CW complexes are well-behaved as topological spaces:

**Theorem 1.1.** Let $X$ be a CW complex. Then

1. The components of $X$ are the path-components (Hatcher, A.4)
2. If $K$ is a compact subset of $X$, then $K$ meets only finitely many cells. (Hatcher, A.1)
3. $X$ is Hausdorff (and even normal) (Hatcher, A.3)

While we are discussing point-set issues, let me mention another important consideration. In algebra, given $R$-modules $M$, $N$, and $P$, there is a bijection

$$\text{Hom}_R(M \otimes_R N, P) \cong \text{Hom}_R(M, \text{Hom}_R(N, P)).$$

In topology, we similarly would like to have a bijection

$$\text{Hom}(X \times Y, Z) \cong \text{Hom}(X, \text{Map}(Y, Z)),$$

where $\text{Map}(Y, Z)$ is the space of continuous maps $Y \to Z$, equipped with the compact-open topology. But the canonical map

$$\text{Hom}(X \times Y, Z) \to \text{Hom}(X, \text{Map}(Y, Z))$$

is not surjective for all spaces $X$, $Y$, and $Z$. There are several ways to fix this problem, and the solution we shall take is to work with **compactly generated weak Hausdorff spaces**.

A space $X$ is **weak Hausdorff** if the image of any compact Hausdorff space is closed in $X$. A weak Hausdorff space is **compactly generated** if a subset $C \subseteq X$ is closed if (and only if) for every continuous map $g : K \to X$ with $K$ compact, the subset $g^{-1}(C)$ is closed in $K$.

Any time from now on that we talk about spaces, we really mean compactly generated weak Hausdorff spaces. There are a couple more modifications that we need. One point is that if $X$ and $Y$ are compactly generated weak Hausdorff, then $X \times Y$ need not be. So we redefine the topology on $X \times Y$ by setting the closed subsets to be those satisfying the compactly generated condition. Similarly, the mapping space $\text{Map}(X, Y)$ is not always compactly generated, so we similarly redefine the topology. It turns out that after these steps, these replacements work as desired, and we end up with a homeomorphism

$$\text{Map}(X \times Y, Z) \cong \text{Map}(X, \text{Map}(Y, Z)).$$

There is also a based variant of this. We write $\text{Map}_*(X, Y)$ for the space of based maps. Then we have

$$\text{Map}_*(X \wedge Y, Z) \cong \text{Map}_*(X, \text{Map}_*(Y, Z)).$$

See Chapter 5 of [May] or Neil Strickland’s notes for more on compactly generated spaces.

**Proposition 1.2.** If $X$ and $Y$ are CW complexes, then so is $X \times Y$. An $n$-cell of $X \times Y$ corresponds to a $p$-cell of $X$ and a $q$-cell of $Y$, where $p + q = n$. 

Proof. The point is that we use the cube models for disks, then we have a homeomorphism

\[ D^{p+q} = I^{p+q} \cong I^p \times I^q = D^p \times D^q, \]

and under this model we get

\[ S^{p+q-1} = \partial D^{p+q} = \partial(D^p) \times D^q \cup D^p \times \partial(D^q) = S^{p-1} \times D^q \cup D^p \times S^{q-1}. \]

I should emphasize here that when we write product, we mean the product in the compactly generated sense. Otherwise, the topology on \( X \times Y \) might not satisfy condition (3) from the definition of a CW complex. See [Hatcher, Theorem A.6]. ■

There are two important generalizations: a relative CW complex \( (X, A) \) is defined in the same way, except that one starts with \( X_0 \) as the space \( A \) disjoint union a discrete set.

A cell complex is a space with an increasing filtration \( X = \bigcup_n X_n \) as before, but there are now no conditions on the dimensions of the cells attached at stage \( n \). For instance, \( X_1 \) might be obtained from \( X_0 \) by attaching a 0-cell and a 3-cell. There is also the notion of a relative cell complex \( (X, A) \).

