CLASS NOTES
MATH 527 (SPRING 2011)
WEEK 4

BERTRAND GUILLOU

1. Mon, Feb. 7

Proposition 1.1. *Let $i : A \longrightarrow X$ be a based map between non-degenerately based spaces. If i is a based cofibration, then it is also an unbased cofibration.*

Proof. Let

$$\begin{array}{ccc} A & \xrightarrow{h} & Y^I \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle ev_0} \\ X & \xrightarrow{f} & Y \end{array}$$

be a test diagram (f and g are not based maps). The space Y does not have a preassigned basepoint, so we choose $y_0 = f(x_0)$ as the basepoint. Then f, but not h, is a based map. Since A is non-degenerately based, the homotopy h is homotopic to a based map. This requires us to know that $h(a_0)$ is in the path-component (in Y^I) of the basepoint. But $h(a_0)$ is a path in Y beginning at y_0, so there is an obvious homotopy to the constant path at y_0. Use of this produces a homotopy of homotopies $H : A \times I \times I \longrightarrow Y$ satisfying

$$H(a_0, 0, s) = y_0 = H(a_0, t, 1).$$

The first of these equalities comes from our choice of contracting homotopy of $h(a_0)$, and the second is the statement that $H(a, t, 1)$ is a based homotopy.

Let us write $h_2(a, s) := H(a, 0, s)$. As we said above, this is a based homotopy. Then we get a lift in the diagram

$$\begin{array}{ccc} A & \xrightarrow{h_2} & Y^I \\ {\scriptstyle i}\downarrow & \overset{F}{\nearrow} & \downarrow \\ X & \xrightarrow{f} & Y. \end{array}$$

Let $\tilde{H} : A \times I \longrightarrow Y^I$ be defined by $\tilde{H}(a, t)(s) = (a, s, t)$ (note the change of order of the variables). We are now thinking of $A \times I$ as based at $(a_0, 1)$, and $\tilde{H}$ is now based. We therefore get a lift in the diagram

$$\begin{array}{ccc} A \times I & \xrightarrow{\tilde{H}} & Y^I \\ {\scriptstyle i\times \mathrm{id}}\downarrow & \overset{G}{\nearrow} & \downarrow \\ X \times I & \xrightarrow{F} & Y. \end{array}$$

The restriction of G to $X \times \{0\}$ is then a lift of h. ∎

Proposition 1.2. *Let $f : (X, x_0) \longrightarrow (Y, y_0)$ be a based map between non-degenerately based spaces. If f is a homotopy equivalence, then it is a* <u>*based*</u> *homotopy equivalence.*

Proof. Let $g : Y \longrightarrow X$ be a homotopy inverse. Since $g \circ f \simeq \mathrm{id}$, it follows that $g(y_0)$ is in the path-component of x_0, so that by the previous result we can replace g up to homotopy by a based map. Let $h : g \circ f \simeq \mathrm{id}$ be a homotopy. This homotopy may not be based.

Let γ be the path $h(x_0, t)$ in X. Since X is well-pointed, we have a lift in the diagram

$$\begin{array}{ccc} x_0 & \xrightarrow{\gamma} & X^I \\ \downarrow & \overset{h'}{\nearrow} & \downarrow ev_0 \\ X & \xrightarrow[\mathrm{id}]{} & X. \end{array}$$

Let $e = h'_1 : X \longrightarrow X$. We claim that $e \circ g \circ f$ is based homotopic to the identity. Define maps

$$J : X \times I \longrightarrow X \qquad K : I \longrightarrow X^I$$

by the formulas

$$J(x, s) = \begin{cases} h'(g \circ f(x), 1 - 2s) & s \leq \frac{1}{2} \\ h(x, 2s - 1) & s \geq \frac{1}{2}, \end{cases}$$
$$K(s, t) = \begin{cases} \gamma(1 - 2s(1 - t)) & s \leq \frac{1}{2} \\ \gamma(1 - 2(1 - s)(1 - t)) & s \geq \frac{1}{2}. \end{cases}$$

