1. Mon, Apr. 4

Now that we understand cohomology as classified by Eilenberg-Mac Lane spaces, we can discuss a process dual to that of CW approximation.

Let $X$ be a space. A **Postnikov tower** for $X$ is a (homotopy commutative) diagram

\[ \begin{array}{ccccccc}
& & & & & & \vdots \\
& & & & & \downarrow & \\
P_3(X) & & & & & \downarrow_{q_2} & \\
& & f_2 & \searrow & & \downarrow & \\
X & \rightarrow & P_1(X) & \rightarrow & & P_0(X) & \\
& f_0 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
& P_1(X) & \rightarrow & P_0(X) & \rightarrow & & \end{array} \]

such that

1. For each $i$, the map $\pi_j(f_i) : \pi_j(X) \longrightarrow \pi_j(P_i(X))$ is an isomorphism for $j \leq i$
2. $\pi_j(P_i(X)) = 0$ for $j > i$.

Note that it follows that the map $\pi_j(P_{i+1}(X)) \xrightarrow{\pi_j(q_i)} \pi_j(P_i(X))$ is an isomorphism for $j \neq i + 1$ and is the zero map when $j = i + 1$. We also deduce that the fiber

\[ F(q_i) \longrightarrow P_{i+1}(X) \longrightarrow P_i(X) \]

is an Eilenberg-Mac Lane space $F(q_i) \simeq K(\pi_{i+1}(X)), i + 1$.

We give two slightly different constructions of Postnikov towers.

**Version I:** (small) Choose a set of generators for $\pi_i+1(X)$, and for each chosen generator, we attach an $i$+2-cell to $X$ to kill that element of homotopy. We call the result $P_i(X)_{i+1} = X = P_i(X)$, so we may take $q_i$ on the $(i+1)$-skeleton.
to be the identity map. On the \((i + 2)\)-skeleton, the map \(P_{i+1}(X)_{i+2} = X \to P_i(X)_{i+2}\) is the inclusion. The \((i + 3)\)-skeleton is the first one requiring any work. Let \(\Phi : D^{i+3} \to P_{i+1}(X)_{i+3}\) be a cell with attaching map \(\varphi : S^{i+2} \to P_{i+1}(X)_{i+2} = X\).

**Version I:** The element \(q_i \circ \alpha\) is null in \(\pi_{i+2}(P_i(X)_{i+2})\), so we may take any extension over the disk as the definition of \(q_i\) on \(\Phi\).

**Version II:** The composite \(q_i \circ \alpha\) is the attaching map for a \((i + 3)\)-cell of \(P_i(X)_{i+2}\), so we can use the inclusion of this cell as the definition of \(q_i\) on the cell \(\Phi\). Note that no choice was made here.

The same argument allows us to define \(q_i\) on the higher skeleta. Using either definition, we get \(q_i \circ f_i \longrightarrow = f_i\).

**Functoriality.** Suppose given a maps \(g : X \longrightarrow Y\) and \(h : Y \longrightarrow Z\).

Using Version I, we can build maps \(g_i : P_i(X) \longrightarrow P_i(Y)\) for each \(i\), by making a choice of null homotopy for each cell. But then this may not produce a map of towers. That is, the best we can say is that we have a homotopy \(g_i \circ f_i^X \simeq f_i^Y \circ g_{i+1}\). Similarly, we can build maps \(h_i : P_i(Y) \longrightarrow P_i(Z)\) by making many choices, and we can also build maps \((h \circ g)_i : P_i(X) \longrightarrow P_i(Z)\). But in general, the best we can hope for is a homotopy \((h \circ g)_i \simeq h_i \circ g_i\).

On the other hand, using Version II produces much better results. The map \(g_i : P_i(X) \longrightarrow P_i(Y)\) defines itself, and we get a map of towers, so that \(g_i \circ f_i^X = f_i^Y \circ g_{i+1}\). Similarly, we get maps \(h_i\) and \((hg)_i\), without making any choices, and the functoriality equation \((hg)_i \simeq h_i \circ g_i\) holds.

**Question.** To what extent can we recover \(X\) from its Postnikov tower? Is \(X\) equivalent to \(\lim P_i(X)\)?

**Proposition 1.1.** For any sequence of fibrations \(\ldots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0\), the canonical maps \(\lambda_i : \pi_i(\lim X_n) \longrightarrow \lim \pi_i(X_n)\) are surjective. Moreover, \(\lambda_i\) is injective if \(\pi_{i+1}(X_n) \longrightarrow \pi_{i+1}(X_{n-1})\) is surjective for sufficiently large \(n\).

