{W})$$

is continuous.

Similarly, the product $\mathbf{Vect}_R \times \mathbf{Vect}_{\mathbb{R}}$ is also enriched over spaces. We will want to consider only vector space constructions that are compatible with this enrichment.

Definition 1.37. We say that a functor $T : \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$ is **enriched** (or **continuous**) if the functions

$$\mathbf{Vect}_{\mathbb{R}}(\mathbf{V}, \mathbf{W}) \xrightarrow{T} \mathbf{Vect}_{\mathbb{R}}(T\mathbf{V}, T\mathbf{W})$$

are continuous for all $\mathbf{V}$ and $\mathbf{W}$ in $\mathbf{Vect}_{\mathbb{R}}$.

Suppose now that $T : \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$ is continuous. Let X be a space and let $\mathbf{Vect}_{\mathbb{R}}(X)$ denote the category of vector bundles over X. We will use this to define a functor

$$\mathbb{T} : \mathbf{Vect}_{\mathbb{R}}(X) \longrightarrow \mathbf{Vect}_{\mathbb{R}}(X).$$

Construction 1.38. Given a bundle $E \xrightarrow{p} X$, define

$$\mathbb{T}(E) = \bigcup_{x \in X} T(E_x).$$

This defines $\mathbb{T}(E)$ as a set and also specifies the vector space structure on each fiber. To specify the topology, we consider first the case of a trivial bundle. Then $\mathbb{T}(E)$ is also a trivial bundle, and we have

$$\mathbb{T}(X \times \mathbf{V}) \overset{\text{def}}{=} X \times T\mathbf{V}.$$

Now for a general bundle $E \in \mathbf{Vect}_R(X)$ with a trivializations over some cover $\{U\}$, we have have already specified $\mathbb{T}(E_{|U})$. Now declare $V \subset \mathbb{T}(E)$ to be open precisely when $V \cap \mathbb{T}(E_{|U})$ is open for all U in the cover. Note that we have then built in the local trivializations of $\mathbb{T}(E)$ into the definition.

We have not used the continuity of T. This is needed to make the above construction into a functor. Thus let $\varphi : E \longrightarrow E'$ be a bundle map over X. We wish to product a bundle map $\mathbb{T}(\varphi) : \mathbb{T}(E) \longrightarrow \mathbb{T}(E')$. But it suffices to specify this continuous map locally. Thus we restrict to some $U \subset X$ on which both E and E' are trivial. Then φ is a bundle map $X \times \mathbf{V} \longrightarrow X \times \mathbf{W}$. But the map to X is necessarily the projection, so φ corresponds to a (linear) map $X \times \mathbf{V} \longrightarrow \mathbf{W}$, or equivalently a map $X \longrightarrow \mathbf{Vect}_{\mathbb{R}}(\mathbf{V}, \mathbf{W})$. For the same reason, a bundle map $X \times T\mathbf{V} \longrightarrow X \times T\mathbf{W}$ corresponds precisely to a map $X \longrightarrow \mathbf{Vect}_{\mathbb{R}}(T\mathbf{V}, T\mathbf{W})$. We take this map to be the composite of *continuous* maps

$$X \longrightarrow \mathbf{Vect}_{\mathbb{R}}(\mathbf{V}, \mathbf{W}) \xrightarrow{T} \mathbf{Vect}_{\mathbb{R}}(T\mathbf{V}, T\mathbf{W}).$$

This specifies the bundle map $\mathbb{T}(\varphi)$, and we leave as an exercise to verify that this does in fact make $\mathbb{T}$ into a functor.

Proposition 1.39. *For any map $X \xrightarrow{f} Y$ and enriched functor $T : \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$, the following square commutes:*

$$\begin{tikzcd}
\mathbf{Vect}_{\mathbb{R}}(Y) \arrow[r, "\mathbb{T}"] \arrow[d, "f^*"'] & \mathbf{Vect}_{\mathbb{R}}(Y) \arrow[d, "f^*"] \\
\mathbf{Vect}_{\mathbb{R}}(X) \arrow[r, "\mathbb{T}"'] & \mathbf{Vect}_{\mathbb{R}}(X).
\end{tikzcd}$$

A similar discussion applies if T starts as an enriched functor $\mathbf{Vect}_{\mathbb{R}} \times \mathbf{Vect}_{\mathbb{R}} \xrightarrow{T} \mathbf{Vect}_{\mathbb{R}}$.

Example 1.40. If we take $T = \oplus : \mathbf{Vect}_{\mathbb{R}} \times \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$, then $\mathbb{T}$ is the direct sum construction of Definition 1.32.

Example 1.41. If we take $T = \otimes : \mathbf{Vect}_{\mathbb{R}} \times \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$, we get $\mathbb{T} = \otimes$, a tensor product construction for bundles.

Example 1.42. If we take $T : \mathbf{Vect}_{\mathbb{R}}^{op} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$ to be $T(\mathbf{V}) = \mathbf{V}^* = \mathbf{Vect}_{\mathbb{R}}(\mathbf{V}, \mathbb{R})$, the dual vector space, we get the dual bundle construction.

Example 1.43. If we take $T = \mathbf{Vect}_{\mathbb{R}}(-,-) : \mathbf{Vect}_{\mathbb{R}}^{op} \times \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$, we get the Hom bundle construction.

In vector spaces, we have an isomorphism $\mathbf{Vect}_{\mathbb{R}}(\mathbf{V}, \mathbf{W}) \cong \mathbf{V}^* \otimes \mathbf{W}$, and this carries over to the analogous bundle constructions.

Mon, Sept. 17

1.6. **Classification of bundles.** Recall the canonical line bundle γ_1^n over $\mathbb{RP}^n$, discussed in Example 1.8. In Example 1.28, we showed that the (trivial) normal bundle on S^n can be expressed as a pullback of the canonical line bundle. In fact, this is true of any line bundle over a reasonable base space, if we allow n to become arbitrarily large. We will write γ_1 for γ_1^∞, the canonical line bundle over $\mathbb{RP}^\infty$.

We have a function

$$\mathcal{C}(X, \mathbb{RP}^n) \longrightarrow \{\text{line bundles on } X\},$$

but it is neither injective nor surjective in general. We really are concerned with <u>isomorphism classes</u> of bundles, and we will write $\mathrm{Vect}^1(X)$ for isomorphism classes of line bundles on X. Thus we will consider

$$\mathcal{C}(X, \mathbb{RP}^n) \longrightarrow \mathrm{Vect}^1(X).$$

In order to make it surjective, we will let n go to ∞. In order to make it injective, we will pass to homotopy classes of maps.

The following results hold if the base space is <u>paracompact</u>. This includes compact Hausdorff spaces, manifolds, and CW complexes. We will give the proofs in the compact Hausdorff case.

Theorem 1.44. *Let X be paracompact. Let $E \xrightarrow{p} B$ be a vector bundle and $f_0, f_1 : X \longrightarrow B$ homotopic maps. Then $f_0^*(E) \cong f_1^*(E)$.*

Proof. We sketch the proof in the case of a compact, Hausdorff base. See [H2, Theorem 1.6] for the paracompact case.

Let $h : X \times I \longrightarrow B$ be a homotopy from f_0 to f_1. Then $f_0^*(E) = h^*(E)_{|X \times \{0\}}$ and similarly for $f_1^*(E)$. So without loss of generality we may replace B by $X \times I$, and we wish to show that the time 0 and time 1 restrictions of a bundle E on $X \times I$ are isomorphic.

Using compactness, one can show that there is a finite cover $\{U_1, \dots, U_n\}$ of X so that the restriction of E to each $U_i \times I$ is trivial. Let $\{\varphi_i\}_{i=1}^n$ be a partition of unity subordinate to the cover $\{U_i\}_{i=1}^n$. For each $0 \leq j \leq n$, define $\Phi_j = \sum_{i=1}^j \varphi_i$. Thus $\Phi_0 = 0$ and $\Phi_n = 1$ on X. For simplicity, we will assume $n = 2$, since that is enough to see the argument. Thus we have

$$\Phi_0 = 0 \leq \Phi_1 = \varphi_1 \leq \Phi_2 = 1$$

on X. For each $0 \leq j \leq n$, we define $X_j \subseteq X \times I$ to be the graph of Φ_j. Thus $X_0 = X \times \{0\}$ and $X_2 = X \times \{1\}$, and each X_j is homeomorphic to X via the projection. Finally, let E_j be the restriction of E to $X_j \cong X$. We claim that $E_0 \cong E_1 \cong E_2$.

To see that $E_0 \cong E_1$, recall that E is trivial on $U_1 \times I$. It follows that the trivialization of E on U_1 restricts to trivializations φ_{U_1} of E_0 and E_1 on U_1. Define $\alpha_{U_1} : (E_0)_{|U_1} \longrightarrow (E_1)_{|U_1}$ to be the composition

$$(E_0)_{|U_1} \xrightarrow{(\varphi_{U_1})_{|E_0}} F \times U_1 \xrightarrow{(\varphi_{U_1})^{-1}_{|E_1}} (E_1)_{|U_1}.$$

Now let $V_1 = X \setminus \mathrm{supp}(\varphi_1)$. Since φ_1 is supported inside U_1, it follows that $U_1 \cup V_1 = X$. Also, we have that $(E_0)_{|V_1} = (E_1)_{|V_1}$, since $X_0 \cap (V_1 \times I) = V_1 \times \{0\} = X_1 \cap (V_1 \times I)$. Now α_{U_1} on $(E_0)_{|U_1}$ glues together with id on $(E_0)_{|V_1}$ to give an isomorphism $E_0 \cong E_1$.

■

Corollary 1.45. *Any vector bundle over a contractible space is trivial.*

We are working our way towards Theorem 1.47 below. We first prove an auxiliary result.

Proposition 1.46. *Let $E \longrightarrow B$ be a line bundle. Then a map $f : B \longrightarrow \mathbb{RP}^\infty$ with an isomorphism $E \cong f^*\gamma_1$ corresponds precisely to a map $g : E \longrightarrow \mathbb{R}^\infty$ which is a linear injection in each fiber.*

Proof. Recall that $E(\gamma_1) \subset \mathbb{RP}^\infty \times \mathbb{R}^\infty$ by definition. Suppose given f and an isomorphism $\varphi : E \cong f^*\gamma_1$. Then we may take g to be the composition $E \xrightarrow{\varphi} f^*\gamma_1 \xrightarrow{p_2} \mathbb{R}^\infty$, which is a linear injection in each fiber. On the other hand, suppose given the map g. Then we define $f(b) = g(p^{-1}(b))$. We get a map of bundles $E \longrightarrow f^*\gamma_1$ by $e \mapsto (p(b), g(b))$. Since it is a linear injection on fibers and both are line bundles, this must be an isomorphism. ■

Wed, Sept. 19

Theorem 1.47. *(Classification of line bundles) Let X be paracompact. Then the map*

$$[X, \mathbb{RP}^\infty] \xrightarrow{\cong} \mathrm{Vect}^1(X)$$
$$f \mapsto f^*\gamma_1$$

is a bijection.

Proof. Again, we suppose for simplicity that X is compact Hausdorff. See [H2, Theorem 1.16] for the more general paracompact case.

We show that this function is surjective. Thus let $E \longrightarrow X$ be a line bundle. By Proposition 1.46, it suffices to produce a map $E \longrightarrow \mathbb{R}^\infty$ that is a linear injection on fibers. We may suppose a finite open cover $\{U_i\}_{i \leq k}$, and a partition of unity $\{\varphi_i\}$ subordinate to this cover, such that E is trivial on each U_i. On each U_i, write g_i for the composite

$$p^{-1}(U_i) \cong U_i \times \mathbb{R} \xrightarrow{p_2} \mathbb{R}.$$

We can scale this by the function s_i defined as

$$p^{-1}(U_i) \xrightarrow{p} U_i \xrightarrow{\varphi_i} [0, 1].$$

Since the support of s_i is contained in $p^{-1}(U_i)$, we can extend the product $s_i \cdot g_i$ by zero outside of $p^{-1}(U_i)$ to get a continuous map $E \longrightarrow \mathbb{R}$. Repeating this for $i = 1, \dots, k$ gives a map

$$sg : E \longrightarrow \mathbb{R}^k.$$

Since each $s_i g_i$ is linear in each fiber, so is sg. Given $e \in E$, suppose that $\varphi_i(p(e)) \neq 0$ (this must be true for at least one i). This implies that $p(e) \in U_i$. Then $s_i g_i$ is a linear injection from $p^{-1}(p(e))$ to $\mathbb{R}$, and it follows that sg is a linear injection on each fiber.

It remains to show injectivity. Thus suppose given $f_0, f_1 : X \longrightarrow \mathbb{RP}^\infty$ and an isomorphism $f_0^*\gamma_1 \cong f_1^*\gamma_1$. Writing $E = f_0^*\gamma_1$, according to Proposition 1.46 the map f_0 corresponds to a map $g_0 : E \longrightarrow \mathbb{R}^\infty$ that is a linear injection on fibers. Similarly, using the isomorphism $E \cong f_1^*\gamma$, the map f_1 corresponds to $g_1 : E \longrightarrow \mathbb{R}^\infty$.

But now we claim that any two maps $g_0, g_1 : E \longrightarrow \mathbb{R}^\infty$ that are linear injections on fibers are necessarily homotopic-through maps which are themselves linear injections on fibers. We do this in three steps

(1) Note that the identity map of $\mathbb{R}^\infty$ is homotopic to the map $(x_1, x_2, x_3, x_4, \dots) \mapsto (x_1, 0, x_2, 0, x_3, \dots)$. The homotopy is given by the straight-line homotopy.

$$L_t(x_1, x_2, x_3, x_4, \dots) = (1-t)(x_1, x_2, x_3, x_4, \dots) + t(x_1, 0, x_2, 0, x_3, \dots)$$

Note that each L_t is a linear injection. Composing this homotopy with g_0 gives a homotopy $g_0 \simeq g_0^L$, where $g_0^L = L_1 \circ g_0$.

