Math 751 – Fall 2020
Equivariant homotopy and cohomology
Worksheet 7

1. In this problem, you will show that $I(G)/I(G)^2$ is isomorphic to the abelianization of G.

   (a) Show that the function $\varphi\colon G \longrightarrow I(G)/I(G)^2$ given by $\varphi(g) = g - e$ is a homomorphism. Note that the group operation in G is written as multiplication, whereas the group operation in $I(G)/I(G)^2$ is addition.

   Hint: Consider the element $(g-e)(h-e) \in I(G)^2$.

   (b) Show that the commutator subgroup $[G,G] \leq G$ is in the kernel of φ.

   (c) Define $\lambda\colon I(G) \longrightarrow G_{ab} = G/[G,G]$ by $\lambda(g-e) = \overline{g}$. Why is this a well-defined homomorphism? Show that $\lambda\left(I(G)^2\right) = 0$.

   (d) Conclude that $I(G)/I(G)^2 \cong G_{ab}$.

2. Any short exact sequence $M \hookrightarrow N \twoheadrightarrow P$ induces a long exact sequence in group homology

$$\longrightarrow \mathrm{H}_n(G;M) \longrightarrow \mathrm{H}_n(G;N) \longrightarrow \mathrm{H}_n(G;P) \longrightarrow \mathrm{H}_{n-1}(G;M) \longrightarrow$$

   Show that the long exact sequence in homology of the group C_2 arising from the short exact sequence of C_2-modules

$$\mathbb{Z} \xrightarrow{e+g} \mathbb{Z}[C_2] \twoheadrightarrow \mathbb{Z}_{\mathrm{sgn}}$$

   is compatible with the answers given in Examples 1.3.4 and 1.3.5 in the notes.

3. Compute the cohomology groups $\mathrm{H}^*(C_2;\mathbb{Z}_{\mathrm{sgn}})$.