Math 654 - Algebraic Topology
Homework 4
Fall 2019

1. Recall the functor $\text{Gl}_n : \text{Comm} \rightarrow \text{Gp}$ from HW1. When $n = 1$, this gives the functor $(-)^\times : \text{Comm} \rightarrow \text{Gp}$ which takes a commutative ring $R$ and gives $R^\times$, the units (invertible elements) in $R$. Show that the determinant yields a natural transformation $\det : \text{Gl}_n \rightarrow (-)^\times$.

2. Let $G$ be a group. Define a category $\star_G$ which has a single object, $\star$, and such that $\text{Hom}(\star, \star) = G$. The identity morphism and composition of morphisms are defined to be the identity element of the group and the group multiplication, respectively.

   (a) Show that a $G$-set $X$ is the same data as a functor $\mathcal{X} : \star_G \rightarrow \text{Set}$.

   (b) Show that if $X$ and $Y$ are $G$-sets, then a $G$-equivariant function $f : X \rightarrow Y$ corresponds precisely to a natural transformation of functors $\mathcal{X} \rightarrow \mathcal{Y}$.

3. Let $\mathcal{I} = \{ \bullet \rightarrow \bullet \}$ be the category with two objects and a single non-identity morphism. Describe the data involved in a natural transformation $\eta : F \Rightarrow G : \mathcal{I} \rightarrow \mathcal{C}$.

4. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Let $G : \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$ be a function, and suppose given an isomorphism $\eta_C : F(C) \cong G(C)$ for each $C \in \mathcal{C}$. Show that there is a unique way to define $G$ on morphisms of $\mathcal{C}$ that makes $\{\eta_C\}$ a natural isomorphism.