(2) Similarly, the identity of $\mathbb{R}^\infty$ is homotopic to the map $(x_1, x_2, x_3, x_4, \dots) \mapsto (0, x_1, 0, x_2, 0, x_3, \dots)$. The homotopy is given by a straight-line homotopy R_t, and then $g_1 \simeq g_1^R$, where $g_1^R = R_1 \circ g_1$.
(3) Finally, we can use the straight-line homotopy m_t between g_0^L and g_1^R. Notice that each m_t is a linear injection.

Now given our homotopy $H_t : g_0 \simeq g_1$, we define $h_t : f_0 \simeq f_1$ by $h_t(x) = H_t(p^{-1}(x))$. ∎

Example 1.48. We can use this to classify line bundles on a sphere. By Theorem 1.47, we have a bijection

$$\mathrm{Vect}^1(S^n) \cong [S^n, \mathbb{RP}^\infty].$$

Beware that this means unbased homotopy classes. But since $\mathbb{RP}^\infty$ is path-connected, the set of unbased homotopy classes can be identified with the set of conjugacy classes in $\pi_n(\mathbb{RP}^\infty)$. This is $\mathbb{Z}/2\mathbb{Z}$ if $n = 1$ and trivial if $n \geq 2$. We conclude that, on S^1, there is precisely one nontrivial line bundle (the Möbius band!), whereas on S^n with $n \geq 2$, every line bundle is trivial.

Mon, Sept. 24

The argument for Theorem 1.47 did not really depend on the fact that we were considering line bundles. More generally, we have

Theorem 1.49. *Let X be paracompact. Then*

$$[X, Gr_n(\mathbb{R}^\infty)] \xrightarrow{\cong} \mathrm{Vect}^n(X)$$
$$f \mapsto f^*\gamma_n$$

is a bijection.

This shows that the Grassmannian classifies rank n vector bundles. In other words, it is a "classifying space" for rank n vector bundles. This space is also often written $BGl_n(\mathbb{R})$.

1.7. **Reduction of structure group.** As we have discussed, an alternative way of describing a vector bundle is by specifying the transition function $g_{U,V} : U \cap V \longrightarrow Gl_n(\mathbb{R})$. If these transition functions happen to land in a subgroup, such as $O(n)$ or $SO(n)$, this tells us more information about the bundle.

Suppose that all transition functions of E land in $O(n)$. A matrix in $O(n)$ is a linear transformation that preserves an inner product. So if the transition function for E land in $O(n)$, we make make compatible choices of inner products on all of the fibers, leading to what is called a "Euclidean bundle".

Proposition 1.50. *Let E be a vector bundle over a paracompact base space B. Then E can be given a Euclidean structure.*

Proof. This can be done directly, starting by choosing inner products on each trivialization, and then combining the various choices via a partition of unity. Another choice is to use Theorem 1.49. According to this, $E \cong f^*\gamma_n$ for some map $f : B \longrightarrow \mathrm{Gr}_n(\mathbb{R}^\infty)$. But the canonical bundle γ_n possesses a Euclidean structure. Recall that $E(\gamma_n)$ is a subspace of $\mathrm{Gr}_n(\mathbb{R}^\infty) \times \mathbb{R}^\infty$. Thus the fibers already come as subspaces of $\mathbb{R}^\infty$. But we have the dot product on $\mathbb{R}^\infty$ (or any other inner product we prefer), and this restricts to give an inner product on any subspace. ∎

In the case where $E = TM$ for some manifold M, a Euclidean structure on TM is also called a **Riemannian metric** on M.

Proposition 1.51. *Any rank n vector bundle E over a paracompact B is isomorphic to a bundle E' over B for which all transition functions lie in $O(n)$.*

Proof. We suppose that E is equipped with a Euclidean structure. We may then use the Gram-Schmidt process to replace the trivializations of E with orthonormal trivializations (i.e. ones respecting the inner product). Since the transition functions are given as compositions of trivializations (and their inverses), it follows that they will take values in $O(n)$. ∎

We say that "the structure group of E' has been **reduced** to the subgroup $O(n)$". So the space $\mathrm{Gr}_n(\mathbb{R}^\infty)$ also classifies vector bundles with group $O(n)$, and we also write $BO(n) \simeq \mathrm{Gr}_n(\mathbb{R}^\infty)$. Topologists prefer $O(n)$ to $Gl_n(\mathbb{R})$ since the orthogonal group is compact.

Note that for a line bundle, this means that all transition functions can be taken to take values in $O(1) = \{\pm 1\}$.

Of more interest is the question of reducing the structure group even further to $SO(n)$.

Definition 1.52. We say that a bundle is **orientable** if its structure group can be reduced to $SO(n)$.

Recall that an **orientation** of a vector space $\mathbf{V}$ is an equivalence class of bases of $\mathbf{V}$, where one basis is equivalent to another if the change-of-basis matrix has positive determinant.

Definition 1.53. An **orientation** of a vector bundle $E \xrightarrow{p} B$ is a choice of orientation on each fiber, such that the local trivializations are orientation-preserving.

The following result justifies the terminology.

Proposition 1.54. *A vector bundle $E \xrightarrow{p} B$ is orientable if and only if an orientation exists.*

Wed, Sept. 26

Proof. We suppose that the structure group of E has already been reduced to $O(n)$. Thus if all local trivializations are orientation-preserving, it follows that the transition maps will have positive determinant and therefore lie in $SO(n)$.

Conversely, if the transition functions take values in $SO(n)$, we may use the local trivializations to specify orientations in the fibers. More precisely, given $b \in B$, choose a U containing b on which we have a trivialization φ_U of E. Then we can use φ_U^{-1} to transport the standard orientation of $\mathbb{R}^n$ to an orientation of the fiber E_b. If V is another such neighborhood of b, by assumption the transition function $g_{U,V} = \varphi_V \circ \varphi_U^{-1}$ is orientation-preserving. Since φ_U^{-1} is orientation-preserving by definition of the orientation on E_b, it follows that φ_V also preserves the orientation. ∎

We will later (Corollary 4.18) see another characterization of orientable bundles in terms of characteristic classes.

Proposition 1.55. *A manifold M is orientable if and only if its tangent bundle TM is orientable.*

Proof. Local trivializations for TM can be obtained as the derivative (Jacobian matrix) of the coordinate charts of the manifold M. So the transition functions for TM are orientation-preserving if and only if the transition functions for the coordinate charts of M are orientation-preserving. ∎

Note that since $SO(1)$ is the trivial group, a (real) line bundle is orientable if and only if it is trivial.

1.8. Complex vector bundles.

There is also an analogous theory of complex vector bundles, in which each fiber is equipped with the structure of a vector space over $\mathbb{C}$. There, the transition functions take values in $Gl_n(\mathbb{C})$.

Theorem 1.56. *Let X be paracompact. Then*

$$[X, Gr_n(\mathbb{C}^\infty)] \xrightarrow{\cong} \mathrm{Vect}^n_{\mathbb{C}}(X)$$
$$f \mapsto f^*\gamma_n$$

is a bijection.

Analogously to Proposition 1.51, the structure group for a complex bundle (over a paracompact base) can be reduced to the unitary group $U(n)$. Thus we write

$$Gr_n(\mathbb{C}^\infty) \simeq BGl_n(\mathbb{C}) \simeq BU(n).$$

Of course, any complex vector space of dimension n can also be regarded as a real vector space of dimension $2n$, and the same is true at the level of vector bundles.

Proposition 1.57. *Let $E \xrightarrow{p} B$ be a rank n $\mathbb{C}$-vector bundle. Then, when considered as a rank $2n$ $\mathbb{R}$-vector bundle, it is automatically orientable.*

Proof. In terms of transition functions, this operation corresponds to the homomorphism

$$U(n) \longrightarrow O(2n)$$

which considers a complex number as a 2×2 real matrix. We claim that this homomorphism lands in the subgroup $SO(2n)$. Recall that $O(2n)$ is the disjoint union of $SO(2n)$ and the subset of matrices with determinant -1. So the claim will follow if we can show that $U(n)$ is path-connected. But any unitary matrix can be diagonalized, so that we get

$$A = SDS^{-1}$$

, where

$$D = \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix}.$$

If we define

$$D_t = \begin{pmatrix} e^{it\theta_1} & & \\ & \ddots & \\ & & e^{it\theta_n} \end{pmatrix},$$

then $A_t = SD_tS^{-1}$ defines a path from the identity matrix to A. ∎

Thus any complex manifold is automatically orientable when considered as a real manifold.

2. $\mathbb{K}$-THEORY

2.1. **Definition.** For any space X, the set $\mathrm{Vect}_{\mathbb{R}}(X)$ of isomorphism classes of (real) vector bundles has two operations that almost make it into a ring: the (Whitney) sum and tensor product. It is only a *semiring* because there are no additive inverses. It is much more convenient to work with rings rather than semirings, so we produce a group in a minimal way.

Definition 2.1. Let M be an abelian semigroup. Then the **group completion** $\widetilde{M}$ is defined as

$$\widetilde{M} = \mathbb{Z}\{M\}/\sim,$$

where $[m_1] + [m_2] \sim [m_1 + m_2]$.

Then $\widetilde{M}$ is an abelian group, and it is the universal example of an abelian group with a homomorphism from M.

Definition 2.2. Given a space X, define

$$KO(X) = \widetilde{\mathrm{Vect}_{\mathbb{R}}(X)}, \qquad \text{and} \qquad KU(X) = \widetilde{\mathrm{Vect}_{\mathbb{C}}(X)}.$$

Mon, Oct. 1

For today, let X be a compact Hausdorff space.

We defined $KO(X)$ and $KU(X)$ to be the group completions of the sets of isomorphism classes of vector bundles on X. We have not specified the rank of the bundle, and in fact in this case we want to allow the rank of the bundle to vary over the various components of X. Notice that the pullback construction makes $KO(-)$ and $KU(-)$ into contravariant functors.

Proposition 2.3. *Let X be compact Hausdorff. Then any vector bundle E on X has a complement, meaning another bundle E' such that $E \oplus E'$ is trivial.*

Proof. Suppose that X is connected. The bundle E is classified by a map $f : X \longrightarrow BU(n) = Gr_n(\mathbb{R}^\infty)$ for some n. By compactness of X, the image $f(X)$ must land in some compact subspace $Gr_n(\mathbb{R}^k)$. But the total space for the canonical bundle over $Gr_n(\mathbb{R}^k)$ is defined as a subspace of $Gr_n(\mathbb{R}^k) \times \mathbb{R}^k$. In other words, the canonical bundle is defined to be a subbundle of $\underline{k}_{Gr_n(\mathbb{R}^k)}$. It follows that the pullback E is a subbundle of the trivial bundle $\underline{k}_X$. By Proposition 1.50, we may assume a Euclidean structure on E. We then define E' to be the orthogonal complement of E in $\underline{k}_X$. ■

Corollary 2.4. *Let X be compact Hausdorff. Then any element of $KO(X)$ may be written in the form $E - k$, where E is a bundle and $k \geq 0$ is a trivial bundle.*

Proof. Consider $E_1 - E_2$ in $KO(X)$. By Proposition 2.3, we know there is an F such that $E_2 \oplus F \cong k$. Then

$$E_1 - E_2 = E_1 + F - E_2 - F = E_1 + F - k,$$

so we take $E = E_1 + F$. ■

Wed, Oct. 3

Recall that we use $BO(n)$ to denote the Grassmannian $\mathrm{Gr}_n(\mathbb{R}^\infty)$ and that we write γ_n for the canonical n-plance bundle over $BO(n)$. Then the $(n+1)$-plane bundle $\gamma_n \oplus \underline{1}$ is classified by some map $\iota_n : BO(n) \longrightarrow BO(n+1)$. We define

$$BO := \operatorname*{colim}_n BO(n).$$

We similarly have

$$BU := \operatorname*{colim}_n BU(n).$$

Definition 2.5. We say that two bundles E and E' over X are *stably equivalent*, or *stably isomorphic*, if

$$E \oplus \underline{n} \cong E' \oplus \underline{k}$$

for some n and k. We will write $\mathrm{StVect}_{\mathbb{R}}(X)$ for the set of stable equivalence classes of (real) bundles on X.

Proposition 2.6. *Let X be compact Hausdorff and connected. Then*

$$\mathrm{StVect}_{\mathbb{R}}(X) \cong [X, BO] \qquad \textit{and} \qquad \mathrm{StVect}_{\mathbb{C}}(X) \cong [X, BU].$$

Proof. We prove the case for real bundles. The argument for complex bundles is the same.

