CLASS NOTES
MATH 527 (SPRING 2011)
WEEK 2

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1. Mon, Jan. 24

There is a functor $\iota : \mathbf{Top} \longrightarrow \mathbf{Ho}(\mathbf{Top})$ which acts as the identity functor on objects (spaces) and sends any continuous map to its homotopy class

Proposition 1.1. *The functor* $\mathbf{Top} \longrightarrow \mathbf{Ho}(\mathbf{Top})$ *is universal among functors that take the homotopy equivalences to isomorphisms. That is, suppose* $F : \mathbf{Top} \longrightarrow \mathscr{C}$ *is a functor such that for any homotopy equivalence* g *in* $\mathbf{Top}$*, the morphism* $F(g)$ *in* $\mathscr{C}$ *is an isomorphism. Then there exists a functor* $\varphi_F : \mathbf{Ho}(\mathbf{Top}) \longrightarrow \mathscr{C}$ *and a natural isomorphism* $\eta : \varphi_F \circ \iota \cong F$*.*

$$\begin{tikzcd} \mathbf{Top} \arrow[r, "\iota"] \arrow[dr, "F"'] & \mathbf{Ho}(\mathbf{Top}) \arrow[d, dashed, "\varphi_F"] \\ & \mathscr{C} \end{tikzcd}$$

Proof. We may define the functor φ_F on objects to agree with F, setting $\varphi_F(X) := F(X)$. Let $\alpha : X \longrightarrow Y$ be a morphism in $\mathbf{Ho}(\mathbf{Top})$ and let $f_\alpha : X \longrightarrow Y$ be a continuous map in the homotopy class α. Define $\varphi_F(\alpha) := F(f_\alpha)$. We must show that the morphism $F(f_\alpha)$ does not depend on the choice of representative f_α for α.

Suppose that $g \simeq f_\alpha$. That is, we have a homotopy $h : X \times I \longrightarrow Y$ such that $h \circ i_0 = f_\alpha$ and $h \circ i_1 = g$. The maps i_0 and $i_1 : X \hookrightarrow X \times I$ and the projection $\pi_X : X \times I \longrightarrow X$ are homotopy equivalences and so F takes these to isomorphisms in $\mathscr{C}$. Furthermore, $\pi_X \circ i_0 = \mathrm{id}_X$ and $\pi_X \circ i_1 = \mathrm{id}_X$, so we have

$$F(\pi_X) \circ F(i_0) = \mathrm{id}_{F(X)} = F(\pi_X) \circ F(i_1).$$

Since $F(\pi_X)$ is an isomorphism, we conclude that $F(i_0) = F(i_1)$. But then

$$F(g) = F(h) \circ F(i_1) = F(h) \circ F(i_0) = F(f_\alpha),$$

so φ_F is well-defined on morphisms. It remains to show that φ_F is a functor, but we leave this as an exercise.

Finally, given our construction of the functor φ_F, the composition $\varphi_F \circ \iota$ is already the functor F on the nose, so that we may simply take the identity natural transformation for η. ∎

In general, if $\mathscr{D}$ is a category and $\mathcal{W}$ is a collection of morphisms in $\mathscr{D}$, then the universal solution to the above problem (a category $\mathscr{E}$ and a functor $\iota : \mathscr{D} \longrightarrow \mathscr{E}$ converting all morphisms in $\mathcal{W}$ into isomorphisms) is called a localization of $\mathscr{D}$ with respect to $\mathcal{W}$ and is denoted $\mathscr{D}[\mathcal{W}]^{-1}$. There is not always a category $\mathscr{D}[\mathcal{W}]^{-1}$ satisfying this universal property but the result above says that $\mathbf{Ho}(\mathbf{Top})$ is a candidate for $\mathbf{Top}[\text{hoequiv}]^{-1}$.

Algebraic topology studies topological spaces up to homotopy equivalence, and many of the techniques involve attaching algebraic invariants to spaces which only depend on the homotopy type of a space. Another way to say this is that one uses functors from $\mathbf{Top}$ to algebraic categories (abelian groups, for instance), which factor through the homotopy category. Such functors are called homotopy-invariant functors.