J specifies a homotopy $e \circ g \circ f \simeq g \circ f$ on the first half of the interval and a homotopy $g \circ f \simeq \mathrm{id}$ on the second half (this is not a based homotopy). The map K is given, for fixed t, by traveling along $\gamma\,|_{[t,1]}$ backwards and then forwards. The important thing is that K takes value x_0 if either $s = 0, 1$ or $t = 1$. The HEP now gives a lift

$$\begin{array}{ccc} I & \xrightarrow{K} & X^I \\ \downarrow & \overset{L}{\nearrow} & \downarrow ev_0 \\ X \times I & \xrightarrow[J]{} & X. \end{array}$$

The restriction of L to the intervals $(0, t)$, $(s, 1)$, and $(1, 1 - t)$ now specifies a based homotopy $e \circ g \circ f$. So, writing $g' = e \circ g$, we have that $g' \circ f$ is based homotopic to the identity of X. We know that $f \circ g' \simeq \mathrm{id}_Y$, but we do not know that there is a based homotopy. But we can repeat the above argument to replace f by a homotopic based map f' so that $f' \circ g' \simeq_* \mathrm{id}_Y$. It is now formal that the left and right homotopy inverses for g' must coincide up to based homotopy, so that we have a based homotopy equivalence. ■

We are headed towards a proof of the Whitehead theorem, but first we will need to discuss **relative homotopy groups**. Suppose given a based map $i : A \longrightarrow X$ (usually an inclusion). In order to define relative homotopy groups, it is convenient to use the models I^n and ∂I^n for D^n and S^{n-1}. Recall that we also have the subspace $J^n \subset \partial^n$ given by $J^n = \partial I^{n-1} \times I \cup I^{n-1} \times \{1\}$ with $\partial I^n / J^n \cong S^{n-1}$. For any $n \geq 1$, we define

$$\pi_n(X, A, a_0) = [(I^n, \partial I^{n-1}, J^n), (X, A, a_0)].$$

That is, the relative homotopy group $\pi_n(X, A, a_0)$ is the set of homotopy classes of diagrams

$$\begin{array}{ccc} J^n & \longrightarrow & a_0 \\ \downarrow & & \downarrow \\ \partial I^n & \xrightarrow{g} & A \\ \downarrow & & \downarrow \\ I^n & \xrightarrow{f} & X, \end{array}$$

where the homotopies are through maps of the same form. Note that when A is simply the basepoint of X, then we get $\pi_n(X, x, x) = \pi_n(X, x)$.

There is another useful description of relative homotopy groups. Given a based map $i : A \longrightarrow X$ as above, define a space $F(i) \subseteq X^I \times A$ (the **homotopy fiber** of i) by

$$F(i) = \{(\gamma, a) \mid \gamma(1) = x_0, \gamma(0) = i(a).$$

The pair (c_{x_0}, a_0) consisting of the constant path at x_0 and the lift a_0 serve as a natural basepoint for $F(i)$.

Proposition 1.3. *For any $n \geq 1$, we have*

$$\pi_n(X, A, a_0) \cong \pi_{n-1}(Fi).$$

Proof. A map $f : I^n \longrightarrow X$ corresponds to a map $I^{n-1} \longrightarrow X^I$. The restriction of g to $I^{n-1} \times \{0\} \longrightarrow A$ gives the second component of a map $\varphi : I^{n-1} \longrightarrow F$. Since the restriction of g to J^n is constant at the basepoint, it follows that φ sends all of ∂I^{n-1} to the basepoint of F. ■

Corollary 1.4. *The set $\pi_n(X, A, a_0)$ is a group for $n \geq 2$ and an abelian group for $n \geq 3$.*

