1. Let $x = \sum_{i=0}^{n} x_i$ be an element of $H^\ast(X; F_2)$, where $x_0 = 1$ and the degree of $x_i$ is $i$. Show $x$ is invertible in $H^\ast(X; F_2)$.

2. Let $\gamma_n$ be the canonical line bundle over $\mathbb{R}P^n$. Note that $\gamma_n$ is defined as a subbundle of $n + 1$. Let $E$ be the orthogonal complement of $\gamma_n$. Find the total Stiefel-Whitney class of $E$.

3. Recall that if $M \subseteq N$ is a submanifold, then the tangent bundle $\tau_N$ of $N$, when restricted to the submanifold $M$, splits as $(\tau_N)|_M \cong \tau_M \oplus \nu$, where $\nu$ is the normal bundle to the embedding $M \hookrightarrow N$.

   We showed in class that $w(\tau_{\mathbb{R}P^n}) = (1 + x)^{n+1}$. Find $w(\nu)$, where $\nu$ is the embedding $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^{n+1}$. Which bundle is $\nu$?

4. An embedding $M \hookrightarrow N$ (or more generally, immersion) induces an injection on tangent bundles. For example, an immersion $\mathbb{R}P^n \hookrightarrow \mathbb{R}^n + k$ gives an inclusion $\tau_{\mathbb{R}P^n} \hookrightarrow n + k$. Use the total Stiefel-Whitney class of $\tau_{\mathbb{R}P^n}$ and the normal bundle $\nu$ to establish the following lower bounds on $k$:

   (a) When $n = 4$, then $k \geq 3$.
   (b) When $n = 8$, then $k \geq 7$.
   (c) More generally, when $n = 2^\ell$, then $k \geq 2^\ell - 1$. 