Of course, there is also a basepointed version of the above story. The based homotopy category $\mathbf{Ho}(\mathbf{Top}_*)$ has objects the based spaces and morphism sets the based homotopy classes of based maps.

An example of a homotopy-invariant algebraic construction on (based) spaces is the fundamental group. This is defined as $\pi_1(X,x) = [S^1,(X,x)]_*$. One might just as well replace the circle S^1 by a sphere S^n of arbitrary dimension. So for any $n \geq 0$, we define

$$\pi_n(X,x) := [S^n,(X,x)].$$

Since $\pi_1(X,x)$ is not just a set but also a group, we might ask whether the same is true of $\pi_n(X,x)$ for arbitrary n. Let us first recall how the group structure is defined on $\pi_1(X)$.

We define the "pinch map" $\mathfrak{p} : S^1 \longrightarrow S^1 \vee S^1$ by the formula

$$\mathfrak{p}(e^{i\theta}) = \begin{cases} \iota_1(e^{2i\theta}) & 0 \leq \theta \leq \pi \\ \iota_2(e^{2i\theta}) & \pi \leq \theta \leq 2\pi, \end{cases}$$

where i_1 and i_2 denote the two inclusions $S^1 \hookrightarrow S^1 \vee S^1$. That is, $\mathfrak{p}$ is the loop that first goes around one loop of the figure eight and then goes around the other loop. If we chose to instead work with the model $S^1 = I/\partial I$, the pinch map would be given by the formula

$$\mathfrak{p}(x) = \begin{cases} \iota_1(2x) & 0 \leq x \leq \frac{1}{2} \\ \iota_2(2x-1) & \frac{1}{2} \leq x \leq 1. \end{cases}$$

We can use $\mathfrak{p}$ to define the multiplication on $\pi_1(X,x)$:

$$\pi_1(X,x) \times \pi_1(X,x) = [S^1,X]_* \times [S^1,X]_* \cong [S^1 \vee S^1, X]_* \xrightarrow{-\circ\mathfrak{p}} [S^1,X]_* = \pi_1(X,x).$$

The unit element for the group structure is the constant based map $S^1 \longrightarrow X$.

We now replace S^1 by S^n. For any $n \geq 0$, we can always consider the constant based map $S^n \longrightarrow X$, so we have the "unit element" in $\pi_n(X,x)$ for any n. Note that when $n = 0$, then a based map $S^0 \longrightarrow X$ is specified by the image of the non-basepoint of S^0, so such a map corresponds exactly to a point of X. Two such maps are homotopic if and only if the specified points lie in the same path component of X, so we conclude that $\pi_0(X,x)$ is the set of path components of X. The "unit element" in this case is just the component of the basepoint. But we do not expect to have any additional multiplication on the set of path components, so $\pi_0(X)$ is just a pointed set.

For $n \geq 1$, the group multiplication would come from a pinch map $S^n \longrightarrow S^n \vee S^n$. Recall that if $n \geq 1$, then $S^n \cong (S^1)^{\wedge n}$. The identity

$$(X \vee Y) \wedge Z \cong (X \wedge Z) \vee (Y \wedge Z),$$

then allows us to define a pinch map for $n \geq 1$ by

$$S^n \cong S^1 \wedge (S^1)^{\wedge n-1} \xrightarrow{\mathfrak{p}\wedge 1} (S^1 \vee S^1) \wedge (S^1)^{\wedge n-1} \cong S^n \vee S^n.$$

One can show that this defines a group structure on $\pi_n(X,x)$ for $n \geq 1$.

If $n > 1$, then the choice of applying the pinch map to the first factor of S^1 in S^n is arbitrary, and we see that we could in fact define n different pinch maps $S^n \longrightarrow S^n \vee S^n$ by choosing different smash factors.

