Steenrod Algebra Suminar: Construction of Steenrod Operations, Part 2

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Last time, we began the discussion of the construction of Steenrod operations. Our first main focus today will be the construction of the "External reduced power operation"

$$P : \mathrm{H}^{2n}(X;\mathbb{F}_p) \to \mathrm{H}^{2np}(X \times B\Sigma_p;\mathbb{F}_p).$$

1. Review of homotopy orbits

In order to define the external reduced power map, we will need to discuss the homotopy orbit construction.

Let G be a finite group. Recall that we write BG for a space $K(G,1)$. We write EG for a universal cover of BG. Then EG is contractible, and G acts *freely* (through deck transformations) on EG. More generally, if W is *any* G-space which is contractible and on which G acts freely, one tends to write EG for W, and the orbit space W/G can be seen to be a $K(G,1)$, so that we may write $BG = W/G$.

If Y is any G-space, we can think of G as acting "diagonally" on $EG \times Y$. That is, $g\cdot(w,y) = (g\cdot w, g\cdot y)$. We write $EG \times_G Y$ for the quotient by the G-action. This is sometimes called the *Borel construction on* Y or the *homotopy orbit space* (and written Y_{hG}). This has the feature that if $Y \longrightarrow Z$ is a G-equivariant map that is also a weak equivalence, then the induced map $Y_{hG} \longrightarrow Z_{hG}$ is also a weak equivalence.

Note: If we regard EG as a space with H-action, then it is a contractible space with a free H-action, so EG is a model for EH too. In particular, there is a natural quotient map $EG/H \longrightarrow EG/G$ that is a model for $BG \longrightarrow BG$. This is the model we had in mind in the last talk.

1.1. The external reduced power map

Assume given a factorization in the following diagram (all coefficients are assumed to be $\mathbb{F}_p$)

$$\begin{tikzcd}
\mathrm{H}^{2n}(X) \arrow[rr, dashed, "\Phi"] \arrow[dr] & & \mathrm{H}^{2pn}(E\Sigma_p \times_{\Sigma_p} X^p) \arrow[d] \\
& \mathrm{H}^{2pn}(X^p) \arrow[r, "\cong"'] & \mathrm{H}^{2pn}(E\Sigma_p \times X^p)
\end{tikzcd}$$

The diagonal arrow is the pth power map, and the vertical arrow is induced by the quotient map. The diagonal map $X \xrightarrow{\Delta} X^p$ is Σ_p equivariant and so gives rise to a map

$$E\Sigma_p \times_{\Sigma_p} X \cong B\Sigma_p \times X \longrightarrow E\Sigma_p \times_{\Sigma_p} X^p.$$

Composing the induced map in cohomology with the map Φ above produces the external reduced power map

$$\mathrm{H}^{2n}(X) \xrightarrow{\Phi} \mathrm{H}^{2pn}(E\Sigma_p \times_{\Sigma_p} X^p) \longrightarrow \mathrm{H}^{2pn}(B\Sigma_p \times X).$$

It remains to define the map Φ. Note that the Yoneda lemma implies that it suffices to do this in the case $X = K(\mathbb{F}_p, 2n)$, in which case we are looking for a particular map

$$E\Sigma_p \times_{\Sigma_p} K(\mathbb{F}_p, 2n)^p \longrightarrow K(\mathbb{F}_p, 2pn).$$

The existence of this map is a strengthening of the statement that the multiplication of (even-dimensional) classes is homotopy commutative. It now suffices to find a free Σ_p-space W that is contractible and a Σ_p-equivariant map

$$W \times K(\mathbb{F}_p, 2n)^p \longrightarrow K(\mathbb{F}_p, 2pn)$$

where Σ_p acts trivially on the right.

Define a new space S (S stands for Segal) by

$$S = \prod_n K(\mathbb{F}_p, 2n).$$

It will be important for the following that we take a model $\tilde{\mathbb{F}}_p(S^{2n})$ for $K(\mathbb{F}_p, 2n)$ that is a topological abelian group (really topological $\mathbb{F}_p$-vector space). Then S becomes a graded topological $\mathbb{F}_p$-vector space. We then define, for each $j \geq 0$, a Σ_j-space $M(j) \subseteq \mathrm{Map}(S^j, S)$ as the space of *multilinear graded* maps $S^j \longrightarrow S$. Finally, we define $\mathscr{O}(j) \subseteq M(j)$ to be the component of the cup product.

