Math 751 – Fall 2020
Equivariant homotopy and cohomology
Worksheet 14

1. Let $S^{k\sigma}$ be the representation sphere for the k-dimensional sign representation over C_2. let C be a G-coefficient system. In this problem, you will show that $S^{k\sigma}/C_2 \simeq \Sigma\mathbb{RP}^{k-1}$ for $k \geq 1$.

   (a) (The case $k = 1$) Show directly that S^{σ}/C_2 is contractible. (Note that $\mathbb{RP}^0$ is a point.)

   (b) Show that $\mathbb{R}^{2\sigma}/C_2$ is equivalent to the cone $C(\mathbb{RP}^1)$ on $\mathbb{RP}^1$. (Hint: the representation $\mathbb{R}^{2\sigma}$ deformation retracts onto the unit disk.)

   (c) Conclude that $S^{2\sigma}/C_2$ is equivalent to $C(\mathbb{RP}^1) \cup_{\mathbb{RP}^1} C(\mathbb{RP}^1)$, also known as the unreduced suspension $S(\mathbb{RP}^1)$ of $\mathbb{RP}^1$.

   (d) Argue more generally that $S^{k\sigma}/C_2$ is homotopy equivalent to $S(\mathbb{RP}^{k-1})$ for $k \geq 2$.

   (e) Now it is a general fact (see chapter 0 of Hatcher) that if W is a CW complex and $A \subset W$ is a contractible subcomplex, then the quotient $W \longrightarrow W/A$ is a homotopy equivalence. In particular, if X is a CW complex with 0-cell x_0 and we take $A = \{x_0\} \times I \subset S(X)$, then the quotient map $S(X) \longrightarrow \Sigma X$ from the unreduced suspension to the (reduced) suspension is a homotopy equivalence.

2. Follow the approach of Example 2.2.12 to compute the $RO(C_3)$-graded cohomology groups

$$\mathrm{H}_{C_3}\underline{\mathbb{F}_3}^{\star}(*)$$

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