**Homotopy Extension Property**

**Definition 1.3.** We say a map \( A \rightarrow X \) satisfies the Homotopy Extension Property (HEP) if, given any map \( X \xrightarrow{f} Y \) and homotopy \( h : A \times I \rightarrow Y \) with \( h_0 = f \circ i \), then there is an extension \( \tilde{h} : X \times I \rightarrow Y \) so that \( \tilde{h} \circ (i \times \text{id}) = h \) and \( \tilde{h}_0 = f \). This can be represented by the following “lifting” diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\tilde{h}} & Y^I \\
\downarrow{i} & & \downarrow{ev_0} \\
X & \xrightarrow{f} & Y.
\end{array}
\]

Another name for a map satisfying the HEP is (Hurewicz) cofibration.

There is a universal example of such a lifting diagram. The data of a map \( X \xrightarrow{i} Y \) and a homotopy \( h : A \times I \rightarrow Y \) beginning at \( f \circ i \) amounts to a single map from the space defined by the pushout diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\iota_0} & A \times I \\
\downarrow{i} & & \downarrow{} \\
X & \xrightarrow{M(i)} & M(i) = X \cup_A A \times I.
\end{array}
\]

This is called the mapping cylinder of the map \( i \). Any lifting diagram for the HEP factors as

\[
\begin{array}{ccc}
A & \xrightarrow{\iota_0} & M(i)^I \rightarrow Y^I \\
\downarrow{i} & & \downarrow{ev_0} \\
X & \xrightarrow{M(i)} & Y,
\end{array}
\]

so it is enough to find a lift in the case of \( Y = M(i) \).

**Proposition 1.4.** The map \( i : A \rightarrow X \) is a cofibration if and only if \( M(i) \) is a retract of \( X \times I \).

**Proposition 1.5.** If \( i : A \rightarrow X \) is a cofibration, then \( i \) is a closed inclusion (this uses that \( X \) is weak Hausdorff).

**Example 1.6.** (1) The inclusion \( \emptyset \hookrightarrow X \) is always a cofibration. Indeed, \( Mi = X \), which is clearly a retract of \( X \times I \).
(2) The inclusion \( \{0\} \hookrightarrow I \) is a cofibration.
(3) The inclusion \( \partial I \hookrightarrow I \) is a cofibration.
(4) The inclusion \( S^{n-1} \hookrightarrow D^n \) is a cofibration.

2. Wed, Feb. 2

SNOW DAY!

3. Fri, Feb. 4

**Proposition 3.1.** The class of cofibrations is closed under

1. composition,
2. pushouts,
3. coproducts,
4. retracts, and
5. sequential colimits.

**Proof.** (1) Suppose \( A \xrightarrow{i} B \xrightarrow{j} C \) are both cofibrations, and consider a test diagram

\[
\begin{array}{c}
A \xrightarrow{h} Y' \\
\downarrow \, i \\
B \\
\downarrow \, j \\
C \\
\downarrow \, h_2 \\
X \xrightarrow{\text{ev}_0} Y
\end{array}
\]

The homotopy \( \tilde{h}_1 \) exists since \( i \) is a cofibration. We then form a lifting diagram with \( \tilde{h}_1 \) as the top horizontal arrow, and \( \tilde{h}_2 \) then exists because \( j \) is a cofibration.

(2) Suppose that \( i : A \to X \) is a cofibration and \( f : A \to Z \) is any map. We wish to show that the induced map \( Z \to Z \cup_A X \) is a cofibration. Consider the test diagram

\[
\begin{array}{c}
A \xrightarrow{h_1} Z \\
\downarrow \, i \\
X \xrightarrow{\text{ev}_0} Y
\end{array}
\]

The homotopy \( \tilde{h}_1 \) exists since \( i \) is a cofibration. The lift \( \tilde{h}_2 \) then exists by the universal property of the pushout.