Since BO is defined as the colimit of the $BO(n)$'s and X is compact, we get

$$[X, BO] = [X, \operatorname*{colim}_n BO(n)] \cong \operatorname*{colim}_n [X, BO(n)] \cong \operatorname*{colim}_n \mathrm{Vect}^n_{\mathbb{R}}(X),$$

where the second bijection is Theorem 1.49. The induced map

$$\mathrm{Vect}^n_{\mathbb{R}}(X) \longrightarrow \mathrm{Vect}^{n+1}_{\mathbb{R}}(X)$$

is $E \mapsto E \oplus \underline{1}$. But, since X is connected any bundle must appear in some $\mathrm{Vect}^n_{\mathbb{R}}(X)$, and the colimit of the $\mathrm{Vect}^n_{\mathbb{R}}(X)$ under these maps is $\mathrm{StVect}_{\mathbb{R}}(X)$. ∎

Proposition 2.7. *Let X be compact Hausdorff. Then*

$$KO(X) \cong [X, BO \times \mathbb{Z}] \qquad \textit{and} \qquad KU(X) \cong [X, BU \times \mathbb{Z}].$$

Proof. We give the proof in the KO case. Since both sides decompose as a product over the components of X, we may assume without loss of generality that X is connected. Note then that

$$[X, BO \times \mathbb{Z}] \cong [X, BO] \times [X, \mathbb{Z}] \cong [X, BO] \times \mathbb{Z}.$$

We define

$$\alpha : KO(X) \longrightarrow [X, BO \times \mathbb{Z}] \cong [X, BO] \times \mathbb{Z}$$

by

$$\alpha(E - \underline{n}) = (f_E, \dim(E) - n).$$

We must first show that α is well-defined. Thus suppose that $E - \underline{n} = E' - \underline{n}'$ in $KO(X)$. This means that

$$E \oplus \underline{n}' \oplus F \cong E' \oplus \underline{n} \oplus F \tag{2.8}$$

in $\mathrm{Vect}(X)$. In particular, $\dim(E) + n' = \dim(E') + n$, so that $\dim(E) - n = \dim(E') - n'$. It follows that the $\mathbb{Z}$ component of α is well-defined.

By adding the complement of F to both sides of (2.8), we see that E is stably isomorphic to E'. According to Proposition 2.6, this means that $[f_E] = [f_{E'}]$ as maps to BO.

Now we may define an inverse β to α by $\beta(f_E, m) = E - \underline{\dim(E)} + \underline{m}$. To see that this is well-defined, suppose that $[f_E] = [f_{E'}]$, so that E is stably equivalent to E' and $E \oplus \underline{k} \cong E' \oplus \underline{k}'$ for some k and k'. Then

$$E - \dim(E) = E \oplus \underline{k} - \dim(E \oplus \underline{k}) = E' \oplus \underline{k}' - \dim(E' \oplus \underline{k}') = E' - \dim(E'),$$

This shows that β is well-defined, and it is straight-forward to show it is inverse to α. ∎

It also follows that for the associated reduced theories, we have for any based space X isomorphisms

$$\widetilde{KO}(X) \cong [X, BO \times \mathbb{Z}]_* \qquad \text{and} \qquad \widetilde{KU}(X) \cong [X, BU \times \mathbb{Z}]_*.$$

Mon, Oct. 8

2.2. **Bott Periodicity.** So far, we have discussed the groups (rings) $KO(X)$ and $KU(X)$. Our next task will be to extend these to full-fledged generalized cohomology theories.

Proposition 2.9. *Let H be the canonical line bundle on $S^2 \cong \mathbb{CP}^1$. Then*

$$H^2 \oplus \underline{1} \cong H \oplus H.$$

Proof. We can cover $S^2 \cong \mathbb{CP}^1$ by its standard cover $U_0 = \mathbb{CP}^1 - \{\infty\}$ and $U_1 = \mathbb{CP}^1 - \{0\}$. Then the transition function $g_{01} : U_0 \cap U_1 \cong \mathbb{C}^\times \longrightarrow Gl_1(\mathbb{C})$ is $z \mapsto z$. It follows that the transition functions for $H^2 \oplus \underline{1}$ and $2H$ are, respectively,

$$G_1 = \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad G_2 = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}.$$

Write

$$A = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix},$$

so that $G_1 = A^2$ and $G_2 = AB$. Now $Gl_2(\mathbb{C})$ (more generally $Gl_n(\mathbb{C})$) is path-connected. To see this use row operations to write $M \in Gl_2(\mathbb{C})$ as a product $M = ED$ of elementary matrices and a diagonal matrix. Each elementary matrix $E_{ij}(a)$ can be deformed to the identity by $E_{ij}(ta)$. And the diagonal matrices live in $(\mathbb{C}^\times)^n$, which is path-connected. Let α_t be a path in $U(2)$ from the identity matrix to the permutation matrix $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $B = PAP$ and $A\alpha_t A\alpha_t$ gives a homotopy from G_1 to G_2. By Theorem 1.44, it follows that $H^2 \oplus \underline{1} \cong H \oplus H$. ■

Notice that this means that we can define a ring homomorphism

$$\mathbb{Z}[H]/(1-H)^2 \longrightarrow KU(S^2).$$

We also have an "external" product

$$KU(X) \otimes KU(Y) \longrightarrow KU(X \times Y)$$

given by

$$E_X \otimes E_Y \mapsto (p_X^* E_X) \otimes (p_Y^* E_Y).$$

Theorem 2.10. *The ring homomorphism*

$$KU(X) \otimes \mathbb{Z}[H]/(1-H)^2 \longrightarrow KU(X) \otimes KU(S^2) \longrightarrow KU(X \times S^2)$$

is an isomorphism for any compact Hausdorff X.

Sketch. We will only give a brief sketch (but see [H2, Theorem 2.2]). The main idea is to describe vector bundles over $X \times S^2$ by describing them as pars of bundles over $X \times D^2$, together with gluing (called "clutching") information over the intersection. Then one works to show that clutching functions can be reduced to particularly nices ones ■

In particular, we conclude that

$$KU(S^2) \cong \mathbb{Z}[H]/(1-H)^2 \cong \mathbb{Z}\{\underline{1}\} \oplus \mathbb{Z}\{H\}.$$

This also has a convenient restatement for the corresponding reduced groups. Recall that if X is based, then we define

$$\widetilde{KU}(X) = \ker(KU(X) \longrightarrow KU(*) \cong \mathbb{Z}).$$

Now Theorem 2.10 implies that $\widetilde{KU}(S^2) \cong \mathbb{Z}\{1-H\}$. Moreover, Theorem 2.10 can be used to show the following

Proposition 2.11. *Let X be a based compact Hausdorff space. Then*

$$\widetilde{KU}(X) \cong \widetilde{KU}(X) \otimes \widetilde{KU}(S^2) \longrightarrow \widetilde{KU}(X \wedge S^2)$$

is an isomorphism.

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It follows that

$$[X, BU \times \mathbb{Z}]_* \cong [\Sigma^2 X, BU \times \mathbb{Z}]_* = [X \wedge S^2, BU \times \mathbb{Z}]_* \cong [X, \Omega^2(BU \times \mathbb{Z})]_*$$

This is what is usually referred to as the Bott periodicity theorem (which we are not proving):

Theorem 2.12. *There is a weak equivalence*

$$\Omega^2(BU \times \mathbb{Z}) \simeq BU \times \mathbb{Z}.$$

The homotopy groups of $\Omega^2(BU \times \mathbb{Z})$ are a double shift of those of $BU \times \mathbb{Z}$.

Proposition 2.13.

$$\widetilde{KU}(S^n) \cong \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Proof. The even case reduces by Proposition 2.11 to $\widetilde{KU}(S^0) \cong KU(*) \cong \mathbb{Z}$. The odd case similarly reduces to $\widetilde{KU}(S^1)$. Now any complex vector bundle on S^1 can be described by clutching/transition functions over S^0. These are maps $S^0 \longrightarrow U(n)$. But $U(n)$ is path-connected, so the clutching function is null-homotopic. Again, this implies the bundle is trivial. ■

Corollary 2.14. *The homotopy groups of $BU \times \mathbb{Z}$ are*

$$\pi_n(BU \times \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

We also have the real analogues of the above discussion:

Proposition 2.15. *Let X be a based compact Hausdorff space. Then*

$$\widetilde{KO}(X) \cong \widetilde{KO}(X) \otimes \widetilde{KO}(S^8) \longrightarrow \widetilde{KO}(X \wedge S^8)$$

is an isomorphism.

The real Bott periodicity theorem (which we are not proving) is :

Theorem 2.16. *There is a weak equivalence*

$$\Omega^8(BO \times \mathbb{Z}) \simeq BO \times \mathbb{Z}.$$

The homotopy groups of $\Omega^8(BO \times \mathbb{Z})$ are an eightfold shift of those of $BO \times \mathbb{Z}$.

Proposition 2.17.

$$\widetilde{KO}(S^n) \cong \begin{cases} \mathbb{Z} & n \equiv 0 \pmod 4 \\ \mathbb{Z}/2\mathbb{Z} & n \equiv 1,2 \pmod 8 \\ 0 & \text{else} \end{cases}$$

The generators of the groups $\widetilde{KO}(S^1)$, $\widetilde{KO}(S^2)$, $\widetilde{KO}(S^4)$, and $\widetilde{KO}(S^8)$ are the canonical bundle on $\mathbb{RP}^1$, $\mathbb{CP}^1$, $\mathbb{HP}^1$, and $\mathbb{OP}^1$, respectively (with a suitable trivial bundle subtracted off to place it in the reduced K-group).

2.3. K**-theory as a cohomology theory.** As we said before, the groups $KU(X)$ and $KO(X)$ will be the zeroth groups of cohomology theories. Let's start with KU. A cohomology theory has a suspension isomorphism, so if we have a (reduced) cohomology theory $\widetilde{KU}^*(X)$, there should be an isomorphism

$$\widetilde{KU}^0(\Sigma^2 X) \cong \widetilde{KU}^{-2}(X).$$

Combining with Bott periodicity gives

$$\widetilde{KU}^0(X) \cong \widetilde{KU}^{-2}(X).$$

More generally, we take

$$\widetilde{KU}^{-n}(X) = \widetilde{KU}^0(\Sigma^n X)$$

for $n \geq 0$. For positive gradings, we take

$$\widetilde{KU}^n(X) = \widetilde{KU}^{n-2k}(X) = \widetilde{KU}(\Sigma^{2k-n} X)$$

for $2k > n$.

If we have a reduced theory, the corresponding unreduced theory is given by

$$KU^n(X) = \widetilde{KU}^n(X_+).$$

In order to get a full theory, we also need the relative groups. We define the relative groups

$$KU^n(X, A) = \widetilde{KU}^n(X \cup CA)$$

as the reduced groups applied to the cofiber. If Y is a commutative topological group up to homotopy (H-space), then $[X, Y]_*$ is an abelian group, and moreover the cofiber sequence $A \longrightarrow X \longrightarrow X \cup CA$ induces an exact sequence

$$[X \cup CA, Y]_* \longrightarrow [X, Y]_* \longrightarrow [A, Y]_*.$$

Since $\widetilde{KU}^*$ is given by maps into $BU \times \mathbb{Z}$ (or its loop space in odd degrees), we get the required long exact sequence for our cohomology theory.

This all works for $\widetilde{KO}$ as well. We define

$$\widetilde{KO}^{-n}(X) = \widetilde{KO}^0(\Sigma^n X)$$

for $n \geq 0$. For positive gradings, we take

$$\widetilde{KO}^n(X) = \widetilde{KO}^{n-8k}(X) = \widetilde{KO}(\Sigma^{8k-n} X)$$

for $8k > n$.

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2.4. **Real division algebras and Hopf invariant one.** We will use K-theory to answer the following question.

Question 2.1. *When does $\mathbb{R}^n$ admit the structure of a real division algebra?*

By a division algebra structure, we simply mean a multiplication

$$m : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

which is bilinear and has no zero divisors. We do not assume a unit element, but using that the vector space $\mathbb{R}^n$ is finite-dimensional, it is not difficult to produce a different division algebra structure with a strict two-sided unit element.

Of course, we know examples when $n = 1, 2, 4,$ and 8 (algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$). Frobenius showed in 1877 that if one asks for *associative* division algebras, then the only ones are $\mathbb{R}$, $\mathbb{C}$, and

$\mathbb{H}$. Hurwitz showed in 1900 that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are the only *normed* division algebras. But the more general division algebra question remained unsolved for several decades.

This algebraic question connects to topology as follows. Consider the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Then we can define a multiplication

$$\mu : S^{n-1} \times S^{n-1} \longrightarrow S^{n-1}$$

by $\mu(x, y) = \frac{m(x,y)}{\|m(x,y)\|}$. Note that the norm is nonzero since we are working in a division algebra. Since our division algebra has a unit element, so does μ (we may need to scale the unit element of our division algebra to find the unit element for μ). The resulting structure on S^{n-1}, namely a multiplication with a unit element, is known as an **H-space** structure. We will answer the following more general question:

Question 2.2. *When does S^{n-1} admit an H-space structure?*

A first, partial, answer is that n must be even (excluding the trivial case $n = 1$). We give the argument showing S^2 does NOT admit and H-space structure. A similar argument handles S^{2k} for $k > 1$. Consider the map $S^2 \times S^2 \xrightarrow{\mu} S^2$. This induces a ring homomorphism

$$KU^0(S^2) \xrightarrow{\mu^*} KU^0(S^2 \times S^2).$$

By Theorem 2.10, this has the form

$$\mathbb{Z}[x]/x^2 \longrightarrow \mathbb{Z}[\alpha, \beta](\alpha^2, \beta^2).$$

Let us consider the image of x under this homomorphism. We have $\mu^*(x) = i + j\alpha + k\beta + \ell\alpha\beta$ for some integers $i, j, k, \ell \in \mathbb{Z}$. If S^2 is an H-space, then the composition

$$S^2 \times \{e\} \hookrightarrow S^2 \times S^2 \xrightarrow{\mu} S^2$$

is the identity. The KU-restriction along $S^2 \times \{e\} \hookrightarrow S^2 \times S^2$ maps β to zero, and we conclude that $i = 0$ and $j = 1$. By using the inclusion $S^2 \times \{e\} \hookrightarrow S^2 \times S^2$, we similarly conclude that $k = 1$. But then

$$0 = x^2 \mapsto (\alpha + \beta + \ell\alpha\beta)^2 = 2\alpha\beta \neq 0$$

in $\mathbb{Z}[\alpha, \beta](\alpha^2, \beta^2)$. This is a contradiction.

