CLASS NOTES
MATH 527 (SPRING 2011)
WEEK 1

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1. Wed, Jan. 19

What is homotopy theory?

- classical homotopy theory
- simplicial homotopy (Goerss-Jardine)
- abstract homotopy theory (Quillen, Hovey, Dwyer-Spalinski)
- homological algebra
- stable homotopy theory
- equivariant homotopy theory

The first part of the course will be concerned with classical homotopy theory.

Definition 1.1. Given maps f and $g : X \longrightarrow Y$, a **homotopy** h between f and g is a map $h : X \times I \longrightarrow Y$ ($I = [0,1]$) such that $f(x) = h(x,0)$ and $g(x) = h(x,1)$. We say f and g are **homotopic** if there exists a homotopy between them (and write $h : f \simeq g$).

Proposition 1.2. *The property of being homotopic defines an equivalence relation on the set of maps $X \longrightarrow Y$.*

Proof. (Reflexive): Need to show $f \simeq f$. Use the **constant homotopy** defined by $h(x,t) = f(x)$ for all t.

(Symmetric): If $h : f \simeq g$, need a homotopy from g to f. Define $H(x,t) = h(x, 1-t)$ (reverse time).

(Transitive): If $h_1 : f_1 \simeq f_2$ and $h_2 : f_2 \simeq f_3$, we define a new homotopy h from f_1 to f_3 by the formula

$$h(x,t) = \begin{cases} h_1(x,2t) & 0 \leq t \leq 1/2 \\ h_2(x,2t-1) & 1/2 \leq t \leq 2. \end{cases}$$

■

We write $[X,Y]$ for the set of homotopy classes of maps $X \longrightarrow Y$.

Proposition 1.3. *(Interaction of composition and homotopy) Suppose given maps $X \xrightarrow{f} Y \xrightarrow{g} Z$. If $f \simeq f'$ and $g \simeq g'$ then $g \circ f \simeq g' \circ f'$.*

We will often choose to work with **based spaces**, that is, spaces X with a specified basepoint $x_0 \in X$. A based map $f : (X, x_0) \longrightarrow (Y, y_0)$ is simply a map such that $f(x_0) = y_0$. There is a corresponding notion of homotopy between based maps. A based homotopy $h : f \simeq g$ between based maps is simply a homotopy in the above sense such that for each $t \in I$, the map $h_t(x) = h(x,t) : X \longrightarrow Y$ is a based map. That is we require $h(x_0, t) = y_0$ for all t. We write $[X,Y]_*$ for the set of based homotopy classes of maps.

Example 1.4. You already know about the fundamental group $\pi_1(X,x)$ of a based space (X,x). This is simply the set $\pi_1(X,x) = [S^1, (X,x)]_*$. Here S^1 is the standard unit circle in $\mathbb{R}^2$, and the basepoint is usually taken to be the point $(1,0)$.

Definition 1.5. A map $f : X \longrightarrow Y$ is said to be a **homotopy equivalence** if there exists a map $g : Y \longrightarrow X$ and homotopies $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. We write $X \simeq Y$ if there is a homotopy equivalence between them and say X and Y are homotopy equivalent.

Example 1.6. For any X, the projection $X \times I \longrightarrow X$ is a homotopy equivalence.

Proposition 1.7. *The property of being homotopy equivalent defines an equivalence relation on spaces. Moreover, homotopy equivalences satisfy the "2-out-of-3" property.*

Proof. The 2-out-of-3 property says that if we are given maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ and we define $h = g \circ f$, then if two of the three maps f, g, and h are homotopy equivalences, then so is the third. Proposiiton 1.3 readily implies that if f and g are homotopy equivalences, then so is h. We will show that if f and h are homotopy equivalences, then so is g.

Let f' and h' be homotopy inverses for f and h, respectively. Consider the map $f \circ h' : Z \longrightarrow Y$. We claim that this is a homotopy inverse for g. First, $g \circ f \circ h' = h \circ h' \simeq \mathrm{id}_Z$ since h' is homotopy inverse to h. Second,

$$\begin{aligned} f \circ h' \circ g &\simeq f \circ h' \circ g \circ f \circ f' \\ &= f \circ h' \circ h \circ f' \\ &\simeq f \circ f' \simeq \mathrm{id}_Y. \end{aligned}$$

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Definition 1.8. Say a space X is **contractible** if it is homotopy equivalent to $*$, the one-point space. Say a map $f : X \longrightarrow Y$ is **null-homotopic** (or simply **null**) if it is homotopic to a constant map.

Example 1.9. The spaces I, D^n, and $\mathbb{R}^n$ are contractible.

