CLASS NOTES
MATH 527 (SPRING 2011)
WEEK 5

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1. Mon, Feb. 14

The same method we used to prove the Whitehead theorem last time also gives the following result.

Theorem 1.1. *Let X be CW and suppose $f : Y \longrightarrow Z$ is a weak equivalence (Y and Z are not assumed to be CW). Then f induces a bijection*

$$[X, Y] \xrightarrow{\cong} [X, Z].$$

Cellular Approximation

Proposition 1.2. *The inclusion of the n-skeleton $X_n \hookrightarrow X$ is n-connected.*

Proof. We must show that given any map of pairs $(f, g) : (D^n, S^{n-1}) \longrightarrow (X, X_n)$, there is a homotopy of f, rel S^{n-1} to a map landing in X_n. Since D^n is compact, $f(D^n)$ meets only finitely many cells in X, and we may assume by induction that $X = X_n \cup e^p$, with $p > n$.

Lemma 1.3. *$f : D^n \longrightarrow X_n \cup e^p$ is homotopic, rel S^{n-1} to a map $f' : D^n \longrightarrow X_n \cup e^p$ which misses a point y of e^p.*

See Hatcher, 4.10 for a proof of this. The space $(X_n \cup e^p) - \{y\}$ deformation retracts onto X_n, so we are done. ■

The above result states that the homotopy groups $\pi_k(X)$ for $k < n$ only depend on the n-skeleton X_n. In particular, if we form $Y = X \cup e^N$, where $N > n$, then the map $X \longrightarrow Y$ is also an n-equivalence, since we have not changed the n-skeleton.

Theorem 1.4. *(Cellular approximation theorem) A map $f : X \longrightarrow Y$ of CW complexes is homotopic to a cellular map.*

Proof. First, since any point of Y is connected by a path to Y_0, we may homotope the map $f\mid_{X_0}$ to a map landing in Y_0. By induction, assume that we have a homotopy $h : X_n \times I \longrightarrow Y$ such that $h_0 = f$ and $g = h_1$ lands in Y_n. Given any attaching map $j : S^n \longrightarrow X_n$ for an $n+1$-cell of X, we apply HELP with the pair (D^{n+1}, S^n) mapping to the $(n+1)$-equivalence $Y_{n+1} \hookrightarrow Y$.

$$\begin{tikzcd}
S^n \arrow[rr, "i_0"] \arrow[dd] & & S^n \times I \arrow[dl, "h\circ j"'] \arrow[dd] & & S^n \arrow[ll, "i_1"'] \arrow[dl, "g\circ j"] \arrow[dd] \\
& Y & & Y_n \arrow[ll] & \\
D^{n+1} \arrow[ur, "f\circ\Phi"] \arrow[rr, "i_0"'] & & D^{n+1} \times I \arrow[ul, dashed] & & D^{n+1} \times I \arrow[ll, "i_1"] \arrow[ul, dashed]
\end{tikzcd}$$

This provides us with a map $g_{n+1} : D^{n+1} \longrightarrow Y_n + 1$ and the desired homotopy. ■

Remark 1.5. The same proof extends to show that if $f : (X, A) \longrightarrow (Y, B)$ is a map of relative CW complexes, then f is homotopic rel A to a cellular map.

So now we know that $\pi_k(S^n) \cong 0$ for $k < n$. Just give S^n the CW structure with one 0-cell and one n-cell. Then any map $S^k \longrightarrow S^n$ is homotopic, rel the basepoint, to the constant based map.

Example 1.6. A famous non-example of cellular map is the diagonal map $\Delta_X : X \longrightarrow X \times X$. For instance, consider $X = S^1$. The 1-skeleton of the torus $S^1 \times S^1$ is $S^1 \vee S^1$, which does not include the diagonal copy of S^1 in the torus.

This causes difficulties when working with the cohomology ring. Recall that the cup product in cohomology can be described as a composition

$$\mathrm{H}^n(X) \otimes \mathrm{H}^p(X) \longrightarrow \mathrm{H}^{n+p}(X \times X) \longrightarrow \mathrm{H}^{n+p}(X),$$

where the second map is induced by the diagonal Δ_X. Cellular cohomology is functorial only with respect to cellular maps, which means that in order to compute the cup product in cellular cohomology, one must first choose a cellular approximation to the diagonal map.