Remark 2.18. Historically, Adém used the Steenrod operations in cohomology to show that n must further be a power of 2. We will not need this, since we will prove a stronger result.

We will relate the H-space question to the homotopy groups of spheres via the following construction.

Construction 2.19. Given a map $\mu : S^{n-1} \times S^{n-1} \longrightarrow S^{n-1}$, we define the **Hopf contruction** on μ to be the map

$$H(\mu) : S^{2n-1} \longrightarrow S^n$$

defined by the map of pushout diagrams

$$\begin{tikzcd}
& S^{n-1} \times S^{n-1} \arrow[dl, hook] \arrow[dr, hook] \arrow[rrr, bend left, "\mu"] & & & S^{n-1} \arrow[dl] \arrow[dr] & \\
D^n \times S^{n-1} \arrow[dr] \arrow[rrr, "1"] & & S^{n-1} \times D^n \arrow[dl] \arrow[rrr, "2"] & D^n \arrow[dr] & & D^n \arrow[dl] \\
& S^{2n-1} \arrow[rrr, dashed, bend right, "H(\mu)"] & & & S^n &
\end{tikzcd}$$

Here the map labeled 1 is the composition

$$C(S^{n-1}) \times S^{n-1} \longrightarrow C(S^{n-1} \times S^{n-1}) \xrightarrow{C(\mu)} C(S^{n-1})$$

where the first map is $(x, t, y) \mapsto (x, y, t)$ and similarly for $\boxed{2}$.

Example 2.20. Identify S^0 with the group $\{\pm 1\} = O(1)$, and consider the multipication $S^0 \times S^0 \xrightarrow{\mu} S^0$. The Hopf construction then produces a map $H(\mu) : S^1 \longrightarrow S^1$. The resulting CW structure on the source S^1 has four 0-cells, whereas the target S^1 has two 0-cells. The restriction of $H(\mu)$ to the 0-skeleton is precisely the map μ, and by inspecting what is forced on the 1-cells, it follows that $H(\mu)$ is the degree 2 map.

The next few examples are not quite as straightforward.

Example 2.21. When $n = 2$, the multiplication in $\mathbb{C}$ gives $S^1 \times S^1 \xrightarrow{\mu} S^1$. The resulting Hopf map is

$$\eta : S^3 \longrightarrow S^2.$$

Example 2.22. When $n = 4$, the multiplication in $\mathbb{H}$ gives $S^3 \times S^3 \xrightarrow{\mu} S^3$. The resulting Hopf map is

$$\nu : S^7 \longrightarrow S^4.$$

Example 2.23. When $n = 8$, the multiplication in $\mathbb{O}$ gives $S^7 \times S^7 \xrightarrow{\mu} S^7$. The resulting Hopf map is

$$\sigma : S^{15} \longrightarrow S^8.$$

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Last time, we produced from an H-space structure on S^{n-1} a map $S^{2n-1} \longrightarrow S^n$. We will attach an invariant to such a map, known as the Hopf invariant.

Thus let $\alpha : S^{2n-1} \longrightarrow S^n$ be a homotopy class. The cofiber $C(\alpha)$ has an n-cell and a $2n$-cell. If we denote by x_n a generator of $\mathrm{H}^n(C(\alpha); \mathbb{Z}) \cong \mathbb{Z}$ and by y_{2n} a generator of $\mathrm{H}^{2n}(C(\alpha); \mathbb{Z}) \cong \mathbb{Z}$, then we must have

$$x_n^2 = h(\alpha) \cdot y_{2n}$$

for some integer (only well-defined up to sign) $h(\alpha)$, known as the Hopf invariant of α.

Proposition 2.24. *If $\mu : S^{n-1} \times S^{n-1} \longrightarrow S^{n-1}$ is an H-space, then the Hopf invariant of the Hopf construction, $h(H(\mu))$, is 1 (or -1).*

Proof. We follow the argument of [H2, Lemma 2.18]. We will write X for the cofiber of $h(H(\mu)) : S^{2n-1} \longrightarrow S^n$. Thus we have $S^n \subseteq X$. Write D^n_ℓ and D^n_r for the two hemispheres of S^n, which appeared in the diagram for Construction 2.19. We will write $\Phi : D^{2n} \cong D^n \times D^n \longrightarrow X$ for the characteristic map of the top cell of X. Consider the diagram

$$\begin{tikzcd}
\widetilde{\mathrm{H}}^n(X) \otimes \widetilde{\mathrm{H}}^n(X) \arrow[rr] & & \widetilde{\mathrm{H}}^{2n}(X) \\
\widetilde{\mathrm{H}}^n(X/D^n_r) \otimes \widetilde{\mathrm{H}}^n(X/D^n_\ell) \arrow[u, "\cong"] \arrow[rr] \arrow[d, "\cong"', "\Phi^*\otimes\Phi^*"] & & \widetilde{\mathrm{H}}^{2n}(X/S^n)] \arrow[u, "\cong"] \arrow[d, "\cong"', "\Phi^*"] \\
\widetilde{\mathrm{H}}^n((D^n \times D^n)/(\partial D^n \times D^n)) \otimes \widetilde{\mathrm{H}}^n((D^n \times D^n)/(D^n \times \partial D^n)) \arrow[d, "\cong"', "1"] & & \widetilde{\mathrm{H}}^{2n}((D^n \times D^n)/\partial(D^n \times D^n)) \\
\widetilde{\mathrm{H}}^n((D^n \times e)/(\partial D^n \times e)) \otimes \widetilde{\mathrm{H}}^n((e \times D^n)/(e \times \partial D^n)) \arrow[urr, "\cong"'] & &
\end{tikzcd}$$

Since S^{n-1} is assumed to be an H-space, the characteristic map Φ carries $D^n \times \{e\}$ homeomorphically onto D^n_ℓ and similarly $\{e\} \times D^n$ to D^n_r. It follows that the composition of $\Phi^* \otimes \Phi^*$ with 1 gives an isomorphism. We conclude that the top horizontal map is an isomorphism since all other outer maps are. ■

If the cells are in even degrees (in other words, if n is even) then the Hopf invariant can alternatively be defined using K-theory. Consider the cofiber sequence

$$S^n \hookrightarrow C(\alpha) \longrightarrow C(\alpha)/S^n \cong S^{2n}.$$

This induces a short exact sequence

$$\widetilde{KU}^0(S^{2n}) \hookrightarrow \widetilde{KU}^0(C(\alpha)) \twoheadrightarrow \widetilde{KU}^0(S^n)$$

This takes the form

$$\mathbb{Z}\{y_{2n}\} \hookrightarrow \mathbb{Z}\{b_{2n}\} \oplus \mathbb{Z}\{a_n\} \twoheadrightarrow \mathbb{Z}\{x_n\},$$

where $y_{2n} \mapsto b_{2n}$ and $a_n \mapsto x_n$. But $x_n^2 = 0$. By exactness, it follows that a_n^2 is in the image of the first map, meaning that it is a multiple of b_{2n}. Thus

$$a_n^2 = h(\alpha) b_{2n}$$

for some integer $h(\alpha)$. This is the Hopf invariant of α.

We will use K-theory to prove

Theorem 2.25. *If $\alpha : S^{2n-1} \longrightarrow S^n$ has Hopf invariant one (or -1), then $n = 1, 2, 4,$ or 8.*

Corollary 2.26. *If R^n has a division algebra structure, then $n = 1, 2, 4,$ or 8.*

Proof. We have discussed how a division algebra structure on $\mathbb{R}^n$ induces an H-space structure μ on S^{n-1}. By Proposition 2.24, the Hopf invariant of the Hopf construction $H(\mu)$ is ± 1. Theorem 2.25 says this only occurs if n is 1, 2, 4, or 8. ■

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In order to prove Theorem 2.25, we will need to employ **Adams operations**.

Theorem 2.27. *There are ring homomorphisms $\psi^k : KU^0(X) \longrightarrow KU^0(X)$ for each $k \geq 0$, such that*

(1) *these are natural in X*
(2) *$\psi^k(L) = L^k$ if L is a line bundle*
(3) *$\psi^k \circ \psi^\ell = \psi^{k\ell}$*
(4) *$\psi^p(\alpha) \equiv \alpha^p \pmod p$ if p is prime.*

(5) *on* $\widetilde{KU}^0(S^{2n})$, *the operation* ψ^k *acts as multiplication by* k^n.

Proof of Theorem 2.25. Suppose that $\alpha : S^{2n-1} \longrightarrow S^n$ has Hopf invariant one. If $n > 1$, then n must be even (by graded-commutativity of cohomology). Thus we assume $n > 1$ and write $n = 2d$. Then, in $\widetilde{KU}(C(\alpha))$, we have

$$\psi^2(a_n) = 2^d a_n + j b_{2n}$$

for some j. By property (4) and the Hopf invariant one assumption, the integer j must be odd. Similarly

$$\psi^3(a_n) = 3^d a_n + \ell b_{2n}.$$

On the other hand, $\psi^2(b_{2n}) = 2^{2d} b_{2n}$ and $\psi^3(b_{2n}) = 3^{2d} b_{2n}$. Now

$$\psi^6(a_n) = \psi^3(\psi^2(a_n)) = \psi^3(2^d a_n + j b_{2n}) = 6^d a_n + (2^d \cdot \ell) b_{2n} + (j \cdot 3^{2d}) b_{2n}$$

and

$$\psi^6(a_n) = \psi^2(\psi^3(a_n)) = \psi^2(3^d a_n + \ell b_{2n}) = 6^d a_n + (3^d \cdot j) b_{2n} + (\ell \cdot 2^{2d}) b_{2n}.$$

By comparing the coefficients of b, we learn that $2^d \ell + 3^{2n} j = 2^{2d} \ell + 3^n j$. In other words,

$$2^d(2^d - 1)\ell = 3^d(3^d - 1)j.$$

Since j is odd, this forces 2^d to divide $3^d - 1$. But this is only true in the cases $d = 1, 2$, or 4. ∎

It remains to prove Theorem 2.27. This relies on

Proposition 2.28 (Splitting Principle). *Given a vector bundle* $E \longrightarrow X$ *with* X *compact Hausdorff, there is a compact Hausdorff space* $F(E)$ *and a map* $f : F(E) \longrightarrow X$ *such that (1)* $f^*(E)$ *splits as a sum of line bundles over* $F(E)$ *and (2) the map* $f^* : KU^*(X) \longrightarrow KU^*(F(E))$ *is injective.*

Sketch. Given a bundle E, we define a space $\mathbb{P}(E)$, the associated projective bundle, as follows. Start with the complement $E - s_0(X)$, where $s_0 : X \longrightarrow E$ is the zero section. We then define

$$\mathbb{P}(E) = (E - s_0(X))/Gl_1(\mathbb{R}),$$

where $Gl_1(\mathbb{R})$ is acting on each (punctured) fiber in E. The space $\mathbb{P}(E)$ is the space of lines in E. The map $E \longrightarrow X$ then induces $g : \mathbb{P}(E) \longrightarrow X$. There is a canonical line bundle γ on $\mathbb{P}(E)$ whose fibers are the lines in E. This line bundle γ admits an injective bundle map $\gamma \hookrightarrow g^*(E)$ given by including each line into E. By Proposition 1.50, this means that $g^*(E)$ splits as $g^*(E) \cong \gamma \oplus E'$ for some bundle E'. Now use induction to further split E'. ∎

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The Splitting Principle implies that the homomorphisms ψ^k of Theorem 2.27 are determined by their behavior on line bundles, which is described in item 2.

Since $\psi^k(-)$ is required to be additive, this will mean that if a bundle E splits as a sum of line bundles $E \cong L_1 \oplus \cdots \oplus L_n$, then we want to have

$$\psi^k(E) \cong L_1^k \oplus \cdots \oplus L_n^k.$$

So the goal is to write down a formula for $\psi^k(E)$ which will agree with the above when E does happen to split (which it will after pulling back to some appropriate space).

One observation is that if we consider the polynomial $x_1^k + \cdots + x_n^k$, this is symmetric in the variables x_i. A classical result (often attributed to Newton) states that any symmetric polynomial can be expressed as a polynomial in the *elementary symmetric functions*, which if n variables are

$$\sigma_1 = x_1 + \cdots + x_n, \qquad \sigma_2 = \sum_{i<j} x_i x_j,$$

$$\sigma_3 = \sum_{i<j<\ell} x_i x_j x_\ell, \qquad \dots \qquad \sigma_n = x_1 \cdots x_n.$$

For example,

$$x_1 + \cdots + x_n = \sigma_1,$$
$$x_1^2 + \cdots + x_n^2 = \sigma_1^2 - 2\sigma_2,$$
$$x_1^3 + \cdots + x_n^2 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3.$$

In general, we write s_k for the appropriate polynomial, so that

$$x_1^k + \cdots + x_n^k = s_k(\sigma_1, \dots, \sigma_n).$$

Going back to bundles, E will play the role of σ_1, but we now want a construction on E that will play the role of the higher σ_i's.

The appropriate construction will be the **exterior power**.

Definition 2.29. The kth **exterior power** of a vector space V is $\Lambda^k(V) = V^{\otimes k}/\sim$, where

$$\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_k \sim \operatorname{sgn}(\tau)\mathbf{v}_{\tau(1)} \otimes \mathbf{v}_{\tau(k)}$$

for any permutation τ.

