Steenrod Algebra Suminar: Construction of Steenrod Operations

Bertrand Guillou

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We will be discussing cohomology operations for cohomology with coefficients in $\mathbb{F}_p$, for p a prime. A cohomology operation of degree i is a natural transformation

$$\mathrm{H}^n(X;\mathbb{F}_p) \longrightarrow \mathrm{H}^{n+i}(X;\mathbb{F}_p).$$

An obvious example is the identity transformation, of degree 0. Another important example is the pth power map $x \mapsto x^p$. Note that this is an additive operation since $(x+y)^p \equiv x^p + y^p \pmod{p}$. Yet another example is the Bockstein homomorphism

$$\beta : \mathrm{H}^n(X;\mathbb{F}_p) \longrightarrow \mathrm{H}^{n+1}(X;\mathbb{F}_p).$$

This is just the connecting homomorphism in the long exact sequence in cohomology arising from the short exact sequence in coefficients

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \longrightarrow 0.$$

1. Basic properties

We will focus on the so-called Steenrod operations. They look a little different depending on whether p is odd or 2, so we list the properties in the two cases separately.

Theorem. *Let p be odd. There are cohomology operations*

$$P^i : \mathrm{H}^n(X;\mathbb{F}_p) \longrightarrow \mathrm{H}^{n+2i(p-1)}(X;\mathbb{F}_p)$$

of degree $2i(p-1)$ satisfying the following properties.

1. *P^0 is the identity*
2. *If $n = 2i$, then $P^i(x) = x^p$*
3. *(Instability) If $n < 2i + \varepsilon$, then $\beta^\varepsilon P^i(x) = 0$ for $\varepsilon \in \{0, 1\}$.*
4. *(Stability) $P^i(\Sigma x) = \Sigma P^i(x)$*
5. *(Cartan formula) $P^k(xy) = \sum_{i+j=k} P^i(x)P^j(y)$ and $\beta(xy) = \beta(x)y + (-1)^{|x|}x\beta(y)$*

6. *(Adem relations)If $\ell < pk$ then*

$$P^\ell P^k = \sum_{i=0}^{\lfloor \ell/p \rfloor} (-1)^{\ell+i} \binom{(p-1)(k-i)-1}{\ell - pi} P^{\ell+k-i} P^i,$$

and if $\ell < pk+1$ then

$$\begin{aligned} P^\ell \beta P^k = & \sum_{i=0}^{\lfloor \ell/p \rfloor} (-1)^{\ell+i} \binom{(p-1)(k-i)}{\ell - pi} \beta P^{\ell+k-i} P^i \\ & + \sum_{i=0}^{\lfloor (\ell-1)/p \rfloor} (-1)^{\ell+i-1} \binom{(p-1)(k-i)-1}{\ell - pi - 1} P^{\ell+k-i} \beta P^i, \end{aligned}$$

For the prime $p = 2$, we have operations

$$\mathrm{Sq}^i : \mathrm{H}^n(X; \mathbb{F}_2) \longrightarrow \mathrm{H}^{n+i}(X; \mathbb{F}_2)$$

satisfying analogous conditions, with the following changes.

1. We add that $\mathrm{Sq}^1 = \beta$
2. $\mathrm{Sq}^n(x) = x^2$ (for any n)
3. $\mathrm{Sq}^i(x) = 0$ if $i > n$
4. The Cartan formula reads $\mathrm{Sq}^k(xy) = \sum_{i+j=k} \mathrm{Sq}^i(x)\,\mathrm{Sq}^j(y)$
5. The Adem relations are that, for $\ell < 2k$,

$$\mathrm{Sq}^\ell\,\mathrm{Sq}^k = \sum_{i=0}^{\lfloor \ell/2 \rfloor} \binom{k-i-1}{\ell-2i} \mathrm{Sq}^{\ell+k-i}\,\mathrm{Sq}^i$$

2. Outline

The homotopical construction of Steenrod operations involves two main steps:

(i) Construct the external reduced power operations

$$P : \mathrm{H}^{2n}(X; \mathbb{F}_p) \to \mathrm{H}^{2np}(X \times B\Sigma_p; \mathbb{F}_p).$$

(ii) Compute $\mathrm{H}^*(X \times B\Sigma_p; \mathbb{F}_p)$ in terms of $\mathrm{H}^*(X; \mathbb{F}_p)$.

We discuss step (ii) today, and we will handle (i), and show how (i) and (ii) give the operations, next time.

3. The cohomology of $B\Sigma_p$

Let G be a finite group, and suppose given a subgroup $H \leq G$. Then, using suitable models for BH and BG, there is a covering map $BH \longrightarrow BG$ of degree $|H : G|$. In cohomology, any covering admits a "transfer map" $\tau : \mathrm{H}^*(BH) \longrightarrow \mathrm{H}^*(BG)$ such that the composition

$$\mathrm{H}^*(BG) \longrightarrow \mathrm{H}^*(BH) \xrightarrow{\tau} \mathrm{H}^*(BG)$$

is multiplication by the degree of the covering. In particular, if H is a p-Sylow subgroup of G and we are working with $\mathbb{F}_p$ as our coefficients, the composition is an isomorphism, so that $\mathrm{H}^*(BG)$ maps injectively into $\mathrm{H}^*(BH)$. In fact, one can identify the image. Let $N_G(H)$ be the normalizer of H in G. This acts on H by conjugation and thus acts on the space BH and also on the cohomology ring $\mathrm{H}^*(BH)$. The inner conjugation action of H on BH is null-homotopic, so the quotient *Weyl group* $W_G(H) := N_G(H)/H$ acts on $\mathrm{H}^*(BH)$, and it turns out that

$$\mathrm{H}^*(BG) = \mathrm{H}^*(BH)^{W_G(H)}.$$

Specializing now to $G = \Sigma_p$, the Sylow p-subgroup is C_p. We know that

$$\mathrm{H}^*(BC_p; \mathbb{F}_p) \cong \mathbb{F}_p[u, v]/(u^2 = 0, \beta(u) = v), \qquad |u| = 1, |v| = 2.$$

Recall that the units $\mathbb{F}_p^\times$ is a cyclic group C_{p-1}. Let $j \in \mathbb{F}_p$ be a generator for $\mathbb{F}_p^\times$. Then multiplication by j in $\mathbb{F}_p$ corresponds to an element $\lambda \in \Sigma_p$ of order $p-1$, and let $\sigma \in \Sigma_p$ correspond to addition by 1 in $\mathbb{F}_p$. Then σ generates a Sylow subgroup C_p, and it turns out that $N_{\Sigma_p}(C_p) = \langle \sigma, \lambda \rangle$. It follows that $W_{\Sigma_p}(C_p) \cong C_{p-1}$. The generator λC_p of $W_{\Sigma_p}(C_p)$ acts on $\mathbb{F}_p$ as the element λ, and it follows that the induced action on the class u is multiplication by j. Since $v = \beta(u)$, the same is true for v. More generally, the action on a class $u^\varepsilon v^i$ is by $j^{\varepsilon+i}$. So a class in $\mathrm{H}^*(BC_p)$ will be fixed by $W_{\Sigma_p}(C_p)$ if and only if it is a scalar multiple of either $v^{(p-1)m}$ or $uv^{(p-1)m-1}$ for some m. It follows that

$$\mathrm{H}^*(B\Sigma_p; \mathbb{F}_p) \cong \mathbb{F}_p[w, z]/(w^2 = 0, \beta(w) = z), \qquad |w| = 2(p-1) - 1, |z| = 2(p-1).$$