Note that the relative homotopy groups are functorial with respect to maps of triples. In particular, the map of triples $(X, x_0, x_0) \longrightarrow (X, A, a_0)$ induces a map

$$j_* : \pi_n(X, x) \longrightarrow \pi_n(X, A, a_0).$$

We also have a "boundary map"

$$\partial : \pi_n(X, A, a_0) \longrightarrow \pi_{n-1}(A, a_0)$$

which assigns to a map (f, g) of triples the restriction of g to $I^{n-1} \times \{0\}$. The further restriction of this to $\partial I^{n-1} \times \{0\} \subseteq J^n$ is constant at the basepoint, so we get an induced based map $I^{n-1}/\partial I^{n-1} \longrightarrow A$.

2. Wed, Feb. 9

Theorem 2.1. *The sequence*

$$\dots \to \pi_n(A, a_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, a_0) \xrightarrow{\partial} \pi_{n-1}(A) \to \dots \xrightarrow{\partial} \pi_0(A) \to \pi_0(X)$$

is a long exact sequence.

Proof. We begin by establishing that

$$\pi_1(A) \xrightarrow{i_*} \pi_1(X) \xrightarrow{j_*} \pi_1(X, A) \xrightarrow{\partial} \pi_0(A) \xrightarrow{i_*} \pi_0(X)$$

is exact.

Exactness at $\pi_0(A)$: Let $a' \in A$ and suppose that $i_*(a') = [x_0]$ in $\pi_0(X)$. Then we have a path γ in X starting at $i(a')$ and ending at x_0. But then the pair (γ, a') specifies a point of F, and $a' = \partial(\gamma, a')$. Conversely, if (γ, a') is a point in F, then the path γ in X establishes that $i_*\partial(\gamma, a') = i_*[a'] = [x_0]$ in X.

Exactness at $\pi_1(X, A)$: Let (γ, a') be a point of F (so γ is a path in X starting at $i(a')$ and ending at x_0) such that $[a'] = [a_0]$ in $\pi_0(A)$. Let α be a path in A starting at a' and ending at a_0. Then $\gamma^{-1}i(\alpha)$ specifies a loop in X based at x_0. Moreover, the corresponding map of triples $(I, \partial I, \{1\}) \longrightarrow (X, A, a_0)$ is homotopic to (γ, a') via the homotopy that simply contracts $i(\alpha)$ to the constant path at $i(a')$.

Exactness at $\pi_1(X)$: Let $\beta : I \longrightarrow X$ be a loop based at x_0, and suppose $j_*(\beta)$ is trivial in $\pi_1(X, A)$. This means that we have a homotopy $h : \beta \simeq c_{x_0}$ to the constant path such that $h(1, t) = x_0$ for all t and such that $h(0, t)$ is the image of a loop α in A. Now the homotopy h specifies a based homotopy $\beta \simeq i(\alpha)$. In other words, $[\beta] = i_*[\alpha]$.

Now we will reinterpret the rest of the terms in the sequence as shifted copies of the terms just discussed. Let $d_i : F(i) \longrightarrow A$ be the projection map.

Lemma 2.2. *The map $\partial : \pi_1(X,A) \longrightarrow \pi_0(A)$ corresponds to $(d_i)_*$ under the isomorphism $\pi_1(X,A) \cong \pi_0(F(i))$.*

Let ΩX denote the based loop space $\mathrm{Map}_*(S^1, X)$ of X. Then

$$\pi_1(X) = [S^1, X]_* \cong [S^0, \Omega X]_* \cong \pi_0(\Omega X).$$

We have a map $\Omega X \longrightarrow F(i)$ which sends a loop γ to the pair (γ, a_0).

Lemma 2.3. *The above map makes the diagram*

$$\begin{array}{ccc} \pi_1(X) & \xrightarrow{j_*} & \pi_1(X,A) \\ \cong\downarrow & & \downarrow\cong \\ \pi_0(\Omega X) & \longrightarrow & \pi_0(Fi) \end{array}$$

commute.