Thus in $\Lambda^3(V)$, we have

$$\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3 \sim -\mathbf{v}_2 \otimes \mathbf{v}_1 \otimes \mathbf{v}_3 \sim \mathbf{v}_2 \otimes \mathbf{v}_3 \otimes \mathbf{v}_1.$$

Notice that under this relation, we have $\mathbf{v} \otimes \mathbf{v} \sim -\mathbf{v} \otimes \mathbf{v}$, forcing $\mathbf{v} \otimes \mathbf{v} = 0$ for any vector $\mathbf{v}$. Thus if V has basis $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$, then $\Lambda^k(V)$ has basis consisting of k-tuples of distinct elements of the basis for V. It follows that

$$\dim \Lambda^k(V) = \binom{n}{k},$$

where $n = \dim V$.

We use Construction 1.38 to extend the exterior product construction to bundles. Now the point is that if $E \cong L_1 \oplus \cdots \oplus L_n$, then

$$\Lambda^k(E) \cong \sigma_k(L_1, \dots, L_n).$$

It follows from the above that we should define

$$\psi^k(E) = s_k(\Lambda^1(E), \dots, \Lambda^n(E)),$$

where $n = \dim(E)$.

Proof of Theorem 2.27. We have defined $\psi^k : \mathbf{Vect}_{\mathbb{R}}(X) \longrightarrow \mathbf{Vect}_{\mathbb{R}}(X)$. It follows from the formula defining ψ^k that it is natural. In order to extend ψ^k to a functor on $KU^0(X)$, it suffices to show that it is additive. But the splitting principle tells us it is enough to check this after splitting the bundles, which does hold. Multiplicativity is similarly checked.

Property (2) follows from the definition, since the higher exterior powers vanish for a line bundle. Property (3) follows from (2) using the splitting principle and additivity. Property (4) follows from the splitting principle and the congruence

$$(x_1 + \cdots + x_n)^p \equiv x_1^p + \cdots + x_n^p \pmod{p}.$$

Wed, Oct. 24 Finally, for property (5), recall that a generator for $\widetilde{KU}^0(S^2)$ is $\gamma - 1$, where γ is the canonical line bundle. We wrote H for this class and showed the relation $H^2 = 2H - 1$. Then

$$\psi^k(H-1) = H^k - 1 = k \cdot H - (k-1) - 1 = k \cdot H - k = k(H-1),$$

where the second equality comes from Worksheet 7. ■

Here is another corollary of the Hopf invariant one theorem (you will work through a proof on Friday's worksheet).

Corollary 2.30. *The sphere S^n is parallelizable only when $n = 0, 1, 3,$ or 7.*

2.5. **Vector fields on spheres.** To say that the sphere S^n is parallelizable means that we can write down n linearly independent sections of the tangent bundle. Recall that a section of the tangent bundle is known as a vector field. We can ask how this generalizes.

Question 2.3. *What is the largest number of linearly independent vector fields on S^n, for a given n?*

Generalizing the Hairy Ball theorem, we have:

Proposition 2.31. *There are no nonzero vector fields on S^n if n is even.*

Proof. Suppose that $s : S^{2k} \longrightarrow TS^{2k}$ is a nowhere vanishing section. We can then normalize s so that all of its values are unit tangent vectors, producing a map $\nu : S^{2k} \longrightarrow S^{2k}$ such that $\nu(x) \perp x$ for all $x \in S^{2k}$. But then the formula

$$h_t(x) = x\cos(\pi t) + \nu(x)\sin(\pi t)$$

defines a homotopy between the identity and the antipodal map. But if n is even, then the anitpodal map, being a composite of $n+1$ reflections, is of degree -1. This is a contradiction. ∎

On the other hand, for odd-dimensional sphere, the formula

$$s(x_1, y_1, \ldots, x_n, y_n) = (y_1, -x_1, y_2, -x_2, \ldots, y_n, -x_n)$$

defines a nowhere vanishing vector field.

To give the general answer, we introduce the **Radon-Hurwitz** numbers. Given a natural number n, write it as

$$n = \text{odd} \cdot 16^d \cdot 2^j$$

for $0 \le j \le 3$. The Radon-Hurwitz number of n is defined to be

$$\rho(n) = 2^j + 8d.$$

The first few values are given in the table below

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\rho(n)$	1	2	1	4	1	2	1	8	1	2	1	4	1	2	1	8

Then the answer is given by

Theorem 2.32. *There are $\rho(n) - 1$ linearly inependent vector fields on S^{n-1}, and no more.*

In particular, this agrees with the previous statement that S^0, S^1, S^3, and S^7 are the only parallelizable spheres.

The existence part of the theorem was known first and comes from algebraic constructions. The fact that there can be no more is due to Adams in 1961.

Define a_k by

k	0	1	2	3	4	5	6	7
a_k	1	2	4	4	8	8	8	8

and $a_{k+8} = 16 \cdot a_k$.

Lemma 2.1. *a_k divides n if and only if $k \le \rho(n) - 1$.*

Then one shows that if there exist k linearly independent vector fields over S^{n-1}, then $n([H] - 1) = 0$ in $\widetilde{KO}^0(\mathbb{RP}^k)$. But Adams calculated these groups to be

Theorem 2.33 (Adams). *$\widetilde{KO}^0(\mathbb{RP}^k) \cong \mathbb{Z}/a_k$. The class $[H] - 1$ is a generator for this group.*

It now follows that n must be a multiple of a_k. But by Lemma 2.1, this shows that $k \le \rho(n) - 1$, establishing $\rho(n) - 1$ as an upper bound.

Mon, Oct. 29

3. Equivariant bundles & $\mathbb{K}$-theory

Let G be a topological group. In many cases, G is given the discrete topology (especially when G is finite). Recall that a G-space means a space X equipped with an action of G, meaning a continuous map $G \times X \longrightarrow X$ which is unital and associative. Recall that if X and Y are G-spaces, then a map $f : X \longrightarrow Y$ is called G-equivariant if $f(g \cdot x) = g \cdot f(x)$. Given a G-space X, we have a quotient map $q : X \longrightarrow X/G$.

Example 3.1. When $G = C_2 = \{\pm 1\}$, we have the antipodal action on S^n, where the group element -1 acts as multiplication by -1 in each coordinate. The quotient is $S^n/C_2 = \mathbb{RP}^n$.

Example 3.2. When $G = S^1 = U(1)$, we have the action as multiplication by complex numbers in each coordinate on $S^{2n+1} \subset \mathbb{C}^{n+1}$. The quotient is $S^{2n+1}/S^1 = \mathbb{CP}^n$.

For the rest of this section, we take G to be finite. Suppose X is a G-space and $E \longrightarrow X/G$ is a vector bundle. Then $q^*(E)$ will be a vector bundle over X. Moreover, G will act on $q^*(E)$ by the formula $g \cdot (x, \mathbf{v}) = (g \cdot x, \mathbf{v})$. The bundle map $q^*(E) \longrightarrow X$ is then G-equivariant. For each $g \in G$, action by g is a homeomorphism $a_g : q^*(E) \longrightarrow q^*(E)$. This restricts to a map $q^*(E)_x \cong q^*(E)_{gx}$ that is a linear isomorphism. We generalize this example to define a G-bundle.

Definition 3.3. A G**-vector bundle** is a G-space E equipped with a G-map $p : E \longrightarrow X$ which is a vector bundle when we forget about the G-action and such that each homeomorphism $E_x \cong E_{gx}$ is a linear isomorphism.

We have shown above that

Proposition 3.4. *If X is a G-space and $E \longrightarrow X/G$ is a vector bundle, then $q^*(E) \longrightarrow X$ is a G-vector bundle.*

But not every G-vector bundle arises in this way, since if $x \in X$ is fixed by g, then the G-action restricts to a G-action on the fiber $q^*(E)_x$, which in this case will be trivial. But that need not be true in a general G-vector bundle.

Example 3.5. A G-vector bundle over a point is a vector space equipped with a linear action of G. This is what is called a (real) representation of G.

Example 3.6. If X is a G-space and V is a G-representation, then the product $X \times V$ becomes a (trivial) G-vector bundle.

Proposition 3.7. *Suppose that G acts* **freely** *on X. Then*

$$q^* : \mathrm{Vect}_{\mathbb{R}}(X/G) \longrightarrow G\mathrm{Vect}_{\mathbb{R}}(X)$$

is a bijection.

The inverse is given by $(E \longrightarrow X) \mapsto (E/G \longrightarrow X/G)$. The point is that since G acts freely on X (and therefore also on E), then $E/G \longrightarrow X/G$ is locally homeomorphic to $E \longrightarrow X$, which makes it possible to obtain a local trivialization for E/G.

Wed, Oct. 31

We also have a classification theorem for G-bundles. Let $\mathbb{R}[G]$ be the "regular" representation of G. This has one basis element for each element of G, and G permutes the basis elements. Then for each k and n, the set $Gr_n(\mathbb{R}[G]^k)$ of n-dimensional subspaces of $\mathbb{R}[G]^k$ has a natural G-action by $g \cdot V = gV = \{g\mathbf{v} \mid \mathbf{v} \in V\}$.

Theorem 3.8. *Let X be a paracompact G-space. Then*

$$[X, Gr_n(\mathbb{R}[G]^\infty)]^G \xrightarrow{\cong} G\mathrm{Vect}^n_{\mathbb{R}}(X)$$
$$f \mapsto f^*\gamma_n$$

is a bijection.

The notation $[-,-]^G$ means G-equivariant homotopy classes.

Example 3.9. When $G = C_2$, then $\mathbb{R}[C_2] = \mathbb{C}$, equipped with the complex conjugation action. It follows that $Gr_n(\mathbb{R}[G]^\infty) = Gr_n(\mathbb{C}^\infty)$, the space of real n-dimensional subspaces of $\mathbb{C}^\infty$, with its complex conjugation action.

We also have equivariant versions of K-theory given by group-completing GVect.

Definition 3.10. Given a G-space X, define

$$KO^0_G(X) = \widetilde{G\mathrm{Vect}_{\mathbb{R}}}(X), \qquad \text{and} \qquad KU^0_G(X) = \widetilde{G\mathrm{Vect}_{\mathbb{C}}}(X).$$

Proposition 3.7 has the following consequence:

Proposition 3.11. *If X is a free G-space, then*

$$KO_G(X) \cong KO(X/G) \qquad \textit{and} \qquad KU_G(X) \cong KU(X/G).$$

At the other end of the spectrum, we can describe equivariant K-theory in the case of a trivial action. For this, we will need the representation ring of a group.

Definition 3.12. Let G be a finite group. The sets $\mathrm{Rep}_{\mathbb{R}}(G)$ and $\mathrm{Rep}_{\mathbb{C}}(G)$ of (isomorphism classes of) real, resp. complex, representations of G is a commutative semiring under direct sum and tensor product, and we define

$$RO(G) = \widetilde{\mathrm{Rep}_{\mathbb{R}}}(G), \qquad RU(G) = \widetilde{\mathrm{Rep}_{\mathbb{C}}}(G).$$

From the definitions, we see that

$$RO(G) = KO^0_G(pt), \qquad RU(G) = KU^0_G(pt).$$

Generalizing these isomorphisms, we have

Proposition 3.13. *If X is a trivial G-space, then*

$$KO_G(X) \cong KO(X) \otimes RO(G) \qquad \textit{and} \qquad KU_G(X) \cong KU(X) \otimes RU(G).$$

To give the equivariant version of complex Bott periodicity, for a complex representation V, define $\lambda(V) \in RU(G) \cong KU^0_G(pt)$ by

$$\lambda(V) = \sum_{i=0}^{\dim V} (-1)^i \Lambda^i(V).$$

Also, if V is a representation, we denote by S^V the one-point compactification. This is then a based, compact G-space. It has (at least) two fixed points, namely 0 and ∞. More generally, a basepoint of a G-space is always required to be G-fixed.

Theorem 3.14 (Equivariant Bott periodicity)**.** *Given a compact based G-space X and a complex representation V, there is a class $b_V \in \widetilde{KU}_G(S^V)$ which restricts along $S^0 \hookrightarrow S^V$ to $\lambda(V)$, and multiplication by b_V is an isomorphism*

$$\widetilde{KU}^0_G(X) \cong \widetilde{KU}^0_G(S^V \wedge X).$$

In particular, this holds when $V = \mathbb{C}$ with trivial G-action, giving that

$$\widetilde{KU}^0_G(X) \cong \widetilde{KU}^0_G(S^2 \wedge X).$$

There is also a periodicity theorem for KO_G. In that case, the representation V is required to have dimension a multiple of 8, and it is further required to be a "spin" representation.

Mon, Nov. 5

4. Characteristic classes

4.1. **Stiefel-Whitney classes.** Recall that line bundles are classified by $BO(1) = \mathbb{RP}^\infty$. But $\mathrm{H}^*(\mathbb{RP}^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[x]$.

Proposition 4.1. *The function* $\mathrm{Vect}^1_R(X) \longrightarrow \mathrm{H}^1(X; \mathbb{F}_2)$*,* $L \mapsto f_L^*(x)$*, is a bijection.*

Proof. We already know (Theorem 1.47) that $\mathrm{Vect}^1_R(X) \cong [X, \mathbb{RP}^\infty]$, so it remains to identify the set of maps with the cohomology group. Given a map f, we have a ring homomorphism $f^* : \mathrm{H}^*(\mathbb{RP}^\infty; \mathbb{F}_2) \longrightarrow \mathrm{H}^*(X; \mathbb{F}_2)$ which is completely determined by $f^*(x)$ since $\mathrm{H}^*(\mathbb{RP}^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[x_1]$ is polynomial on a class in degree 1.

