Problem 1. (i) Show that if $E \xrightarrow{p} B$ is a fibration and $Z$ is any space, then $\text{Map}(Z, E) \rightarrow \text{Map}(Z, B)$ is also a fibration.

(ii) Show that if $A \xrightarrow{i} X$ is a cofibration and $Z$ is any space, then $\text{Map}(X, Z) \rightarrow \text{Map}(A, Z)$ is a fibration.

(iii) Show that if $A \xrightarrow{i} X$ is a cofibration and $E \xrightarrow{p} B$ is a fibration, then the natural map

$$\text{Map}(X, E) \rightarrow \text{Map}(A, E) \times_{\text{Map}(A, B)} \text{Map}(X, B)$$

is a fibration.

**Hint:** The key is to show that one can find a lift in a diagram of the form

$$
\begin{array}{ccc}
M(j) & \xrightarrow{E} & \rightarrow X \\
\downarrow & \downarrow \downarrow & \downarrow \downarrow \\
X \times I & \xrightarrow{r} & \rightarrow M(j)
\end{array}
$$

This can be done by establishing that the inclusion $M(j) \hookrightarrow X \times I$ is a retract of the inclusion $i_0 : X \times I \hookrightarrow (X \times I) \times I$. One argument for the latter fact is as follows.

We have previously shown that if $j : A \rightarrow X$ is a cofibration, then the inclusion $M(j) \hookrightarrow X \times I$ admits a retraction $r : X \times I \rightarrow M(j)$. This can be improved: define a homotopy $h : X \times I \times I \rightarrow X \times I$ by

$$h(x, t, s) = (r_1(x, t(1-s)), st + (1-s)r_2(x, t))$$

($r_1$ and $r_2$ are the two components of the map $r$). Then $h$ defines a homotopy from $r$ to the identity, $\text{rel } M(j)$. Thus $M(j)$ is a deformation retract of $X \times I$.

By Theorem 6.4 of [May], there is a continuous $u : X \rightarrow I$ such that $u^{-1}(0) = A$. Define $v : X \times I \rightarrow I$ by $v(x, t) = t \cdot u(x)$, and note that $v^{-1}(0) = M(j)$. Define now a new homotopy $\Phi : X \times I \times I \rightarrow X \times I$ by

$$\Phi(x, t, s) = \begin{cases} 
h(x, t, s \cdot v(x, t)^{-1}) & s < v(x, t) \\
h(x, t, 1) = (x, t) & s \geq v(x, t)
\end{cases}$$

(you should convince yourself that $\Phi$ is continuous). Finally, define $\Lambda : X \times I \rightarrow X \times I \times I$ by $\Lambda(x, t) = (x, t, v(x, t))$. Check that the following diagram is a retract diagram

$$
\begin{array}{ccc}
M(j) & \xrightarrow{E} & X \times I \\
\downarrow & \downarrow r & \downarrow \downarrow \\
X \times I & \xrightarrow{\Lambda} & X \times I \times I
\end{array}
\xrightarrow{i_0} \begin{array}{ccc}
\downarrow & \downarrow \downarrow & \downarrow \downarrow \\
M(j) & \xrightarrow{\Phi} & X \times I
\end{array}
$$

Problem 2. (The Hopf invariant) Let $k \geq 2$ and let $f : S^{2k-1} \rightarrow S^k$ be a map. Then the cofiber $C(f)$ has a natural CW structure with cells in dimensions $0$, $k$, and $2k$. As $k \geq 2$, the cellular chain complex has trivial differentials, and the cohomology of $C(f)$ is $\mathbb{Z}$ in dimensions $0$, $k$, and $2k$. Let $x$ be the generator of $H^k(C(f))$ corresponding to the top cell of $S^k$, and let $y$ be the
generator of $H^{2k}(C(f))$ corresponding to the cell attached via $f$. Then $x^2 = h(f)y$ for some integer $h(f)$, which is called the Hopf invariant of the map $f$.

(i) Show that if $k$ is odd, then $h(f) = 0$.

(ii) Let $η : S^3 \to S^2$ be the Hopf map. Show that $h(η) = 1$.

(iii) Let $k = 2n$ and let $ι : S^{2n} \to S^{2n}$ be the identity map. Show that the Whitehead product $[ι, ι] \in π_{4n-1}(S^{2n})$ has Hopf invariant 2. (Hint: Use the diagram

$\begin{array}{ccc}
S^{2n} ∨ S^{2n} & \to & S^{2n} \times S^{2n} \\
↓ & & ↓ \\
S^{2n} & \to & C([ι, ι]) \to S^{4n}
\end{array}$

to compute $h([ι, ι])$.

By part (iii) and the fact that the Hopf invariant is a homomorphism (see Hatcher, 4B.1), for each $n$, the map $h : π_{4n-1}(S^{2n}) \to Z$ is either surjective or has image 2$Z$. In either case, we have a surjective homomorphism from $π_{4n-1}(S^{2n})$ onto an infinite cyclic group, and such a map always has a section. We conclude that $π_{4n-1}(S^{2n})$ always has a summand of $Z$. J. P. Serre proved that the complement is finite and that every other homotopy group of spheres is finite. Frank Adams proved that maps of Hopf invariant one exist only for $k = 2n = 2$, 4, or 8 (examples are the Hopf maps $η$, $ν$, and $σ$).

**Problem 3.** Let $η \in π_3(S^2)$ be the Hopf map and let $ι \in π_2(S^2)$ be correspond to the identity of $S^2$. Show that for any integers $a, b ∈ Z$ we have

$$(aι) \circ (bη) = a^2 bη.$$  

This shows that the map $π_2(S^2) \to π_3(S^2)$ induced by precomposition with $η$ is not a homomorphism.

(Hint: Consider a commutative diagram

$\begin{array}{ccc}
S^2 & \to & C(bη) \to S^4 \\
\downarrow aι & & \downarrow id \\
S^2 & \to & C(bη ∘ aι) \to S^4
\end{array}$

and the Hopf invariants.)