Lemma 2.4. *There is a homeomorphism making the following diagram commute:*

$$\begin{array}{ccc} F(\Omega i) & \xrightarrow{\cong} & \Omega F(i) \\ & {\scriptstyle d_{\Omega i}}\searrow \quad \swarrow{\scriptstyle \Omega d_i} & \\ & \Omega A. & \end{array}$$

Proof. We define the required map by sending the pair (h, γ) to the map

$$t \mapsto (ev_t \circ h, \gamma(t)).$$

It is not difficult to see this is a homeomorphism and that the images of these elements under the maps to ΩA are both γ. ■

As a result of the above lemmas, we get exactness of the long sequence at three more spots to the left. The above tells us that the maps

$$\pi_2(A) \xrightarrow{i_*} \pi_2(X) \xrightarrow{j_*} \pi_2(X,A) \xrightarrow{\partial} \pi_1(A) \xrightarrow{i_*} \pi_1(X)$$

may be reinterpreted as the maps in

$$\pi_1(\Omega A) \xrightarrow{i_*} \pi_1(\Omega X) \xrightarrow{j_*} \pi_1(\Omega X, \Omega A) \xrightarrow{\partial} \pi_0(\Omega A) \xrightarrow{i_*} \pi_0(\Omega X),$$

so we are done. ■

3. Fri, Feb. 11

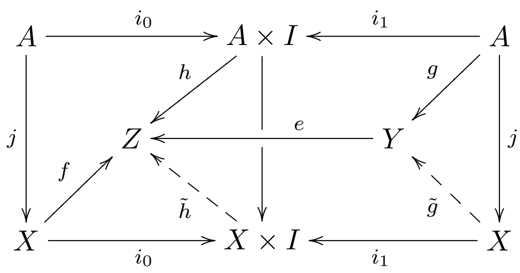
Definition 3.1. We say that a map $f : X \longrightarrow Y$ is an n**-equivalence** if for every choice of basepoint $x \in X$, the map $\pi_i(X,x) \longrightarrow \pi_i(Y, f(x))$ is an isomorphism for $i < n$ and a surjection for $i = n$.

Proposition 3.2. *A map $f : X \longrightarrow Y$ is an n-equivalence if and only the relative homotopy groups $\pi_i(Y,X)$ vanish for $i \leq n$ and $\pi_0(X) \longrightarrow \pi_0(Y)$ is surjective*

Because of this, an n-equivalence is also sometimes called an n-connected map.

One of the key tools in working with CW complexes is the Homotopy Extension and Lifting Property (HELP).

Theorem 3.3. *(HELP, May 10.3) Let (X,A) be a relative CW complex of dimension $\leq n$ and let $e : Y \longrightarrow Z$ be an n-equivalence. Then, given maps $f : X \longrightarrow Z$, $g : A \longrightarrow Y$, and $h : A \times I \longrightarrow Z$ such that $f|_A = h \circ i_0$ and $e \circ g = h \circ i_i$ in the following diagram, there are maps $\tilde{g}$ and $\tilde{h}$ that make the entire diagram commute:*



Proof. The proof is by induction on the cells. It thus suffices to consider the case of attaching a single cell e^d of dimension $d \leq n$ to A. Since then $X = A \cup_{S^{d-1}} e^d$, by the universal properties of pushouts, it is enough to consider the case $S^{d-1} \hookrightarrow D^d$. We treat this case separately below. ∎

Proposition 3.4. *The HELP holds for the inclusion $S^{d-1} \hookrightarrow D^d$ for any $d \leq n$.*

Proof. We have already seen that the inclusion $S^{d-1} \hookrightarrow D^d$ satisfies the HEP. That is, the homotopy h defined on S^{d-1} extends to one $\hat{h}$ defined on D^d. This is not yet the desired homotopy $\tilde{h}$, as there is no reason for the endpoint of the homotopy, $h(-,1)$, to lift to a map to Y.