Proposition 1.2. *(Eckmann-Hilton argument) Let A be a set with a distinguished point $0 \in A$. Suppose given two operations*

$$\star : A \times A \longrightarrow A \qquad\qquad \square : A \times A \longrightarrow A$$

such that 0 acts as a unit element for both operations and such that

$$(a \star b)\square(c \star d) = (a \square c) \star (b \square d)$$

for all $a, b, c, d \in A$. Then

$$a \star b = a \Box b = b \star a$$

for all $a, b \in A$.

Proof. The 4-tuple $a, 0, 0, b$ gives

$$a \Box b = (a \star 0) \Box (0 \star b) = (a \Box 0) \star (0 \Box b) = a \star b.$$

Similarly, the 4-tuple $0, a, b, 0$ gives

$$a \Box b = (0 \star a) \Box (b \star 0) = (0 \Box b) \star (a \Box 0) = b \star a.$$

■

Proposition 1.3. *Let $n > 1$. Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be pinch maps $S^n \longrightarrow S^n \vee S^n$ arising from pinching different smash factors of S^n. The resulting maps $\mathfrak{p}_1^*$ and $\mathfrak{p}_2^* : \pi_n(X, x) \times \pi_n(X, x) \longrightarrow \pi_n(X, x)$ satisfy the assumptions of the Eckmann-Hilton argument up to homotopy.*

Proof. We give the proof for $n = 2$. Viewing S^2 as $I^2/\partial I^2$, we may view one pinch map as a horizontal pinch and the other as a vertical pinch. The Eckmann-Hilton property amounts to the condition that the diagram

$$\begin{tikzcd}
S^2 \arrow[r, "\mathfrak{p}_1"] \arrow[d, "\mathfrak{p}_2"'] & S^2 \vee S^2 \arrow[dr, "\mathfrak{p}_2 \vee \mathfrak{p}_2"] & \\
S^2 \vee S^2 \arrow[dr, "\mathfrak{p}_1 \vee \mathfrak{p}_1"'] & & S^2 \vee S^2 \vee S^2 \vee S^2 \\
& S^2 \vee S^2 \vee S^2 \vee S^2 \arrow[ur, "\mathrm{id} \vee \tau \vee \mathrm{id}"'] &
\end{tikzcd}$$

commutes up to homotopy. Using the model $S^2 = I^2/\partial I^2$, the composition $(\mathfrak{p}_2 \vee \mathfrak{p}_2) \circ \mathfrak{p}_1$ is the map

$$(x, y) \mapsto \begin{cases} \iota_1(2x, 2y) & x, y \leq \frac{1}{2} \\ \iota_2(2x, 2y - 1) & x \leq \frac{1}{2} \leq y \\ \iota_3(2x - 1, 2y) & y \leq \frac{1}{2} \leq x \\ \iota_4(2x - 1, 2y - 1) & \frac{1}{2} \leq x, y, \end{cases}$$

and the composition $(\mathfrak{p}_1 \vee \mathfrak{p}_1) \circ \mathfrak{p}_2$ is the map

$$(x, y) \mapsto \begin{cases} \iota_1(2x, 2y) & x, y \leq \frac{1}{2} \\ \iota_2(2x - 1, 2y) & y \leq \frac{1}{2} \leq x \\ \iota_3(2x, 2y - 1) & x \leq \frac{1}{2} \leq y \\ \iota_4(2x - 1, 2y - 1) & \frac{1}{2} \leq x, y, \end{cases}$$

Composing the latter with the permutation $\mathrm{id} \vee \tau \vee \mathrm{id}$ gives precisely the former, on the nose. ■

Corollary 1.4. *The homotopy groups $\pi_n(X, x)$ are abelian for $n \geq 2$.*

2. Wed, Jan. 26

Example 2.1. One of the first computations of fundamental groups that one learns is that $\pi_1(S^1) \cong \mathbb{Z}$. This generalizes, as we shall see, to the result

$$\pi_n(S^n) \cong \mathbb{Z}$$

for $n \geq 1$. When $n = 0$, $\pi_0(S^0)$ is the set of path components of S^0, namely the two point set S^0 itself.

One method of computing fundamental groups is the use of covering spaces, but it turns out that this is not as helpful for computing higher homotopy groups because of the following result.

