HOMEWORK I
MATH 527
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Problem 1. Suppose given based maps $S^n \overset{f}{\underset{g}{\rightrightarrows}} X \xrightarrow{h} Y$ with $n \geq 1$. Show that

$$[h] \circ ([f] + [g]) = [h] \circ [f] + [h] \circ [g]$$

in $\pi_n(Y)$. Conclude that $h_* : \pi_n(X) \longrightarrow \pi_n(Y)$ is a homomorphism.

Problem 2. An H-space (named after Hopf) is a based space (X, e) equipped with a multiplication $\mu : X \times X$ for which e serves as a unit. Show that if X is an H-space, then $\pi_1(X, e)$ is abelian.

Problem 3. For each $n \geq 0$, consider S^n as a subspace of S^{n+1} via the embedding

$$(x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_n, 0).$$

Let S^∞ be the union $\bigcup_n S^n$. Show that S^∞ is contractible.

Problem 4. Show that a based homotopy $h : X \times I \longrightarrow Y$ is the same as a based map $\overline{h} : X \wedge (I_+) \longrightarrow Y$.

Problem 5. Show that for spaces X and Y, there is a natural isomorphism

$$(X_+) \wedge (Y_+) \cong (X \times Y)_+.$$

Problem 6. Show that if X and Y are based spaces such that Y is based contractible, then $X \wedge Y$ is also (based) contractible.

Problem 7. Let **Set** denote the category whose objects are sets and whose morphisms are functions. There is a forgetful functor $U : \textbf{Top} \longrightarrow \textbf{Set}$ which takes a space X to the underlying set of X (so it forgets about the topology on X).

(i) Show that there is a functor $F : \textbf{Set} \longrightarrow \textbf{Top}$ such that F is *left* adjoint to the functor U.

(ii) Show that there is a functor $G : \textbf{Set} \longrightarrow \textbf{Top}$ such that G is *right* adjoint to the functor U.

(iii) Let $L : \mathscr{C} \rightleftarrows \mathscr{D} : R$ be an adjoint pair of functors (so L is the left adjoint, R the right adjoint). Suppose that products exist in $\mathscr{D}$ and also in $\mathscr{C}$. Show that R necessarily preserves products. That is, show that there is a natural isomorphism $R(X \times_{\mathscr{D}} Y) \cong R(X) \times_{\mathscr{C}} R(Y)$.

Together with the dual to (iii), which says that left adjoints always preserve coproducts, the above statements show that the categorical products and coproducts of topological spaces must have the same underlying sets as the products and coproducts in sets.

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