We will use the following lemma:

Lemma 3.5 (Compression)**.** *If a map of triples $(I^d, \partial I^d, J^d) \xrightarrow{f,g} (Z, Y, y_0)$ represents zero in $\pi_d(Z, Y, y_0)$, then the map $I^d \longrightarrow Z$ is homotopic, rel ∂I^d, to a map that lifts to Y.*

Proof. Suppose H is a homotopy from the map (f, g) of triples to the constant map. Thus H corresponds to a map $H_1 : I^d \times I \longrightarrow Z$ and a lift $H_2 : \partial I^d \times I \longrightarrow Y$ of $H_1 \mid_{\partial I^d \times I}$. The restriction of H_1 to $J^d \times I$ is constant at the basepoint z_0 and similarly with the restriction to $I^d \times \{1\}$. So both of these restrictions lift to a constant map to Y. The restriction of H_1 to $I^{d-1} \times \{0\} \times I$ lifts to Y by hypothesis. But now the point is that the union

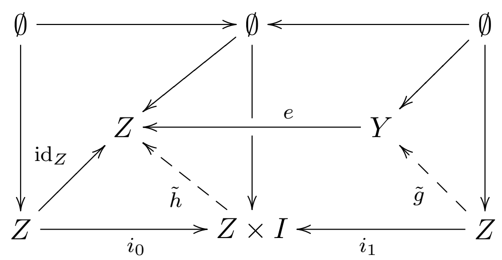
$$(J^d \times I) \cup (I^d \times \{1\}) \cup (I^{d-1} \times \{0\} \times I)$$

is another model for the disk D^d. The boundary is $\partial I^d \times \{0\}$, the same as that of the disk $I^d \times \{0\}$. It follows that the map H_1 specifies a homotopy from f to the map $e \circ H_2 \mid_{I^{d-1} \times \{0\} \times I}$ and that this homotopy is constant on the chosen model for S^{d-1}. ∎

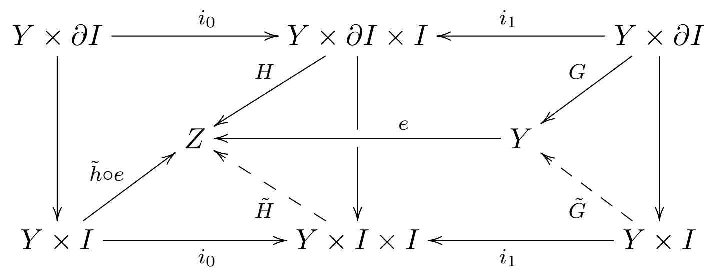
As $e : Y \longrightarrow Z$ is an n-equivalence, the relative homotopy group $\pi_d(Z, Y)$ vanishes, so that the map of pairs $h(-,1) : (D^d, S^{d-1}) \longrightarrow (Z, Y)$ is homotopic, rel S^{d-1}, to a map that lifts to Y by the lemma. This new homotopy may be glued to $\hat{h}$ to obtain $\tilde{h}$. (Draw a picture) ∎

Theorem 3.6 (Whitehead's theorem)**.** *Let $e : Y \longrightarrow Z$ be a weak equivalence between cell complexes. Then e is a homotopy equivalence.*

Proof. Applying HELP with $A = \emptyset$, $X = Z$, and $f = \mathrm{id}_Z$ gives a map $\tilde{g} : Z \longrightarrow Y$ and a homotopy $\tilde{h} : \mathrm{id}_Z \simeq e \circ \tilde{g}$.



To see that $\tilde{g} \circ e$ is also homotopic to the identity, use HELP with $A = Y \times \partial I$, $X = Y \times I$, G the map $\tilde{g} \circ e \coprod \mathrm{id}_Y$ and H the constant homotopy.



The desired homotopy is then given by $\tilde{G}$. ■