**Proof.** (Surjectivity) Let \((\gamma_n) \in \lim \pi_i(X_n)\). So for all \(n\), we have a homotopy \(h_n : q_{n+1} \circ \gamma_{n+1} \simeq \gamma_n\). Since \(q_{n+1}\) is a fibration by assumption, we can lift the diagram \(\lambda_i : \pi_i(\lim X_n) \longrightarrow \lim \pi_i(X_n)\) are surjective. Moreover, \(\lambda_i\) is injective if \(\pi_{i+1}(X_n) \longrightarrow \pi_{i+1}(X_{n-1})\) is surjective for sufficiently large \(n\).

**Invertivity** Suppose now that \(\pi_{i+1}(q_k)\) is surjective for \(k > n\). Suppose given \(S^i \longrightarrow \lim X_n\) such that \(\lambda_i(\gamma) = 0\). Thus \(S^i \longrightarrow \lim X_n \longrightarrow X_n\) is null for each \(n\). Let \(\delta_n : D^{i+1} \longrightarrow X_n\) be a null-homotopy for the composite. We then get a lift in the diagram

Thus \(h_1\) has image in \(F(q_{n+1})\). Since \(\pi_{i+1}(q_{n+1})\) is surjective, the map \(\pi_i(F(q_{n+1})) \longrightarrow \pi_i(X_{n+1})\) is injective. But \(q_{n+1} \circ h_1\) is constant, so \(h_1\) is null in \(\pi_i(F(q_{n+1}))\). Writing

\[
D^{i+1} \simeq (S^i \times I) \cup_{S^i \times \{1\}} D^{i+1} \times \{1\},
\]
we may glue the homotopy $h$ to a null-homotopy for $h_1$ to define a map $\delta_{n+1} : D^{i+1} \longrightarrow X_{n+1}$ lying over $\delta_n$. In the same way, we obtain a null-homotopy $\delta_{n+2} : D^{i+1} \longrightarrow X_{n+2}$ lying over $\delta_{n+1}$, and so on. Taking all of the maps $\delta_k$ together provides a map $D^{i+1} \longrightarrow \lim X_n$ exhibiting a null-homotopy for $\gamma$. ■

2. Wed., Apr. 6

By the result at the end of last time, if we replace all of the maps in the Postnikov tower by fibrations, then the natural map $f : X \longrightarrow \lim P_i(X)$ is a weak equivalence. Incidentally, the limit of a tower obtained by first replacing all of the maps by fibrations is called the homotopy limit of the tower. So we have $X \simeq \text{holim} P_i(X)$.

As we said earlier, Postnikov towers should be viewed as dual to CW complexes. The sort of duality we have in mind is sometimes called Eckmann-Hilton duality. CW complexes are built out of cells (spheres) using cofiber sequences, and Postnikov towers are built out of cocells (Eilenberg-Mac Lane spaces) using "cocells".

In what sense are spheres dual to Eilenberg-Mac Lane spaces? A space $X$ is equivalent to $K(A,n)$ if and only if

$$[S^i, X]_* \cong \begin{cases} A & i = n \\ 0 & i \neq n. \end{cases}$$

Similarly, a (simply connected) space $X$ is equivalent to $S^n$ if and only if

$$[X, K(A,n)]_* \cong \begin{cases} A & i = n \\ 0 & i \neq n. \end{cases}$$

**Example 2.1.** (1) $X = K(A,n)$. Then of course $P_1(X) = *$ if $i < n$, and $P_i(X) = X$ for $i \geq n$.

(2) $X = K(A,n) \times K(B,m)$ with $m < n$. Then similarly $P_1(X) = *$ for $i < m$. We have $P_1(X) = K(B,M)$ for $m \leq i < n$ and $P_i(X) = X$ for $i \geq n$.

Granted, these are not terribly exciting examples. Unfortunately, it is difficult to find examples of spaces with Postnikov towers that are simultaneously more interesting and completely understood, for the following reason. Any simply connected finite complex $X$ that is not contractible has infinitely many nontrivial homotopy groups. So any finite Postnikov stage must be an infinite complex.