On the other hand, suppose given $\alpha \in \mathrm{H}^1(X; \mathbb{F}_2)$. By the Universal Coefficients Theorem,

$$\mathrm{H}^1(X; \mathbb{F}_2) \cong \mathrm{Hom}(\pi_1(X), \mathbb{F}_2).$$

Thus we may suppose given a homomorphism $\alpha : \pi_1(X) \longrightarrow \mathbb{F}_2$. We wish to build a corresponding map $f : X \longrightarrow \mathbb{RP}^\infty$. We first define f on the 1-skeleton $\mathrm{sk}_1 X$ as follows. The 1-skeleton is a graph, and by collapsing a maximal tree, it is equivalent to a wedge of circles. Thus we define

$$f_1 : \mathrm{sk}_1 X \simeq \bigvee S^1 \xrightarrow{\phi} \mathbb{RP}^2,$$

where the retriction of ϕ to each circle is determined by α. Next, we wish to extend f_1 over the 2-skeleton. This requires knowing that the attaching map for each 2-cell is sent to zero in $\pi_1(\mathbb{RP}2) \cong \mathbb{F}_2$ by f_1. But this follows from the fact that f_1 was defined by $\alpha : \pi_1(X) \longrightarrow \mathbb{F}_2$.

Having defined f on the 2-skeleton, it remains to extend it to higher skeleta. But $\mathbb{RP}^\infty$ has no higher homotopy groups. It follows that the attaching maps for all higher cells are automatically annihilated by f, so that there are no "obstructions" to extending f_2 over the higher skeleta. ■

So we have seen that line bundles are completely controlled by a cohomology class.

Example 4.2. Since $\mathrm{H}^1(S^1; \mathbb{F}_2) = \mathbb{F}_2$, we see that there is exactly one nontrivial line bundle on S^1 (it's the Möbius band.)

Example 4.3. Since $\mathrm{H}^1(S^n; \mathbb{F}_2) = 0$ for $n \geq 2$, we see that there are no nontrivial line bundles on S^n if $n \geq 2$.

Example 4.4. Since $\mathrm{H}^1(\mathbb{RP}^n; \mathbb{F}_2) = \mathbb{F}_2$ for $n \geq 2$, we see that there is exactly one nontrivial line bundle on $\mathbb{RP}^n$ (it's the canonical line bundle.)

Example 4.5. Since $\mathrm{H}^1(T^2; \mathbb{F}_2) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$, there are three nontrivial line bundles on T^2. We have the pullback of the Mobius bundle from each coordinate. The tensor product of these two is the third bundle.

This cohomology class associated to a line bundle L is called the **first Stiefel-Whitney class** $w_1(L)$, and it is the first example of a characteristic class.

Wed, Nov. 7

Theorem 4.6. *There is a (unique) sequence $w_1, w_2, \ldots,$ of natural tranformations $\mathrm{Vect}^*_{\mathbb{R}}(X) \longrightarrow \mathrm{H}^*(X;\mathbb{F}_2)$ such that*

(1) $w_n(E_1 \oplus E_2) = \sum_{i+j=n} w_i(E_1) \cdot w_j(E_2)$
(2) $w_i(E) = 0$ *if* $i > \dim(E)$
(3) $w_1(\gamma_1) \neq 0$ *in* $\mathrm{H}^1(\mathbb{RP}^1;\mathbb{F}_2) \cong \mathbb{F}_2$.

In formula (1), we use the notation $w_0(E) = 1$. It is often more convenient to rewrite (1) in terms of the **total Steifel-Whitney class** $w(E) = \sum_{i\geq 0} w_i(E)$. Then (1) says $w(E_1 \oplus E_2) = w(E_1)w(E_2)$.

By (2), for a line bundle, the first Stiefel-Whitney class is the only nonzero one.

Proposition 4.7. *If $E = \underline{n}_B$, then $w_i(E) = 0$ for $i > 0$.*

Proof. A trivial bundle is pulled back from a bundle over a point. But a point has no higher cohomology, so we are done by naturality. ■

Proposition 4.8. $w_i(E \oplus \underline{n}_B) = w_i(E)$.

This follows from (1).

Example 4.9. Recall (Example 1.34) that the tangent bundle of the sphere is stably trivial. It follows that $w_i(TS^n) = 0$ for all $i > 0$.

Proposition 4.10. *Suppose E is a rank n bundle with a nowhere-vanishing section. Then $w_n(E) = 0$.*

Proof. We may assume E has a Euclidean structure. The nowhere-vanishing section gives an inclusion of a rank one trivial bundle, and the Euclidean structure allows us to split this as $E \cong \underline{1}_B \oplus E_2$, where E_2 has rank $n-1$. Then $w_n(E) = w_1(\underline{1}_B)w_{n-1}(E_2) = 0$. ■

More generally, by repeating the argument, we see that if E has k linearly independent sections, then

$$w_n(E) = w_{n-1}(E) = \cdots = w_{n-k+1}(E) = 0.$$

Thus the Stiefel-Whitney classes can be viewed as obstructions to the existence of sections.

Example 4.11. The total Steifel-Whitney class of γ_1^n on $\mathbb{RP}^n$ is $1+x$. The higher classes must vanish by (2), and the first class cannot vanish by (3), as γ_1^n restricts to γ_1^1.

Example 4.12. Consider now the tangent bundle $\tau_{\mathbb{RP}^n}$ on $\mathbb{RP}^n$. Note that under the correspondence $C_2\mathrm{Vect}_{\mathbb{R}}(S^n) \cong \mathrm{Vect}_{\mathbb{R}}(\mathbb{RP}^n)$, we have $\tau_{S^n} \leftrightarrow \tau_{\mathbb{RP}^n}$. We also have $\underline{1}_{S^n} \leftrightarrow \underline{1}_{\mathbb{RP}^n}$ and $\underline{\sigma}_{S^n} \leftrightarrow \gamma_n^1$, where σ is the sign representation. Nonequivariantly, the inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$ gave rise to an isomorphism $\tau_{S^n} \oplus \nu \cong \underline{n+1}$. Equivariantly, S^n includes into $\mathbb{R}^{n+1}$ equipped with the sign action. In other words, we have an inclusion $S^n \hookrightarrow \sigma^{n+1}$, where σ denotes the sign representation. The normal bundle ν on S^n is trivial, and this is also true equivariantly. Equivariantly, we have an isomorphism

$$\tau_{S^n} \oplus \underline{1} \cong \sigma^{n+1}.$$

On $\mathbb{RP}^n$, this gives rise to the bundle identity

$$\tau_{\mathbb{RP}^n} \oplus \underline{1}_{\mathbb{RP}^n} \cong (\gamma_1^n)^{\oplus n+1}.$$

If follows that

$$w(\tau_{\mathbb{RP}^n}) = (1+x)^{n+1},$$

where $x^{n+1} = 0$ in $\mathrm{H}^*(\mathbb{RP}^\infty; \mathbb{F}_2)$. Thus $w(\tau_{\mathbb{RP}^n}) = 1$ if and only if $n+1$ is a power of 2, so that $\mathbb{RP}^n$ cannot be parallelizable if $n+1$ is not a power of 2. Note that this is consistent with the Hopf Invariant One theorem.

Mon, Nov. 12

We have seen that the first Stiefel-Whitney class $w_1(-)$ for line bundles comes from the cohomology of $Gr_1(\mathbb{R}^\infty) = \mathbb{RP}^\infty$. We will see that the higher classes similarly come from the cohomology of $Gr_n(\mathbb{R}^\infty)$.

Since any n-plane bundle E is the pullback of the universal n-plane bundle γ^n, it suffices to define the Stiefell-Whitney classes $w_i(\gamma^n)$ of the universal bundle. Our goal will be to show that the cohomology ring of $Gr_n(\mathbb{R}^\infty)$ is polynomial in the classes $w_i(\gamma^n)$.

First, we will show that the $w_i(\gamma^n)$ are algebraically independent (here we are fixing n and allowing i to vary). In order to show this, it suffices to see that they are independent after restricting them to some simpler space. The simpler space that we use is $X = (\mathbb{RP}^\infty)^{\times n}$. We have $\mathrm{H}^*(X; \mathbb{F}) \cong \mathbb{F}[a_1, \dots, a_n]$, where the generators a_j each live in degree 1. Consider the bundle E on X which is n-fold external product of γ^1 with itself. Then axiom (1) of Theorem 4.6 gives

$$w(E) = \prod_j w(p_j^*(\gamma^1)) = \prod_j (1 + a_j).$$

From this we see that $w_i(E) = \sigma_i(a_1, \dots, a_n)$ is the nth elementary symmetric function in the a_j's. But the elementary symmetric functions σ_i are algebraically independent, which implies that the same must be true of $w_i(\gamma^n)$.

It follows that we have an injection $\mathbb{F}[w_1(\gamma^n), \dots, w_n(\gamma^n)] \hookrightarrow \mathrm{H}^*(Gr_n(\mathbb{R}^\infty); \mathbb{F})$. We wish to show this is an isomorphism. In order to see this, we will discuss a cell structure on the Grassmannian $Gr_n(\mathbb{R}^\infty)$.

We start with the finite Grassmannian $Gr_n(\mathbb{R}^k)$. Consider $X \in Gr_n(\mathbb{R}^k)$. Consider also the complete flag

$$0 \subseteq \mathbb{R}^1 \subseteq R^2 \subseteq R^k,$$

where we think of $\mathbb{R}^i$ as the span of the first i standard basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_i$. Then consider the sequence of non-negative integers

$$0 \le \dim(X \cap \mathbb{R}^1) \le \dim(X \cap \mathbb{R}^2) \le \cdots \le \dim(X \cap \mathbb{R}^k) = \dim(X) = n.$$

In this sequence, any two consecutive integers differ by at most one. Let us write s_i for the position in which the ith increase in dimension happens. There are n such increases. The n-tuple $(s_1, \dots, s_n)$ is an increasing sequence of positive integers with

$$1 \le s_1 < s_2 < \cdots < s_n \le k$$

We refer to this n-tuple as a **Schubert symbol**.

For each Schubert symbol $s = (s_1, \dots, s_n)$, we define $e(s) \subseteq Gr_n(\mathbb{R}^k)$ to be the set of X for which the jump sequence of the dimensions of the $X \cap \mathbb{R}^i$ is given by s. We refer to $e(s)$ as a **Schubert cell**.

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Proposition 4.13. *The subset $e(s)$ is an open cell of dimension*

$$d(s) = (s_1 - 1) + (s_2 - 2) + \cdots + (s_n - n).$$

The point is that any $X \in e(s)$ has a unique basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ for which $\mathbf{v}_i \in X \cap \mathbb{R}^i$, the s_i-th coordinate of $\mathbf{v}_i$ is equal to 1, and the s_i-th coordinate of the other $\mathbf{v}_j$'s is zero. If we form a matrix whose rows are given by this basis of X, the matrix is in a reduced echelon form

$$A = \begin{pmatrix} * & 1 & 0 & 0 & 0 \\ * & 0 & * & 1 & 0 \\ * & 0 & * & 0 & 1 \end{pmatrix}$$

In row i (or basis element $\mathbf{v}_i$), we have $s_i - i$ "free" coordinates, which gives the dimension count. That the topology is right can be seen by thinking of $Gr_n(k)$ as a quotient of $Gl_k(\mathbb{R})$ (via the transitive action of this group on the set of subspaces on $\mathbb{R}^k$).

To see that this really gives a CW structure on $Gr_n(\mathbb{R}^k)$, we would want to know that if $X \in \overline{e(s)} - e(s)$ for some Schubert cell $e(s)$, then X lies in some cell of lower dimension. For example, consider the Schubert cell $e(1,3) \subseteq Gr_2(\mathbb{R}^3)$. This consists in the planes in $\mathbb{R}^3$ which have one basis vector on the x-axis and another basis vector oustide of the xy-plane. Then the xy-plane itself, which lives in the cell $e(1,2)$, is in $\overline{e(1,3)}$ since we can express it as a limit of planes living in $e(1,3)$.

Example 4.14. Consider $Gr_2(\mathbb{R}^4)$. This has Schubert cells

$$\begin{aligned} &e(1,2)\ (\dim = 0), \qquad e(1,3)\ (\dim = 1), \qquad e(1,4)\ (\dim = 2), \\ &e(2,3)\ (\dim = 2), \qquad e(2,4)\ (\dim = 3), \qquad e(3,4)\ (\dim = 4). \end{aligned}$$

Example 4.15. Consider $Gr_3(\mathbb{R}^5)$. This has Schubert cells

$$\begin{aligned} &e(1,2,3)\ (\dim = 0), \qquad e(1,2,4)\ (\dim = 1), \qquad e(1,2,5)\ (\dim = 2), \\ &e(1,3,4)\ (\dim = 2), \qquad e(1,3,5)\ (\dim = 3), \qquad e(1,4,5)\ (\dim = 4), \\ &e(2,3,4)\ (\dim = 3), \qquad e(2,3,5)\ (\dim = 4), \qquad e(2,4,5)\ (\dim = 5), \end{aligned}$$

and

$$e(3,4,5)\ (\dim = 6).$$

Theorem 4.16. *The cohomology ring of the Grassmannian $Gr_n(\mathbb{R}^\infty)$ is*

$$\mathrm{H}^*(Gr_n(\mathbb{R}^\infty); \mathbb{F}) \cong \mathbb{F}[w_1(\gamma^n), \dots, w_n(\gamma^n)].$$

Proof. As we discussed last time, we have an injection

$$\mathbb{F}[w_1(\gamma^n), \dots, w_n(\gamma^n)] \hookrightarrow \mathrm{H}^*(Gr_n(\mathbb{R}^\infty); \mathbb{F}).$$

It remains to show it is surjective.

