1. Suppose that $S^n$ is parallelizable (has trivial tangent bundle), with trivialization $\varphi : S^n \times \mathbb{R}^n \cong TS^n$. Given $x \in S^n$, write

$$p_x : S^n \xrightarrow{\cong} \mathbb{R}^n = \mathbb{R}^n \cup \{\infty\}, \quad q_x = p_x^{-1} : \mathbb{R}^n \xrightarrow{\cong} S^n$$

for stereographic projection and its inverse, so that $p_x(-x) = 0$ and $p_x(x) = \infty$. Define $\varphi : TS^n \to S^n$ by letting $\varphi(x, y)$ be the point on the line through $-x$ and $x + y$ (recall that $y \perp x$) that lies on $S^n$. Define $\nu : S^n \times \mathbb{R}^n \to S^n$ by the composition

$$S^n \times \mathbb{R}^n \cong TS^n \xrightarrow{\varphi} S^n.$$

Then this extends to a map $\overline{\nu} : S^n \times \mathbb{R}^n \to S^n$. Now pick some $e \in S^n$ and define $\mu : S^n \times S^n \to S^n$ by the formula

$$S^n \times S^n \xrightarrow{id \times p \circ e} S^n \times \mathbb{R}^n \xrightarrow{\overline{\nu}} S^n.$$

Show that this makes $S^n$ an $H$-space with unit $e$. (The Hopf invariant one theorem now implies that $S^n$ can only be parallelizable when $n = 0, 1, 3, 7$.)

2. Suppose that $\mathbb{R}^n$ is a division algebra (has a bilinear product $\mu$ with no zero divisors). Pick a nonzero vector $e \in \mathbb{R}^n$. Show that there is a division algebra structure in which $e$ is the unit. Pick a linear isomorphism $f : \mathbb{R}^n \cong \mathbb{R}^n$ such that $f(\mu(e, e)) = e$. Then $\overline{\mu} = f \circ \mu$ takes $(e, e)$ to $e$. Write $\ell(x) = \overline{\mu}(x, e)$ and $r(x) = \overline{\mu}(e, x)$. Show that

$$\nu(x, y) = \overline{\mu}(\ell^{-1}(x), r^{-1}(y))$$

is a division algebra structure on $\mathbb{R}^n$ with unit element $e$.

3. Consider the function $c : \text{Vect}_\mathbb{R}(X) \to \text{Vect}_\mathbb{C}(X)$ which takes a real bundle and tensors each fiber with $\mathbb{C}$ to produce a complex bundle. This map induces a map on $K$-theory which fits into a long exact sequence (I am not asking you to prove this part) of the form

$$\cdots \xrightarrow{c} \widetilde{KU}^{n-1}(X) \to \widetilde{KO}^{n+1}(X) \to \widetilde{KO}^n(X) \xrightarrow{c} \widetilde{KU}^n(X) \to \cdots.$$

Use Bott periodicity to analyze this long exact sequence in the case $X = S^0$. 