We have discussed the fact that $X$ can be recovered from its Postnikov tower. What data is needed to build the tower? Certainly, knowledge of the homotopy groups is not enough. For the first Postnikov section $P_1(X)$, we only need to know the fundamental group (let’s assume $X$ is connected). But the space $P_2(X)$ sits in a fiber sequence $K(\pi_2(X), 2) \longrightarrow P_2(X) \longrightarrow P_1(X)$, and the groups $\pi_1(X)$ and $\pi_2(X)$ do not alone determine the space $P_2(X)$.

Suppose that the fiber sequence extends to the right, so that we have a fiber sequence

$$P_2(X) \longrightarrow P_1(X) \xrightarrow{k_1} K(\pi_2(X), 3)$$

for some map $k_1$. Then the space $P_2(X)$ would be described by the group $\pi_1(X)$ and by a cohomology class $k_1 \in H^3(P_1(X); \pi_2(X))$.

This leads to the question of when fiber sequences extend to the right in this way. We say that a fiber sequence $F \longrightarrow E \longrightarrow B$ is a **principal fibration** if the fibration is pulled back from a path loop fibration. That is, we assume a diagram
in which the bottom square is a (homotopy) pullback diagram. We will give an equivalent description of principal fibrations shortly, but let us first recall some background. Recall that any path \( \gamma : b_0 \rightarrow b_1 \) in \( B \) lifts to a homotopy equivalence \( F(b_0) \xrightarrow{\sim} F(b_1) \). In particular, a loop in \( B \) gives a self homotopy equivalence of \( F \). If \( F \) is simply connected, this gives a well-defined action of \( \pi_1(B) \) on \( \pi_n(F) \) for all \( n \).

**Proposition 2.2.** Let \( E \) and \( B \) be connected. A fiber sequence \( K(A,n) \rightarrow E \xrightarrow{f} B \) is a principal fiber sequence if and only if \( \pi_1(B) \) acts trivially on \( \pi_n(K(A,n)) \)

**Proof.**  
(\( \Rightarrow \)) Suppose the fibration is principal. Then we can identify the action of \( \pi_1(B) \) on \( \pi_n(K(A,n)) \) in the given fibration with the action of \( \pi_1(B) \) on \( \pi_{n+1}(K(A,n)) \) in the fibration \( E \rightarrow B \rightarrow K(A,n+1) \). But this factors through the action of \( \pi_1(K(A,n+1)) \sim = 0 \), so the action must be trivial.

(\( \Leftarrow \)). The map \( f \) is an \( n \)-equivalence, so the map \( \Omega C(f) \rightarrow \Omega \mathcal{C}(f) \) is an \( (n-1) \)-equivalence. But \( \Omega C(f) \) is \( (n-1) \)-connected, so \( \Omega \mathcal{C}(f) \) must also be \( (n-1) \)-connected, so that \( \mathcal{C}(f) \) must be \( n \)-connected. The Hurewicz theorem now tells us that

\[
\pi_{n+1}(C(f)) \cong H_{n+1}(C(f)) \cong H_{n+1}(B,E).
\]

We need a stronger result, namely a relative Hurewicz theorem. To state it, note that given a pair \((X,A)\), there is an action of \( \pi_1(A) \) on \( \pi_n(X,A) \) (see Hatcher, p. 345).

**Theorem 2.3** (Relative Hurewicz). Suppose \((X,A)\) is an \((n-1)\)-connected pair \((n > 2)\). Then there is a short exact sequence

\[
0 \rightarrow \{ \text{subgroup generated by } [\gamma \cdot f] - [f] \} \rightarrow \pi_n(X,A) \rightarrow H_n(X,A) \rightarrow 0
\]

In particular, if \( \pi_1(A) \) acts trivially on \( \pi_n(X,A) \), then the relative Hurewicz map \( \pi_n(X,A) \rightarrow H_n(X,A) \) is an isomorphism. We now have

\[
\pi_{n+1}(C(f)) \cong H_{n+1}(B,E) \cong \pi_{n+1}(B,E) \cong \pi_n(F) = A.
\]

Since \( C(f) \) is \( n \)-connected, we may attach cells of dimension \( n+3 \) and higher to build a \( K(A,n+1) \).