Consider a monomial $w_1^{r_1} w_2^{r_2} w_3^{r_3}$ in the w_i's. This is a cohomology class of degree $r_1 + 2r_2 + 3r_3$. This monomial corresponds to the non-decreasing sequence of nonnegative integers

$$r_3 \leq r_3 + r_2 \leq r_3 + r_2 + r_1,$$

which corresponds to the strictly increasing sequence of positive integers

$$r_3 + 1 < r_3 + r_2 + 2 < r_3 + r_2 + r_1 + 3.$$

The latter is a Schubert symbol, and this process is reversible. So we have shown that the dimension (over $\mathbb{F}_2$) of our polynomial ring in degree r is equal to the number of cells in our CW structure on $Gr_n(\mathbb{R}^\infty)$. Since this is an upper bound on the dimension of the (co)homology, we conclude that (1) the differentials in the mod 2 cochain complex all vanish and (2) our map is an isomorphism. ∎

Mon, Nov. 19

We saw previously (Proposition 4.1) that line bundles L are controlled by a class in $H^1(X;\mathbb{F})$, which we now know to be $w_1(L)$. What does $w_1(E)$ correspond to when E is rank larger than 1?

Proposition 4.17. *Let E be a rank n bundle. Then $w_1(E)$ corresponds to the top exterior power $\Lambda^n(E)$, in the sense that $w_1(E) = w_1(\Lambda^n(E))$.*

Proof. By the splitting principle, it suffices to consider the case of a split bundle $E = L_1 \oplus L_2 \oplus \cdots \oplus L_n$. Now we know that

$$w_1(L_1 \oplus \cdots \oplus L_n) = w_1(L_1) + \cdots + w_1(L_n).$$

You also showed on Worksheet 11-2 that $w_1(L_1 \otimes L_2) = w_1(L_1) + w_1(L_2)$. It follws that w_1 takes an n-fold tensor product of line bundles to the n-fold sum of their w_1's. In other words,

$$w_1(E) = w_1(L_1 \oplus \cdots \oplus L_n) = w_1(L_1) + \cdots + w_1(L_n) = w_1(L_1 \otimes \cdots \otimes L_n) = w_1(\Lambda^n(E)).$$

■

Corollary 4.18. *A rank n vector bundle E is orientable if and only if $w_1(E) = 0$.*

Proof. An orientation of E is a compatible family of orientations of all fibers. This amounts to a nowhere zero section of the line bundle $\Lambda^n(E)$. Thus E is orientable if and only if $\Lambda^n(E)$ is trivial. ■

Recall that a manifold is orientable exactly when its tangent bundle is orientable.

Example 4.19. All spheres are orientable, as follows from Example 4.9.

Example 4.20. The canonical bundle on $\mathbb{RP}^n$, or more generally on $Gr_n(\mathbb{R}^k)$, is never orientable.

Example 4.21. Since $w(\tau_{\mathbb{RP}^n}) = (1+x)^{n+1}$, it follows that $\mathbb{RP}^n$ is orientable exactly when n is odd.

Example 4.22. The tangent bundle of the torus $\mathbb{T} = S^1 \times S^1$ is the external product of the (trivial) tangent bundles of S^1 and is therefore trivial. In particular, the torus is orientable.

4.2. **The Thom isomorphism and Euler class.**

Definition 4.23. Let E be a bundle on X. We assume a Euclidean metric on E and define $S(E) \subseteq D(E) \subseteq E$ to be the unit sphere and disk bundles. That is, we take the unit sphere or disk in each fiber of E. We then define the **Thom space** of E to be

$$Th(E) = D(E)/S(E).$$

This is also denoted X^E. It is also the cofiber of the projection $S(E) \longrightarrow X$.

Example 4.24. Let E be a trivial rank n bundle. Then $D(E) = X \times D^n$ and $S(E) = X \times S^{n-1}$. Then

$$Th(X \times \mathbb{R}^n) = (X \times D^n)/(X \times S^{n-1}) = (X_+) \wedge (D^n/S^{n-1}) = \Sigma^n X_+.$$

So the Thom space construction is a generalization of suspension. We can think of it as a twisted suspension, twisted by the bundle.

Example 4.25. Consider the canonical bundle on $\mathbb{RP}^1$. Recall from Worksheet 10-3 that γ_n^1 is the normal bundle to the inclusion $\mathbb{RP}^n \hookrightarrow \mathbb{RP}^{n+1}$. The sphere bundle turns out to be S^n, so that

$$Th(\gamma_n^1) = \mathrm{cofib}(S^n \longrightarrow \mathbb{RP}^n) \simeq \mathbb{RP}^{n+1}.$$

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Suspension interacts nicely with cohomology, under the suspension isomorphism. There is a similar result for the Thom construction.

Definition 4.26. Let E be a rank n bundle on X and R a commutative ring. Then an R-**orientation** for E, also known as a **Thom class**, is a class $\mu \in \widetilde{\mathrm{H}}^n(Th(E); R)$ such that its restriction to each fiber $D(E)_x / S(E)_x \cong S^n$ is an R-module generator.

Proposition 4.27. *Any bundle has an $\mathbb{F}_2$-Thom class. A bundle E has a $\mathbb{Z}$-Thom class if and only if it is orientable.*

In particular, for a manifold M, the tangent bundle τ_M has a $\mathbb{Z}$-Thom class if and only if M is orientable.

Theorem 4.28 (Thom isomorphism)**.** *Let $\mu \in \widetilde{\mathrm{H}}^n(Th(E); R)$ be a Thom class for E. Then the map*

$$\Phi : \mathrm{H}^k(X; R) \longrightarrow \mathrm{H}^{n+k}(D(E), S(E); R) \cong \widetilde{\mathrm{H}}^{n+k}(Th(E); R)$$

given by $\Phi(\alpha) = p^*(\alpha) \cdot \mu$ is an isomorphism for all $k \in \mathbb{Z}$.

Sketch. We give a sketch in the case that the base X is compact. If we restrict our attention to some $U \subseteq X$ on which E is trivial, then the Thom isomorphism is simply the suspension isomorphism. Suppose that $X = U \cup V$, where E is trivial over U and V. Then the Mayer-Vietoris sequence relates the cohomology of X to the cohomology of U, V, and $U \cap V$. The statement of the theorem holds over U, V, and $U \cap V$, and the five-lemma gives the conclusion over X.

An induction argument now gives the result if X is a finite union of U_i's over which E is trivial. But this must be the case if X is compact. ■

Thom isomorphisms in other cohomology theories also play an important role. For instance, there is similarly a Thom isomorphism in K-theory.

We have seen Stiefel-Whitney classes, which are characteristic classes in $\mathrm{H}^*(-; \mathbb{F}_2)$. One can also ask for characteristic classes with $\mathbb{Z}$-coefficients.

Definition 4.29. Let E be an orientable rank n bundle over X. Define the **Euler class** $e(E) \in \mathrm{H}^n(X; \mathbb{Z})$ of the bundle to be the image of the Thom class μ_E under the composition

$$\widetilde{\mathrm{H}}^n(Th(E); \mathbb{Z}) \cong \mathrm{H}^n(D(E), S(E); \mathbb{Z}) \longrightarrow \mathrm{H}^n(D(E); \mathbb{Z}) \cong \mathrm{H}^n(X; \mathbb{Z}).$$

Proposition 4.30. *Let E be an orientable rank n bundle over X.*

(1) *$e(E_1 \oplus E_2) = e(E_1) \cdot e(E_2)$ if E_1 and E_2 are orientable.*
(2) *The Euler class is a lift of the top Stiefel-Whitney class, in the sense that $e(E) \mapsto w_n(E)$ under the reduction of coefficients $\mathbb{Z} \longrightarrow \mathbb{F}_2$.*
(3) *If n is odd, then $2 \cdot e(E) = 0$.*
(4) *If E admits a nowhere-vanishing section, then $e(E) = 0$*

This last statement says that the Euler class is an obstruction to the existence of a nowhere vanishing section.

Proof. (2) It suffices to show this in the case of the universal oriented n-plane bundle. We have not yet discussed universal oriented bundles, but this story is not so different from what we discussed previously. There is a space $\widetilde{Gr_n(\mathbb{R}^\infty)}$ which classifies oriented bundles. A point of $\widetilde{Gr_n(\mathbb{R}^\infty)}$ is an n-dimensional subspace of $\mathbb{R}^\infty$ with a chosen orientation. In fact, $\widetilde{Gr_n(\mathbb{R}^\infty)}$ is the universal cover of $Gr_n(\mathbb{R}^\infty)$. The universal oriented bundle $\tilde{\gamma}^n$ is the pullback of γ^n under the covering map $\widetilde{Gr_n(\mathbb{R}^\infty)} \longrightarrow Gr_n(\mathbb{R}^\infty)$. Moreover there is an analogue of Theorem 4.16:

$$\mathrm{H}^*(\widetilde{Gr_n(\mathbb{R}^\infty)}; \mathbb{F}) \cong \mathbb{F}[w_2(\gamma^n), \ldots, w_n(\gamma^n)],$$

corresponding to the statement (Corollary 4.18) that orientable bundles have trivial first Stiefel-Whitney class.

The restriction of $\tilde{\gamma}^n$ to $Gr_{n-1}(\mathbb{R}^\infty)$ under the map $V \mapsto 1 \oplus V$ is $\underline{1} \oplus \tilde{\gamma}^{n-1}$. It follows that $\underbrace{e(\tilde{\gamma}^n)}$ restricts to $e(\underline{1})\underbrace{e(\tilde{\gamma}^{n-1})} = 0$. Thus $e(\tilde{\gamma}^n)$ is in the kernel of the restriction $\mathrm{H}^n(Gr_n(\mathbb{R}^\infty);\mathbb{F}_2) \longrightarrow \mathrm{H}^n(Gr_{n-1}(\mathbb{R}^\infty);\mathbb{F}_2)$. But this restriction map sends w_i to w_i if $2 \le i \le n-1$, so the kernel is 1-dimensional, generated by $w_n(\gamma^n)$. Since $e(\tilde{\gamma}^n) \neq 0$ (else all Euler classes would vanish—we will see that this is not the case), we conclude the claimed equality.

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(3) This follows from the fact that the Thom isomorphism satisfies $\Phi(e(E)) = \mu_E^2$. The result now follows from graded-commutativity.

(4) As we have seen before, a nowhere-vanishing section corresponds to an injection $\underline{1} \hookrightarrow E$. Using a metric, we can split this off, so that we have $E \cong \underline{1} \oplus E'$. But then (1) gives $e(E) = e(\underline{1})e(E') = 0$ since $e(\underline{1}) = 0$.

■

This implies that $e(\tau_{S^n}) = 0$ if n is odd (all spheres are orientable), since there are no nontrivial 2-torsion classes in $\mathrm{H}^n(S^n;\mathbb{Z})$. More generally, we have

Proposition 4.31. *[[MS, Cor. 11.12]] Let M be a connected, compact, oriented manifold. Then $e(\tau_M)$ is $\chi(M)$ times a generator of $\mathrm{H}^n(M;\mathbb{Z}) \cong \mathbb{Z}$.*

Example 4.32. Thus for an even sphere, it follows that $e(\tau_M)$ is twice a generator. This implies that the tangent bundle admits no nonzero section (vector field) and that S^n is not parallelizable (when n is even).

Example 4.33. For $M = \mathbb{RP}^n$ for n odd, we have $e(\tau_{\mathbb{RP}^n}) = 0$.This is compatible with our previous computation that $w(\tau_{\mathbb{RP}^n}) = (1+x)^{n+1}$, so that $w_n(\tau_{\mathbb{RP}^n}) = (n+1)x^n \equiv 0$.

4.3. **Chern classes.** The characteristic classes for complex vector bundles are given by

Theorem 4.34. *There is a (unique) sequence c_1, c_2, $\dots$, of natural tranformations $\mathrm{Vect}^*_{\mathbb{C}}(X) \longrightarrow \mathrm{H}^*(X;\mathbb{Z})$, with $c_i(E) \in \mathrm{H}^{2i}(X;\mathbb{Z})$ such that*

(1) $c_n(E_1 \oplus E_2) = \sum_{i+j=n} c_i(E_1) \cdot c_j(E_2)$
(2) $c_i(E) = 0$ *if* $i > \dim(E)$
(3) $c_1(\gamma_1)$ *is a generator of* $\mathrm{H}^2(\mathbb{CP}^1;\mathbb{Z}) \cong \mathbb{Z}$.

Proposition 4.35. *Let E be a rank n complex bundle on X. Then we may regard E as a rank $2n$ oriented real bundle, and $c_n(E) = e(E)$.*

Proof. The proof is similar to Proposition 4.30(2). ■

Example 4.36. The total Chern class for $\tau_{\mathbb{CP}^n}$ is

$$c(\tau_{\mathbb{CP}^n}) = (1+z)^{n+1},$$

where z is a generator of $\mathrm{H}^2(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}$.

Chern classes are related to Stiefel-Whitney classes by the following result.

Proposition 4.37. *Let $E \longrightarrow X$ be a rank n complex bundle. Then we may regard E as a rank $2n$ real bundle. Then all odd Stiefel-Whitney classes vanish, and the even ones are the mod 2 reductions of the Chern classes. That is,*

$$w_{2i+1}(E) = 0, \qquad w_{2i}(E) \equiv c_i(E).$$

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4.4. **The Chern Character.** The total chern class is an operation that takes a sum of complex bundles to a product in $\mathrm{H}^*(X;\mathbb{Z})$. And we explored on a worksheet how this behaves on a tensor product. What we might hope for is a a characteristic class which takes a sum of bundles to a sum of classes and a tensor product of bundles to the product of classes. Again, the total Chern class cannot be the answer, since it takes sums to products.

Let's start with line bundles (the splitting principle will tell us it suffices to specify such on operation on line bundles). Recall that

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2).$$

If we want to get a product as the output rather than a sum, we should feed this through some algebraic operation that converts sums to products. The most famous such operation is the exponential function.

