Math 751
Equivariant Homotopy and Cohomology
Homework 1
Fall 2020

1. Recall from the Dihedral Representations video that the irreducible representations of $\Sigma_3 = D_3$ are $\mathbf{1}$, $\sigma = \mathbf{1}_{\text{sgn}}$, and ν_1. Determine the isomorphism types of $\sigma \otimes \nu_1$ and $\nu_1 \otimes \nu_1$, and use this to write down the representation ring $RO(\Sigma_3)$ in terms of generators and relations.

2. The **quaternion group** Q_8 of order 8 has generators -1, i, j, and k, subject to the relations

$$(-1)^2 = 1 \qquad \text{and} \qquad i^2 = j^2 = k^2 = ijk = -1.$$

We are writing 1 for the identity element here. The elements i, j, and k generate cyclic subgroups of order 4.

(a) Show that $Q_8/\langle -1 \rangle \cong K_4$, and use this to determine four irreducible one-dimensional representations of Q_8.

(b) Recall that there is a four-dimensional (noncommutative) division algebra $\mathbb{H}$ over $\mathbb{R}$, also know as the quaternions. It has a basis consisting of $\{1, i, j, k\}$ satisfying the same multiplicative rules as the group Q_8, and Q_8 sits inside $\mathbb{H}$ as a multiplicatively closed subset. Left multiplication by the elements of Q_8 on $\mathbb{H}$ therefore defines a four-dimensional real representation of Q_8. Show that it is irreducible.

The sum of the dimensions of these five irreducible representations is 8, and so it follows that the regular representation $\mathbb{R}[Q_8]$ splits as a sum of these five irreducibles.

(c) **(Optional)** Determine the multiplicative structure of $RO(Q_8)$.