(3) Exercise

(4) Suppose that \( i : A \to X \) is a retract of \( j : B \to Z \); that is, we have a diagram

\[
\begin{array}{c}
A \xrightarrow{i} B \xrightarrow{j} A \\
\downarrow \, i \\
X \xrightarrow{\text{Id}} Z \xrightarrow{\text{Id}} X
\end{array}
\]

in which both horizontal compositions are the identity. Consider the test diagram

\[
\begin{array}{c}
A \xrightarrow{h} Y' \\
\downarrow \, i \\
B \\
\downarrow \, j \\
X \\
\downarrow \, \text{ev}_0 \\
Y
\end{array}
\]

The displayed lift exists because \( j \) is a cofibration, and the desired lift is obtained by composing with the given map \( X \to Z \).

(5) Exercise
Corollary 3.2. If $A \hookrightarrow X$ is a relative CW complex, the inclusion is a cofibration.

Proposition 3.3. If $A \hookrightarrow X$ is a cofibration and $Z$ is any space, then $A \times Z \hookrightarrow X \times Z$ is a cofibration.

Proof. The two following lifting diagrams are equivalent, and we know there is a lift in the second:

$\begin{array}{ccc}
A \times Z & \xrightarrow{\sim} & Y^I \\
\downarrow & & \downarrow \\
X \times Z & \xrightarrow{} & Y
\end{array}$

Remark 3.4. There is also a notion of based cofibration, in which one starts with a test diagram of based maps (including a based homotopy on $A$) and asks for a based homotopy $\tilde{h}$.

Proposition 3.5. If a based map $A \rightarrow X$ is an unbased cofibration, then it is a based cofibration.

Proof. Given a based lifting diagram, we have a lift $\tilde{h} : X \rightarrow Y \times I$ if we forget about basepoints. But this lift must be a based homotopy, since the basepoint is in $A$, and the initial homotopy on $A$ was assumed to be based.

We say a based space $(X, x)$ is non-degenerately based (or well-pointed) if the inclusion of the basepoint is an unbased cofibration (note that it is vacuously a based cofibration). Given any based space $(X, x)$, one may "attach a whisker" to force $X$ to be well-pointed. That is, form the space $X' = X \vee I$, with new basepoint at the endpoint 1 of the interval (we glue $I$ to $X$ at the endpoint 0). Then it is easy to show that $\ast \rightarrow X'$ is a cofibration. This is a special case of the following result:

Proposition 3.6. (Replacing a map by a cofibration) Any map $f : X \rightarrow Y$ factors as a composition $X \xrightarrow{i} M(f) \xrightarrow{p} Y$, where $i$ is a cofibration and $p$ is a homotopy equivalence.

Proof. The map $i$ includes $X$ at time 1, and the map $p$ is defined by $p(x, t) = f(x)$ for $t > 0$ and $p(y, 0) = y$.

To see that $i$ is a cofibration, we need to provide a retraction to the inclusion $M(i) \hookrightarrow M(f) \times I$. The map $r : M(f) \times I \rightarrow M(i)$ defined by

$$r(x, s, t) = \begin{cases} 
(x, s(1 + t), 0) & s \leq 1/(1 + t) \\
(x, 1, s(1 + t) - 1) & s \geq 1/(1 + t)
\end{cases}$$

does the trick.

We now check that $p$ is a homotopy equivalence. Define $q : Y \rightarrow M(f)$ to be the inclusion at time 0. Then $p \circ q = \text{id}_Y$ and $q \circ p(x, t) = (f(x), 0)$. A homotopy $h : q \circ p \simeq \text{id}_M$ is given by

$$h(x, t, s) = (x, ts) \quad h(y, s) = y.$$

Proposition 3.7. If $X$ is non-degenerately based and $Y$ is a path-connected based space, then any map $f : X \rightarrow Y$ is homotopic to a based map.

Proof. Let $h : \ast \rightarrow Y^I$ specify a path from $f(*)$ to the basepoint $y$. By the HEP for the inclusion $\ast \hookrightarrow X$, this extends to a homotopy $\tilde{h} : X \rightarrow Y^I$, and the map $\tilde{h}(0, 1) : X \rightarrow Y$ is based by construction.