Definition 4.38. Let L be a complex line bundle on X. We define the **Chern character** of L to be

$$\mathrm{ch}(L) = e^{c_1(L)} = 1 + c_1(L) + c_1(L)^2/2 + c_1(L)^3/6 + \dots$$

in $\mathrm{H}^*(X;\mathbb{Q})$.

Note that the denominators used to define the exponential function force us to work with rational coefficients.

Example 4.39. Let $L = \gamma^1$ be the canonical complex line bundle on $\mathbb{CP}^1 \cong S^2$. Then $\mathrm{ch}(\gamma^1) = 1 + c_1(L) = c(L)$.

Example 4.40. Let $L = \gamma^1_n$ be the canonical complex line bundle on $\mathbb{CP}^n$, where $n \geq 2$. Recall that $\mathrm{H}^*(\mathbb{CP}^n;\mathbb{Q}) = \mathbb{Q}[x]/x^{n+1}$. Then

$$\mathrm{ch}(\gamma^1_n) = 1 + c_1(L) + c_1(L)^2/2 + \cdots + c_1(L)^n/n! = 1 + x + x^2/2 + \cdots + x^n/n!.$$

We follow the same procedure we used to define the Adams operations ψ^k to now define $\mathrm{ch}(-)$ on a general bundle. If $E \cong L_1 \oplus \cdots \oplus L_n$, then

$$c(E) = \prod_i (1 + c_1(L_i)) = 1 + \sigma_1\Big(c_1(L_1), \dots, c_1(L_n)\Big) + \cdots + \sigma_n\Big(c_1(L_1), \dots, c_1(L_n)\Big).$$

In other words, $c_i(E) = \sigma_i(c_1(L_1), \dots, c_1(L_n))$. We then want

$$\mathrm{ch}(E) = \sum_i \mathrm{ch}(L_i) = 1 + \sum_i c_1(L_i) + \sum_i c_1(L_i)^2/2 + \dots.$$

Recall from just above Definition 2.29 the polynomials s_k such that

$$s_k(\sigma_1, \dots, \sigma_n) = x_1^k + \cdots + x_n^k.$$

Putting all of this together, we see that we should define

$$\mathrm{ch}(E) = n + \sum_{k \geq 1} s_k(c_1(E), \dots, c_n(E))/k!.$$

We have essentially proved

Proposition 4.41. *The Chern character defines a ring homomorphism*

$$ch : KU^0(X) \longrightarrow \mathrm{H}^*(X;\mathbb{Q}).$$

Proposition 4.42. *When $X = S^{2n}$, the reduced Chern character*

$$\mathrm{ch} : \widetilde{KU}^0(S^{2n}) \longrightarrow \widetilde{\mathrm{H}}^*(S^{2n};\mathbb{Q})$$

is injective with image $\widetilde{\mathrm{H}}^*(S^{2n};\mathbb{Z})$.

Proof. First consider the case $n = 1$. Then $\widetilde{KU}(S^2) \cong \mathbb{Z}$, generated by $\gamma^1 - 1$. Now $\mathrm{ch}(\gamma^1) = 1 + x$, as we already saw, and $\mathrm{ch}(\gamma^1 - 1) = x$, which is the generator of $\widetilde{\mathrm{H}}^2(S^2; \mathbb{Z})$.

The general n case now follows by combining Bott periodicity, the suspension isomorphism in cohomology, and the $n = 1$ case. ∎

In this case, we see that we get

$$\mathrm{ch} : KU^0(S^{2n}) \otimes \mathbb{Q} \xrightarrow{\cong} \mathrm{H}^{\mathrm{even}}(S^{2n}; \mathbb{Q}).$$

The Chern character also maps the odd K-groups to the odd cohomology groups (since they both can be thought of as even groups of a suspension).

Proposition 4.43. [H2, Proposition 4.5] *The map* $\mathrm{ch} : KU^*(X) \otimes \mathbb{Q} \longrightarrow \mathrm{H}^*(X; \mathbb{Q})$ *is an isomorphism for all finite complexes.*

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Let's look at an example.

Example 4.44. Let E be a complex bundle over S^6. Since the cohomology of S^6 is concentrated in degrees 0 and 6, it must be the case that $c_1(E) = 0$ and $c_2(E) = 0$. So then

$$\mathrm{ch}(E) = \dim(E) + \frac{s_3(c_1(E), c_2(E), c_3(E))}{3!} = \dim(E) + \frac{s_3(0, 0, c_3(E))}{3!}.$$

But recall from just above Definition 2.29 that $s_3(c_1, c_2, c_3) = c_1^3 - 3c_1c_2 + 3c_3$. We conclude that

$$\mathrm{ch}(E) = \dim(E) + \frac{3c_3(E)}{3!} = \dim(E) + \frac{c_3(E)}{2}.$$

Then Proposition 4.42 tells us that $\frac{c_3(E)}{2}$ is an integral class. In other words, the chern class $c_3(E)$ is necessarily even. Morever, the image of $c_3 : KU^0(S^6) \longrightarrow \mathrm{H}^6(S^6; \mathbb{Z})$ is $2\mathbb{Z}$.

This last example generalizes to the statement that on S^{2n}, the chern class $c_n(E)$ is divisible by $(n-1)!$.

4.5. **The J homomorphism.** These ideas have bearing on the stable homotopy groups of spheres.

Definition 4.45. Consider the composition

$$O(n) \longrightarrow \mathrm{Homeo}(\mathbb{R}^n, \mathbb{R}^n) \longrightarrow \mathrm{Homeo}_*(S^n, S^n) \subset \mathrm{Map}_*(S^n, S^n),$$

where the first map is given by the action of $O(n)$ on $\mathbb{R}^n$ and the second is the one-point compactification functor (recall that any *proper* map induces a map on one-point compactifications). This induces a homomorphism in homotopy groups

$$J : \pi_i(O(n)) \longrightarrow \pi_i(\mathrm{Map}_*(S^n, S^n)) \cong \pi_{i+n}(S^n)$$

known as the (unstable) **J-homomorphism**. We also get the **stable J-homomorphism**

$$\pi_i(O) \longrightarrow \operatorname*{colim}_n \pi_{i+n}(S^n) = \pi_i^s$$

by passage to colimits over n.

Bott periodicity tells us the homotopy groups of the infinite orthogonal group O. These are the groups listed in Proposition 2.17, shifted by one. Thus

$$\pi_i(O) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & i \equiv 0, 1 \pmod 8 \\ \mathbb{Z} & i \equiv 3, 7 \pmod 8 \\ 0 & \text{else.} \end{cases}$$

The J-homomorphism maps these well-understood groups to the stable homotopy groups of spheres, which we would love to know. In fact, the image is well-understood as well.

We'll start by using the simpler complex J-homomorphism. This is given by precomposing the (real) J-homomorphism with the map $U(n) \longrightarrow O(2n)$ which considers a complex $n \times n$ matrix as a real $2n \times 2n$-matrix. Complex Bott periodicity tells us that $\pi_i(U)$ is $\mathbb{Z}$ if i is odd and vanishes when i is even. We will study the complex J-homomorphism by introducing yet another homomorphism, Adams' e-invariant.

Consider $f : S^{2k-1} \longrightarrow S^{2n}$. This fits into a cofiber sequence

$$S^{2k-1} \xrightarrow{f} S^{2n} \longrightarrow Cf \longrightarrow \Sigma S^{2k-1} \cong S^{2k}.$$

Feeding this into K-theory and the Chern character gives a map of short exact sequences

$$\begin{tikzcd}
\widetilde{KU}^0(S^{2k}) \arrow[r] \arrow[d, "\mathrm{ch}"'] & \widetilde{KU}^0(Cf) \arrow[r] \arrow[d, "\mathrm{ch}"'] & \widetilde{KU}^0(S^{2n}) \arrow[d, "\mathrm{ch}"'] \\
\widetilde{\mathrm{H}}^{ev}(S^{2k};\mathbb{Q}) \arrow[r] & \widetilde{\mathrm{H}}^{ev}(Cf;\mathbb{Q}) \arrow[r] & \widetilde{\mathrm{H}}^{ev}(S^{2n};\mathbb{Q})
\end{tikzcd}$$

Let $\alpha \in \widetilde{KU}^0(S^{2k})$ be a generator and let $a = \mathrm{ch}(\alpha) \in \mathrm{H}^{2k}(S^{2k};\mathbb{Z})$. Then a generates the integral cohomology group (and also the rational one). Let $\beta \in \widetilde{KU}^0(Cf)$ be an element mapping to a generator of $\widetilde{KU}^0(S^{2n})$ and let $b \in \mathrm{H}^{2n}(Cf;\mathbb{Z})$ be a generator. Commutativity of the diagram ensures that $\mathrm{ch}(\beta) = b + ra$ for some $r \in \mathbb{Q}$. Moreover, there is no canonical choice for the element β in $\widetilde{KU}^0(Cf)$; adding any integral multiple of α would give another perfectly good choice. This, in turn, implies that r is only well-defined in the quotient group $\mathbb{Q}/\mathbb{Z}$.

Definition 4.46. We define

$$e : \pi_{2k-1}(S^{2n}) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

by $e(f) = r$.

This is still only well defined up to ± 1, but we won't worry about that.

Proposition 4.47. *The function $e : \pi_{2k-1}(S^{2n}) \longrightarrow \mathbb{Q}/\mathbb{Z}$ is a homomorphism making*

$$\begin{tikzcd}
\pi_{2k-1}(S^{2n}) \arrow[rr, "\Sigma^2"] \arrow[dr, "e"'] & & \pi_{2k+1}(S^{2n+2}) \arrow[dl, "e"] \\
& \mathbb{Q}/Z &
\end{tikzcd}$$

commute.

It follows that e extends to a homomorphism $e : \pi^s_{odd} \longrightarrow \mathbb{Q}/\mathbb{Z}$.

Theorem 4.48. *If $f \in \pi_{2k-1}(U(n))$ is a generator of $\pi_{2k-1}(U)$, then*

$$e(J_{\mathbb{C}} f) = \pm \beta_k / k,$$

where β_k is the kth Bernoulli number, defined by

$$\frac{x}{e^x - 1} = \sum_k \beta_k \frac{x^k}{k!}.$$

Corollary 4.49. *The image of $J_{\mathbb{C}}$ in π^s_{2k-1} has order divisible by the denominator of β_k / k.*

The series defining the Bernoulli numbers is

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} - \frac{x^8}{1209600} + \cdots.$$

The first few Bernoulli numbers are

k	0	1	2	3	4	5	6	7	8	9	10	11
β_k	1	$\frac{1}{2}$	$\frac{1}{6}$	0	$\frac{1}{30}$	0	$\frac{1}{42}$	0	$\frac{1}{30}$	0	$\frac{5}{66}$	0
β_k/k	NA	$\frac{1}{2}$	$\frac{1}{12}$	0	$\frac{1}{120}$	0	$\frac{1}{252}$	0	$\frac{1}{240}$	0	$\frac{1}{132}$	0

These denominators come close to predicting the order of π^s_{2k-1}.

Example 4.50. Since $\beta_1/1 = \frac{1}{2}$, Corollary 4.49 implies that π^s_1 has order at least two (it does have order 2).

Example 4.51. Since $\beta_2/2 = \frac{1}{12}$, Corollary 4.49 implies that π^s_3 has order at least 12 (it has order 24).

Example 4.52. Since $\beta_4/4 = \frac{1}{120}$, Corollary 4.49 implies that π^s_7 has order at least 120 (it has order 240).

Example 4.53. Since $\beta_6/6 = \frac{1}{252}$, Corollary 4.49 implies that π^s_{11} has order at least 252 (it has order 504).

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Last time, we introduced the Bernoulli numbers (these agree up to sign with a reindexing of the classical Bernouilli numbers). In fact, there is a nice formula for their denominators.

Proposition 4.54. *For k even, the denominator of β_k is the product of the primes p such that $p-1$ divides k.*

We saw that the complex J-homomorphism came close to capturing some of the stable homotopy groups. The real J-homomorphism does an even better job.

Theorem 4.55. [JX-IV] *Let $i > 0$ and $i \equiv 0, 1 \pmod 8$. Then*

$$J : \pi_i(O) \cong \mathbb{Z}/2\mathbb{Z} \longrightarrow \pi^s_i$$

is a monomorphism, and the image is a direct summand.

Theorem 4.56. [JX-IV] *Let $i = 2k - 1 \equiv 3 \pmod 4$. Then the image of*

$$J : \pi_i(O) \cong \mathbb{Z} \longrightarrow \pi^s_i$$

is cyclic of order $\frac{\beta_k}{2k}$ and is a direct summand in π^s_{2k-1}.

In the case $2k - 1 \equiv -1 \pmod 8$, the generator of this cyclic summand in π^s_{2k-1} is sometimes called ρ_{2k-1}. The elements of order two of Theorem 4.55 are then $\eta \cdot \rho_{2k-1}$ and $\eta^2 \cdot \rho_{2k-1}$. In the case $2k - 1 \equiv 3 \pmod 8$, the generator is known as ζ_{2k-1}.

If we focus on the 2-torsion, then the element ζ_{2k-1} is of order 8. Indeed, the denominator of β_k has a single factor of 2. The integer k is congruent to 2 modulo 4, so multiplying by k adds one more factor of 2, so that the denominator of $\beta_k/2k$ is 8 times an odd integer.

Remark 4.57. The numerators are also of interest in topology. Kervaire and Milnor showed that the number of diffeomorphism classes on the $(4n-1)$-sphere that are the boundary of parallelizable manifolds is given by

$$2^{2n-2} \cdot (2^{2n-1} - 1) \cdot \text{the numerator of } 4\beta_{2n}/n.$$

But the numerators are more difficult to compute.

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