Proposition 1.10. *If X is any space and Y is contractible, then the projection $X \times Y \longrightarrow X$ is a homotopy equivalence.*

Proposition 1.11. *A space X is contractible if and only if the identity map $\mathrm{id}_X : X \longrightarrow X$ is null.*

2. Fri, Jan. 21

Proposition 2.1. *If $f : X \longrightarrow Y$ and either X or Y is contractible, then f is null-homotopic.*

Proposition 2.2. *If $f : S^n \longrightarrow Y$, then there is an extension $\tilde{f} : D^{n+1} \to Y$ if and only if f is null-homotopic.*

Proof. Proposition 2.1 above shows that if $\tilde{f}$ exists, then f must be null. Suppose now that we have a null homotopy $h : S^n \times I \longrightarrow Y$. Then the restriction of h to $S^n \times \{1\}$ is constant, so h factors through the space $S^n \times I/S^n \times \{1\}$. This space is homeomorphic to D^{n+1}! (Think of the time coordinate t as corresponding to $1 - r$, where r is the radius) ■

In fact, for general X, the construction $X \times I/X \times \{1\}$ is an important one. It is called the **cone** on X (or **mapping cone**) and denoted CX. The result above generalizes to the following:

Proposition 2.3. *If $f : X \longrightarrow Y$, then there is an extension $\tilde{f} : CX \longrightarrow Y$ if and only if f is null-homotopic.*

The spaces S^n and D^n will figure prominently in the rest of the course, so we mention now a few other models for these spaces:

- $S^n \cong D^n/S^{n-1}$

- As D^n is contractible, any other contractible space will do, but an often convenient choice is I^n (the n-fold product of I with itself)
- In the model I^n for D^n, the replacement for S^{n-1} is ∂I^n, the set of n-tuples $(t_1, \ldots, t_n)$ such that one of the coordinates t_i is either 0 or 1.
- $S^n \cong I^n/\partial I^n$
- Let $J^n \subset \partial I^n$ be the subset $\partial I^{n-1} \times I \cup I^{n-1} \times \{1\}$. Then $I^n/J^n \cong D^n$ and $\partial I^n/J^n \cong S^{n-1}$.

As an organizational principle, it is convenient to specify and analyze the categories in which we are working. On the one hand, we have the category **Top** of topological spaces and (continuous) maps. On the other hand, we are also interested in the category $\mathbf{Top}_*$ of based spaces and based maps.

Proposition 2.4. *The coproduct of X in Y in* **Top** *is given by their disjoint union $X \coprod Y$ and their product is given by the cartesian product $X \times Y$.*

The coproduct of (X, x_0) and (Y, y_0) in $\mathbf{Top}_*$ is given by the wedge $X \vee Y$, obtained from $X \coprod Y$ by imposing the relation $x_0 \sim y_0$. The (equivalence class of) x_0 in $X \vee Y$ is the basepoint. The product of (X, x_0) and (Y, y_0) in $\mathbf{Top}_*$ is again the cartesian product $X \times Y$, pointed at (x_0, y_0).

There is another important construction involving based spaces.

Definition 2.5. Given based spaces (X, x_0) and (Y, y_0), their **smash product** $X \wedge Y$ is defined to be the based space $(X \times Y)/(X \vee Y)$. The (class of the) point (x_0, y_0) is the basepoint.

It is tempting to think that $X \wedge Y$ is the categorical product of X and Y in $\mathbf{Top}_*$, but this is false. In general, there are not even well-defined "projection" maps $X \wedge Y \longrightarrow X$ or $X \wedge Y \longrightarrow Y$.

One reason to care about the smash product construction is the following.

Proposition 2.6. *For any m and $n \geq 0$, $S^m \wedge S^n \cong S^{m+n}$.*

Proof. It is most convenient to prove this using the model $S^n := I^n/\partial I^n$. Then

$$\begin{aligned} S^m \wedge S^n &= (S^m \times S^n)/(S^m \vee S^n) \\ &= (I^m \times I^n)/[(\partial I^m \times I^n) \cup (I^m \times \partial I^n)] \\ &\cong I^{m+n}/\partial I^{m+n} = S^{m+n}. \end{aligned}$$

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Given any based space (X, x_0), we can simply forget about the basepoint and consider the underlying space. This defines a functor $u : \mathbf{Top}_* \longrightarrow \mathbf{Top}$. There is also a functor in the other direction: given any space X, we can define a based space X_+ by adjoining a disjoint basepoint to X.

Proposition 2.7. *The functor $X \mapsto X_+$ is left adjoint to $u : \mathbf{Top}_* \longrightarrow \mathbf{Top}$.*

This means that we have a natural bijection

$$\mathbf{Top}_*(X_+, (Y, y_0)) \cong \mathbf{Top}(X, u(Y, y_0))$$

for any space X and based space (Y, y_0).

In addition to the categores **Top** and $\mathbf{Top}_*$, we will also be interested in the associated homotopy categories. We let **Ho**(**Top**) denote the category whose objects are spaces and whose set of morphisms from X to Y is the set of homotopy classes of maps. (**Check that this really defines a category!**)

What are the "isomorphisms" in **Ho**(**Top**)? An isomorphism $\alpha : X \longrightarrow Y$ is a homotopy class of maps such that there is a homotopy class of maps in the other direction and such that both compositions are in the homotopy class of the corresponding identity maps. This is precisely a homotopy equivalence. In fact, something stronger is true, as we will see next time.