Math 751 – Fall 2020
Equivariant homotopy and cohomology
Worksheet 11

1. Recall that the Borel construction $EG \times_G X$ on a (left) G-space X is defined as the orbits with respect to the diagonal G-action on $EG \times X$.

   (a) If Y is a left G-space, show that the prescription
   $$y \cdot g := g^{-1} \cdot y$$
   defines a right G-action on Y.

   (b) For a right G-space Y and a left G-space X, define an equivalence relation on $Y \times X$ by
   $$(y \cdot g, x) \sim (y, g \cdot x).$$
   Show that the quotient space $(Y \times X)/\sim$ is homeomorphic to $Y \times_G X$, where we are using (a) to consider Y as either a left or right G-space.

2. Recall that BG is a path-connected space, with $\pi_1(BG) \cong G$ and no nontrivial higher homotopy groups.

   (a) For a path-connected based space X, consider the surjective function
   $$\Lambda \colon \pi_1(X) \longrightarrow [S^1, X]$$
   which sends based homotopy class of maps to unbased homotopy classes of maps. Recall from MA 651 that if X is a based space, then for based loops α and β in $\pi_1(X)$, we have $\Lambda(\alpha) = \beta$ if and only if α and β are conjugate in $\pi_1(X)$.
   For fixed $g \in G$, let $BG \xrightarrow{c_g} BG$ be the map induced by the conjugation $c_g \colon G \longrightarrow G$. Show that the induced map
   $$[S^1, BG] \xrightarrow{(c_g)_*} [S^1, BG]$$
   is the identity map on unbased homotopy classes.

   (b) Recalling that BG has a CW structure with a single 0-cell, conclude that the composition $\mathrm{sk}_1 BG \hookrightarrow BG \xrightarrow{c_g} BG$ is homotopic to $\mathrm{sk}_1 BG \hookrightarrow BG$.

   (c) Use the fact that $\pi_n(BG) = 0$ for $n \geq 2$ to conclude that the homotopy in (b) can be extended to a homotopy $c_g \simeq \mathrm{id}$ as maps $BG \longrightarrow BG$.
   (Hint: Using the homotopy from (b), you now have a map
   $$(\partial I \times BG) \cup_{\partial I \times \mathrm{sk}_1 BG} (I \times \mathrm{sk}_1 BG) \longrightarrow BG. \qquad (\star)$$
   It remains to extend this to a map from $I \times BG$. In what dimensions are the cells of $I \times BG$ that are not in the source of $(\star)$?)