Evidently this space splits up as a product of spaces $\mathscr{O}(j)[n_1, \dots, n_j]$ parametrizing multilinear maps

$$K(\mathbb{F}_p, 2n_1) \times \cdots \times K(\mathbb{F}_p, 2n_j) \to K\big(\mathbb{F}_p, 2(n_1 + \cdots + n_j)\big)$$

inducing the cup product.

Claim: $\mathscr{O}(j)$ is contractible. It suffices to show that any $\mathscr{O}(j)[n_1, \dots, n_j]$ is contractible, though for simplicity of notation we will only consider $\mathscr{O}[n, \dots, n]$. Indeed, multilinear maps

$$K(\mathbb{F}_p, 2n)^j \to K(\mathbb{F}_p, 2jn)$$

correspond to linear maps

$$K(\mathbb{F}_p, 2n)^{\otimes j} \to K(\mathbb{F}_p, 2jn).$$

But note that

$$K(\mathbb{F}_p, 2n)^{\otimes j} = \tilde{\mathbb{F}}_p(S^{2n})^{\otimes j} \cong \tilde{\mathbb{F}}_p\big((S^{2n})^{\wedge j}\big) \cong \tilde{\mathbb{F}}_p S^{2jn}.$$

Thus

$$\begin{aligned}\mathrm{Map}_{multilin}\big(K(\mathbb{F}_p,2n)^j,K(\mathbb{F}_p,2jn)\big) &\cong \mathrm{Map}_{TopVect}\big(K(\mathbb{F}_p,2n)^{\otimes j},K(\mathbb{F}_p,2jn)\big)\\ &\cong \mathrm{Map}_{TopVect}\big(K(\mathbb{F}_p,2jn),K(\mathbb{F}_p,2jn)\big)\\ &\cong \mathrm{Map}_*\big(S^{2jn},K(\mathbb{F}_p,2jn)\big)\simeq \mathbb{F}_p\end{aligned}$$

Our space $\mathscr{O}[n,\dots,n]$ is clearly a component of this mapping space and is therefore contractible.

Lemma (Kozlowsi)**.** *The group Σ_j acts freely on $\mathscr{O}(j)$.*

Thus the space $\mathscr{O}(p)$ is a model for $E\Sigma_p$. Since $\mathscr{O}(p)$ is a subspace of $\mathrm{Map}_*(S^p,S)$, there is a natural Σ_p-equivariant map

$$\mathscr{O}(p)\times S^p\longrightarrow S.$$

By construction, it restricts to give an equivariant map

$$\mathscr{O}(p)\times K(\mathbb{F}_p,2n)^p\longrightarrow K(\mathbb{F}_p,2pn).$$

and we get the desired map.

2. Putting it all together

Combining the computation from last time with the Kunneth isomorphism gives the computation

$$\mathrm{H}^*(B\Sigma_p\times X;\mathbb{F}_p)\cong \mathrm{H}^*(X;\mathbb{F}_p)p[w,z]/(w^2=0,\beta(w)=z),\qquad |w|=2(p-1)-1,|z|=2(p-1).$$

Then if P denotes the external reduced power operation

$$\mathrm{H}^{2n}(X)\xrightarrow{P}\mathrm{H}^{2pn}(B\Sigma_p\times X),$$

we can express the class $P(x)$ as a polynomial in the classes w and z. We define classes $P^i(x)$ and $B^i(x)$ as the coefficients:

$$\begin{aligned}P(x)=P^n(x)+B^{n-1}(x)w+P^{n-1}(x)z+B^{n-2}(x)wz+P^{n-2}(x)z^2+\dots\\ +B^1(x)wz^{n-2}+P^1(x)z^{n-1}+B^0(x)wz^{n-1}+P^0(x)z^n.\end{aligned}$$

Technically, the above only defines the reduced power on even dimensional classes. For $y\in\mathrm{H}^{2n+1}(X)$, we may suspend to get an even dimensional class $\Sigma y\in\mathrm{H}^{2(n+1)}(\Sigma X)$. Then $P^i(\Sigma y)$ is a well-defined class in $\mathrm{H}^{2(n+1)+2i(p-1)}(\Sigma X)$, corresponding to a well-defined class in $\mathrm{H}^{2n+1+2i(p-1)}(X)$. We define $P^i(y)$ to be this class.

Note that from the above construction it is fairly easy to see that $P^n(x)=x^p$ if $|x|=2n$, but it is nontrivial to check that $P^0(x)=x$. If one first shows that $\beta P=0$, it is then also easy to see, using that $\beta(w)=z$, that the class $B^i(x)$ is none other than $\beta P^i(x)$; in particular, $B^0(x)=\beta(x)$. Moreover, by definition, there are no operations P^i defined on x if $i>n$.