Problem 1. (The Hopf fibration) Let $S^3 \subseteq \mathbb{C}^2 \cong \mathbb{R}^4$ be the unit sphere. Stereographic projection provides an identification $S^2 \cong \mathbb{C}P^1$. The composition $S^3 \to \mathbb{C}^2 - \{0\} \to \mathbb{C}P^1$, where the second map is the natural quotient map, is called the “Hopf map” and is usually denoted by $\eta$.

(i) Show that $\eta^{-1}(1) \cong S^1$.

(ii) Let $F(\eta)$ be the homotopy fiber of $\eta : S^3 \to S^2$. There is a natural map $\eta^{-1}(1) \to F(\eta)$ which assigns to any point $x$ the pair $(c_1, x)$, where $c_1$ is the constant path at 1 in $S^2$. Show that this map is a homotopy equivalence.

(iii) Use the long exact sequence in homotopy for the map $\eta$ to show that $\pi_3(S^2) \cong \mathbb{Z}$ generated by the element $[\eta]$, and that $\pi_n(S^3) \cong \pi_n(S^2)$ for $n \geq 3$. You may assume that $\pi_3(S^3) \cong \mathbb{Z} \cong \pi_2(S^2)$ and that $S^3$ and $S^2$ have no nontrivial lower homotopy groups.

Problem 2. (Whitehead products) For each $n \geq 0$, equip $S^n$ with the CW structure having one cell in dimension 0 and one cell in dimension $n$. Then the product $S^p \times S^q$ has four cells, and if $p, q \geq 1$ the attaching map for the top cell takes the form $S^{p+q-1} \to S^p \vee S^q$. For any based space $X$, the resulting map

$$\pi_p(X) \times \pi_q(X) \to \pi_{p+q-1}(X)$$

is called the Whitehead product.

(i) When $p = q = 1$, the Whitehead product takes the form $\pi_1(X) \times \pi_1(X) \to \pi_1(X)$. What is this map? More generally, what is another description of this map when $p = 1$ (but $q$ is arbitrary)?

(ii) Show that a path-connected H-space has trivial Whitehead products.

Problem 3. Recall that if $X$ is a based space, then $\Omega X$ denotes the space of based loops in $X$.

(i) Show that concatenation of loops gives $\Omega X$ the structure of a homotopy associative H-space

(ii) (Moore loops) Let $\Omega_M(X)$ denote the space of “Moore” based loops. Such a loop consists of a pair $(\gamma, r)$, where $r \geq 0$ and $\gamma$ is a based loop in $X$ thought of as a map $\gamma : [0, r] \to X$. Note that if $r = 0$, then $\gamma$ is necessarily the constant loop. The operation

$$(\beta, r) \ast (\gamma, s) = (\beta \ast \gamma, r + s)$$

defines a strictly associative operation on the space of Moore loops.

Show that the projection map $\Omega_M(X) \to \Omega(X)$ which drops the parameter $r$ is a homotopy equivalence.

Problem 4. Let $i : A \to X$ be a based map. Recall that the map $p : F(i) \to A$ is defined by $p(\gamma, a) = a$. Show that $\Omega X \simeq F(F(i) \xrightarrow{p} A)$.

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