Proposition 2.2. *Let $p : E \longrightarrow B$ be a covering map. Then for any $e \in E$, the map p induces an isomorphism $\pi_n(E, e) \cong \pi_n(B, p(e))$ for any $n \geq 2$.*

Proof. For injectivity, we will assume the "homotopy lifting property" for covering spaces ([Hatcher, 1.30]). That is, we suppose that given a space Y, a map $f : Y \longrightarrow E$ and a homotopy $h : Y \times I \longrightarrow B$ with $h_0 = p \circ f$, then there is a unique lift $\tilde{h} : Y \times I \longrightarrow E$ with $\tilde{h}_0 = f$ and $p \circ \tilde{h} = h$.

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "h"'] \arrow[ur, dashed, "\exists!\,\tilde{h}"] & B. \end{tikzcd}$$

Taking $Y = S^n$, let $\alpha : S^n \longrightarrow E$ be a pointed map. Suppose that $p_*(\alpha) = 0 \in \pi_n(B, b)$. Then we have a homotopy $h : p \circ \alpha \simeq *$. By the homotopy lifting property, we have a homotopy $\tilde{h} : \alpha \simeq \beta$, where $\beta : S^n \longrightarrow E$ is a map whose image lies in $p^{-1}(b)$. Since $p^{-1}(b)$ is discrete and S^n is connected ($n > 0$), β must be a constant map. But we have not yet verified that $\beta(*) = e$. Consider the restriction of the homotopy h to $\{*\} \times I \subset S^n \times I$. This restriction is constant at the basepoint $b \in B$, so a choice of lift of this (starting at e) would be the constant path at e. Since the lift is unique, we conclude that the restriction of $\tilde{h}$ to $\{*\} \times S^n$ is constant at e, and it follows that our homotopy $\tilde{h}$ is based and that β is based as well.

For surjectivity, we use the following lifting property for covering spaces: ([May, §3.7] or [Hatcher, Prop 1.33]) a based map $f : X \longrightarrow B$ lifts to a based map $\tilde{f} : X \longrightarrow E$ if and only if $f_*(\pi_1(X, x)) \subset p_*(\pi_1(E, e))$. In particular, taking $X = S^n$ with $n \geq 2$, then such a lift always exists since S^n is simply connected. ∎

Example 2.3. Since the exponential map $\mathbb{R} \longrightarrow S^1$ is a covering (the universal cover), we find that $\pi_n(S^1) \cong \pi_n(\mathbb{R}) = 0$ for $n \geq 2$, so all higher homotopy groups of S^1 vanish.

We shall see later that in addition $\pi_k(S^n)$ always vanishes for $k < n$. However, the same is not true for $k > n$. For instance, we shall find that $\pi_3(S^2) \cong \mathbb{Z}$. All other higher homotopy groups of S^2 are finite abelian, but there are infinitely many nonzero ones, and they have not all been computed.

2.1. **Dependence on the basepoint.** How does all of the above depend on basepoints? For instance, we might consider the unbased homotopy classes $[S^n, X]$. There is a "forgetful" map

$$[S^n, (X, x)]_* \longrightarrow [S^n, X].$$

When is this surjective? We would want to know if a map $\beta : S^n \longrightarrow X$ is homotopic to a based map. Any such homotopy would provide a path from $\beta(*)$ to the basepoint of X, so at the very least, we would want X to be path connected. In fact, this is sufficient as well.

Let $\beta : S^n \longrightarrow X$ and write $b = \beta(*)$. Suppose that γ is a path in X from b to x. We define a homotopy $H : S^n \times I \longrightarrow X$ such that at time t, $H_t : S^n \longrightarrow X$ satisfies $H_t(*) = \gamma(t)$. (Pictures from page 341 of Hatcher)

We may apply the above construction even if β is based. When $n = 1$, the resulting map

$$\pi_1(X, x) \times \pi_1(X, x) \longrightarrow \pi_1(X, x)$$

is the conjugation action on $\pi_1(X, x)$ on itself:

$$(\gamma, \beta) \mapsto \gamma \cdot \beta \cdot \gamma^{-1}.$$

For $n > 1$, we also get an action

$$\pi_1(X, x) \times \pi_n(X, x) \longrightarrow \pi_n(X, x)$$

of the fundamental group on the higher homotopy groups.