Denote by \( k \) the composite

\[
k : B \rightarrow C(f) \rightarrow K(A,n+1).
\]

Replacing \( k \) by a fibration \( k' \), we get the following diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\sim} & B' \\
\downarrow & & \downarrow k \\
F(k') & \xrightarrow{k} & K(A,n+1).
\end{array}
\]

It remains to check that \( E \rightarrow F(k') \) is an equivalence. Consider the long exact sequences in homotopy
We are now done by the 5-lemma.

3. Fri, Apr. 8

The upshot of the discussion last time is

**Proposition 3.1.** $X$ has a Postnikov tower of principal fibrations if and only if $X$ is simple.

If this is the case, the maps $P_i(X) \xrightarrow{k_i} K(\pi_{i+1}, i + 2)$ correspond to cohomology classes $k_i \in H^{i+2}(X; \pi_{i+1}(X))$, which are known as the “$k$-invariants of $X$.”

**Obstruction Theory Revisited:**

Let $X$ be a CW complex and $Y$ be simple. The construction of maps $X \rightarrow Y$ is governed by obstruction theory. Since $Y$ is simple, it has a Postnikov tower of principal fibrations. We may therefore build a map $f : X \rightarrow Y$ by building a compatible family of maps $f^i : X \rightarrow P_i(Y)$. As before, we build maps $X \rightarrow P_i(Y)$ inductively over the skeleta of $X$. Let us assume given a commutative diagram as follows:

$$
\begin{array}{ccc}
X_n & \xrightarrow{f_n} & P_i(Y) \\
\downarrow{i_n} & & \downarrow{k_{i-1}} \\
X_{n+1} & \xrightarrow{f_{n+1}^{-1}} & P_{i-1} \xrightarrow{k_i} K(\pi_i(Y), i + 1)
\end{array}
$$

When have an exact sequence

$$
[X_{n+1}, P_i(Y)] \xrightarrow{f_{n+1}^{-1}} [X_{n+1}, P_{i-1}Y] \rightarrow H^{i+1}(X_{n+1}, \pi_iY).
$$

The class $o(f_{n+1}^{-1})$ is an obstruction class, and if it vanishes we can lift $f_{n+1}^{-1}$ to a map $f_{n+1} : X_{n+1} \rightarrow P_iY$.

Of course, we would like this lift to be compatible with $f_n$, so we want a lift rel $X_n$.

The commutativity of the diagram tells us that $o(f_{n+1}^{-1})|_{X_n} = 0$, so the obstruction class lifts to a relative obstruction class

$$
o(f_{n+1}^{-1}, X_n) \in H^{i+1}(X_{n+1}, X_n; \pi_iY)
$$

which controls relative lifts. Note that this group vanishes unless $i = n$. Thus there is a single layer in the Postnikov tower at which lifting is nontrivial.

Next, we will discuss an important family of spaces whose $k$-invariants are always trivial. This is equivalent to saying that the space is weakly equivalent to a product of Eilenberg-Mac Lane spaces. Such a space is sometimes referred to as a Generalized Eilenberg-Mac Lane space, or GEM for short.

**Theorem 3.2 (J. Moore).** A topological abelian monoid $X$ is a GEM:

$$
X \simeq \prod_{n \geq 0} K(\pi_n(X), n).
$$
Proof. Recall that for each abelian group \( A \) and \( n \geq 0 \), there is a Moore space \( M(A, n) \) with

\[
\tilde{H}_i(M(A, n); \mathbb{Z}) \cong \begin{cases} A & i = n \\ 0 & i \neq n. \end{cases}
\]

This space is built as a cofiber

\[
\bigvee S^n \to \bigvee S^n \to M(A, n).
\]

For each \( n \), it is easy to build a map

\[
M(\pi_n(X), n) \xrightarrow{\theta_n} X
\]

giving a \( \pi_n \)-isomorphism (by Hurewicz, \( \pi_n(M(A, n)) \cong H_n(M(A, n); \mathbb{Z}) \cong A \)).