Example 2.4. Let $X = S^1 \vee S^2$. The universal cover of X is the space Y given by the real line $\mathbb{R}$ with a copy of S^2 attached to each integer point. Assuming the result $\pi_2(S^2) \cong \mathbb{Z}$, one can further show that each distinct copy of S^2 in Y contributes a summand of $\mathbb{Z}$ to $\pi_2(Y)$. Furthermore, the action of $\pi_1(X) \cong \mathbb{Z}$ on $\pi_2(X) \cong \pi_2(Y)$ can be identified with the deck transformations acting on Y.

3. Fri, Jan. 28

Another application of the above ideas is as follows. Let X be a path-connected space, and let x and y be two choices of basepoints in X. Let γ be a path in X from x to y. Then the above procedure specifies a map (action by the path γ)

$$\pi_n(X, x) \xrightarrow{\gamma\cdot} \pi_n(X, y).$$

This is a homomorphism, and action by the inverse path γ^{-1} specifies an inverse homomorphism

$$\pi_n(X, y) \xrightarrow{\gamma^{-1}\cdot} \pi_n(X, x).$$

That is, the homotopy groups of a based space do not depend, up to isomorphism, on the basepoint. So we will often be sloppy and drop the basepoint from the notation.

Definition 3.1. A map $f : X \longrightarrow Y$ is called a **weak homotopy equivalence** if for <u>every choice</u> of basepoint $x \in X$, the induced map of homotopy groups

$$f_* : \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$$

is an isomorphism for every n.

Proposition 3.2. *A homotopy equivalence $f : X \longrightarrow Y$ is a weak homotopy equivalence.*

Proof. For convenience, we will assume that X and Y are path-connected, so that we don't have to worry about π_0. Let g be a homotopy inverse for f and suppose given a basepoint $x \in X$. Consider the composition

$$\pi_n(X, x) \xrightarrow{f} \pi_n(Y, f(x)) \xrightarrow{g} \pi_n(X, gf(x))$$

We know that $g \circ f \simeq \mathrm{id}_X$, but this is not a based homotopy ($g \circ f$ is not even a based map). Let γ be the path $h(x, t)$ from $gf(x)$ to x. Then we have

$$\pi_n(X, x) \xrightarrow{f} \pi_n(Y, f(x)) \xrightarrow{g} \pi_n(X, gf(x)) \xrightarrow{\gamma\cdot} \pi_n(X, x)$$

and we claim that this composition is the identity. The point is that we can find a based homotopy $\gamma \cdot g \circ f \simeq \mathrm{id}_X$ (draw a picture).

Since we have shown that $g_* \circ f_* : \pi_n(X, x) \longrightarrow \pi_n(X, gf(x))$ is an isomorphism, it follows that f_* is injective. A similar argument, using the homotopy $f \circ g \simeq \mathrm{id}_Y$, shows that f_* is surjective. ∎

The converse is <u>not</u> true, as you will see on the homework.

Proposition 3.3. *The weak homotopy equivalences satisfy the 2-out-of-3 property.*

Proof. Suppose given maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ and let $h = g \circ f$. The only part of the requirement needing any work is the statement that if f and h are weak homotopy equivalences, then so is g. Let $y \in Y$ be a chosen basepoint. If we knew that y were in the image of f, we would be home free, but of course this need not be the case. However, we know that f induces a bijection $\pi_0(X) \cong \pi_0(Y)$. So we know there is a point y_2 in the path component of our chosen y such that y_2 is in the image of f. Let α be a path from y to y_2. Then the conjugation action from above specifies an isomorphism

$$\alpha \cdot (-) : \pi_n(Y, y) \cong \pi_n(Y, y_2)$$

and similarly

$$g(\alpha) \cdot (-) : \pi_n(Z, g(y)) \cong \pi_n(Z, g(y_2)).$$

The following diagram commutes, by construction, which finishes the proof

$$\begin{tikzcd}
\pi_n(Y, y) \arrow[r, "g"] \arrow[d, "\alpha\cdot"', "\cong"] & \pi_n(Z, g(y)) \arrow[d, "\cong"', "g(\alpha)\cdot"] \\
\pi_n(Y, y_2) \arrow[r, "\cong", "g"'] & \pi_n(Z, g(y_2)).
\end{tikzcd}$$

■

Recall that the classical homotopy category of spaces was formed by taking homotopy classes of maps of spaces. We saw that this was equivalent to formally inverting the homotopy equivalences. The **homotopy category of spaces** is defined by formally inverting the weak homotopy equivalences. But we don't know yet that such a category exists! We will see later another construction of this category.

We will largely restrict our attention to spaces which are built up in a particularly nice way.

Definition 3.4. A **CW structure** on a space X is an increasing filtration $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots X$ such that

(1) X_0 is a discrete set
(2) For each $n \geq 0$, there is a collection $\mathcal{C}_{n+1}$ of maps $\alpha : S^n \longrightarrow X_n$ (called the "attaching maps for the $n+1$-cells of X") and a pushout diagram

$$\begin{tikzcd}
\coprod_{\alpha \in \mathcal{C}_{n+1}} S^n \arrow[r] \arrow[d] & \coprod_{\alpha \in \mathcal{C}_{n+1}} D^{n+1} \arrow[d] \\
X_n \arrow[r] & X_{n+1}
\end{tikzcd}$$

(3) The topology on X is the "weak topology" of the union $X = \bigcup_n X_n$. That is $A \subseteq X$ is closed if and only if $A \cap X_n \subseteq X_n$ is closed for all n.

We say that X is a **CW complex** if it admits a CW structure.

The space X_n is called the n-skeleton of the CW complex X. Each map $D^n \xrightarrow{e_n} X_n$ is called an n-cell.

A **finite** CW complex is one with finitely many cells. A CW complex is of **finite type** if it has finitely many cells in each dimension. A CW complex is **finite-dimensional** if $X = X_n$ for some n.

Example 3.5. (1) S^n has a CW structure in which the 0-skeleton is the basepoint. A single n-cell is then attached via a trivial map $S^{n-1} \longrightarrow *$. That the resulting space is S^n is equivalent to the homeomorphism $D^n/S^{n-1} \cong S^n$.

(2) There is an alternative CW structure on S^n. One starts with S^0 for the 0-skeleton. We then attach two 1-cells via two copies of the identity map $S^0 = S^0$ as attaching map. The 1-skeleton is a model for S^1. Inductively, we attach two cells at each stage so that the k-skeleton is S^k. This CW structure on S^n has $2n+2$ cells.

(3) Any CW structure on S^n yields one on D^{n+1}. We attach a single cell to S^n to obtain D^{n+1}. This realizes D^{n+1} as a subcomplex of the second CW structure on S^{n+1} (if we also use this CW structure on S^n).

(4) Let $\mathbb{RP}^n$ be the quotient of S^n by the relation $x \sim (-x)$. Using the larger CW structure on S^n from above, this equivalence relation identifies the two k-cells, for each $0 \leq k \leq n$. There results a CW structure on $\mathbb{RP}^n$ having $n+1$-cells, one in each dimension up n.

(5) Complex projective space $\mathbb{CP}^n$ is similarly a quotient of S^{2n+1}, thought of as the unit sphere in $\mathbb{C}^{n+1}$, by an action of S^1, thought of as the complex numbers of unit norm (the group $SU(1)$). We start with a single point for the 0-skeleton. Suppose inductively that we have defined the $2k$-skeleton (which is a model for $\mathbb{CP}^k$). Then we do not attach any $2k+1$-cells, so that the $2k+1$-skeleton is the same as the $2k$-skeleton. We attach a single $2k+2$-cell via the attaching map $S^{2k+1} \longrightarrow \mathbb{CP}^k$ which is the defining quotient map for $\mathbb{CP}^n$. The resulting CW structure on $\mathbb{CP}^n$ has $(n+1)$-cells, all in even dimensions.