Now for any space \( Y \), the space \( B(Y, \mathbb{N}) \) is a topological abelian monoid\(^1\) — in fact, it is the free topological abelian monoid on \( Y \). Any map of spaces \( Y \to X \) extends to a map of topological monoids \( B(Y, \mathbb{N}) \to X \). Another way to say this is that the functor \( B(-, \mathbb{N}) \) sits in an adjoint pair

\[
B(-, \mathbb{N}) : \text{Top} \rightleftarrows \text{TopAbMon} : U,
\]

where \( U \) is the forgetful functor. We need the basepointed version, which sits in an adjoint pair

\[
\tilde{B}(-, \mathbb{N}) : \text{Top} \rightleftarrows \text{TopAbMon} : U,
\]

Recall, for example, that \( \tilde{B}(S^n, \mathbb{Z}) \) is a model for \( \tilde{K}(\mathbb{Z}, n) \). It turns out\(^2\) that the natural map \( B(Y, \mathbb{N}) \to B(Y, \mathbb{Z}) \) is a “topological group completion”. This means, in particular, that the map is a weak equivalence if \( Y \) is connected. Moreover, the earlier statement that \( \tilde{B}(S^n, \mathbb{Z}) \simeq K(\mathbb{Z}, n) \) can be generalized, using the fact that \( \tilde{B}(-, \mathbb{Z}) \) converts cofiber sequences to fiber sequences, to the statement that \( \tilde{B}(M(A, n), \mathbb{Z}) \simeq K(A, n) \). Putting these results together, we learn that \( \tilde{B}(M(A, n), \mathbb{N}) \simeq K(A, n) \).

We may assemble the maps \( \theta_n \) to get a map of spaces

\[
\bigvee_{n \geq 0} M(\pi_n(X), n) \to X.
\]

This corresponds to a map of topological abelian monoids

\[
\tilde{B} \left( \bigvee_{n \geq 0} M(\pi_n(X), n), \mathbb{N} \right) \to X.
\]

The domain can be identified as

\[
\tilde{B} \left( \bigvee_{n \geq 0} M(\pi_n(X), n), \mathbb{N} \right) \cong \bigoplus_n \tilde{B}(M(\pi_n(X), n), \mathbb{N}) \cong \bigoplus_n K(\pi_n(X), n).
\]

Note that the natural map

\[
\bigoplus_{n \geq 0} K(\pi_n(X), n) \to \prod_{n \geq 0} K(\pi_n(X), n)
\]

is an equivalence since each factor has homotopy in a single degree.

As we have said, each factor induces an isomorphism

\[
\pi_n(K(\pi_n(X), n)) \cong \pi_n(X),
\]

\(^1\)By \( \mathbb{N} \), we really mean the monoid \( \mathbb{Z}_{\geq 0} \)

\(^2\)This is the Dold-Thom Theorem.
so the map
\[
\prod_{n \geq 0} K(\pi_n(X), n) \sim X
\]
is an equivalence.

This result says that topological abelian monoids (and therefore also topological abelian groups) are completely determined by their homotopy groups.

The space \( B(Y, N) \) from above is also known as the “infinite symmetric product” and denoted \( \text{SP}(Y) \) or \( \text{Symm}(Y) \) (see Hatcher, Appendix 4K). This is the union \( \text{Symm}(Y) = \bigcup_n \text{Symm}_n(Y) \), where
\[
\text{Symm}_n(Y) = Y^n/\Sigma_n \cong \{ \text{unordered } n\text{-tuples of points in } Y \}.
\]
For example, we think of the class of \( (y_1, y_3, y_1, y_1, y_3, y_2) \) as representing \( 3y_1 + y_2 + 2y_3 \).

**Example 3.3.** Let \( Y = \mathbb{CP}^1 \cong S^2 \). We claim that \( \text{Symm}_n(S^2) \cong \mathbb{CP}^n \). To see this, first note that we can identify \( \mathbb{CP}^n \) with the set of nonzero \( \mathbb{C} \) polynomials of degree at most \( n \), up to scalar multiple. The correspondence identifies the point \([y_0 : \cdots : y_n]\) of \( \mathbb{CP}^n \) with \( f(t) = y_n t^n + y_{n-1} t^{n-1} + \cdots + y_0 \).

Now define \( \mathbb{CP}^1 \to \mathbb{CP}^n \) by
\[
(z_1, \ldots, z_n) \mapsto f(t) = \prod_i (t - z_i).
\]
If any of the coordinates is \( z_i = \infty \), the corresponding factor \( t - z_i \) is omitted from \( f(t) \). This function is clearly symmetric and so induces a map \( \text{Symm}_n(\mathbb{CP}^1) \to \mathbb{CP}^n \).

The fact that polynomials are determined by their roots means that this map is a bijection. Both spaces are compact Hausdorff, so it is a homeomorphism as well. It then follows that \( \text{Symm}(S^2) \cong \mathbb{CP}^\infty \simeq K(\mathbb{Z}, 2) \).