Let $\mathbf{Ho}(CW)$ be the full subcategory of $\mathbf{Ho}(\mathbf{Top})$ whose objects are the CW complexes, and let us denote by $\iota_C W$ the inclusion $\iota_C W : \mathbf{Ho}(CW) \longrightarrow \mathbf{Ho}(\mathbf{Top})$.

2. Wed, Feb 16

Theorem 2.1. *(CW approximation) There is a functor* $\Gamma : \mathbf{Top} \longrightarrow CW$ *and a natural transformation* $\gamma : \iota_{CW} \circ \Gamma \Rightarrow \mathrm{id}$ *whose components are weak equivalences.*

Proof. Let X be a space. The CW complex ΓX will be a "CW approximation for the space X." The plan will be to build the CW complex ΓX by inductively defining the skeleta. We will have a system of maps

$$\begin{tikzcd}
X_0 \arrow[r, "i_0"] \arrow[drr, "\gamma_0"'] & X_1 \arrow[r, "i_1"] \arrow[dr, "\gamma_1"] & \cdots \arrow[r] & X_n \arrow[r, "i_n"] \arrow[dl, "\gamma_n"'] & X_{n+1} \arrow[r] \arrow[dll, "\gamma_{n+1}"] & \cdots \\
& & X, & & &
\end{tikzcd}$$

in which each γ_n is an n-equivalence.

We begin by setting X_0 to be the set of points of X, with the discrete topology. The natural map $X_0 \longrightarrow X$ is surjective on path components, i.e. a 0-equivalence.

Thus assume inductively that we have $\gamma_n : X_n \longrightarrow X$ satisfying the above conditions. Let K_n denote the set of all diagrams of maps

$$\begin{tikzcd}
S^n \arrow[r, "g"] \arrow[d] & X_n \arrow[d, "\gamma_n"] \\
D^{n+1} \arrow[r, "h"] & X,
\end{tikzcd}$$

where the map D^{n+1} is thought of as a (based) null homotopy $\gamma_n \circ g \sim *$. Define

$$X_{n+1} = X_n \cup_{\amalg_{K_n} S^n} \amalg_{K_n} D^{n+1}.$$

The map $\gamma_{n+1} : X_{n+1} \longrightarrow X$ is then given on each D^{n+1} by the specified map h. This makes $\gamma_{n+1} \circ i_n = \gamma_n$, and γ_{n+1} is an n-equivalence since γ_n does (2-out-of-3). It remains to show that $\pi_n(X_{n+1}) \longrightarrow \pi_n(X)$ is injective and $\pi_{n+1}(X_{n+1}) \longrightarrow \pi_{n+1}(X)$ is surjective.

The injectivity follows from cellular approximation. Any map $\alpha : S^n \longrightarrow X_{n+1}$ is homotopic to one landing in the n-skeleton, which is X_n. But if γ_n of this is null in the homotopy of X, we have killed it in the homotopy of X_{n+1} by construction.

For surjectivity, let $\beta : S^{n+1} \longrightarrow X$ be a class in X. We can represent this is a diagram

$$\begin{tikzcd} S^n \arrow[r, "g(*)"] \arrow[d] & X_n \arrow[d] \\ D^{n+1} \arrow[r, "g"] & X, \end{tikzcd}$$

where the top horizontal map is the constant map to the image $g(*)$ of the basepoint. We thus get a map $D^{n+1} \longrightarrow X_{n+1}$ which collapses the boundary, i.e. a map $S^{n+1} \longrightarrow X_{n+1}$ representing a lift of the class in X.

We now define $\Gamma(X) = \operatorname{colim}_n X_n$, and we have an induced map $\gamma : \Gamma(X) \longrightarrow X$. The inclusion $X_n \hookrightarrow \Gamma(X)$ is an n-equivalence, and we conclude that $\gamma : \Gamma(X) \longrightarrow X$ is an n-equivalence. As this is true for each n, γ is a weak equivalence.

It remains to consider functoriality. Let $f : X \longrightarrow Y$ be a map. We define $\Gamma(f) : \Gamma(X) \longrightarrow \Gamma(Y)$ by defining appropriate maps $f_n : X_n \longrightarrow Y_n$. The map $f_0 : X_0 \longrightarrow Y_0$ is just the map f again. If we have already built f_n, then we specify the map f_{n+1} by sending the cell D^{n+1} labeled as (g, h) isomorphically onto the cell of Y_{n+1} labeled by $(f \circ g, f \circ h)$. There results a map $\Gamma(f)$ making the following diagram commute:

$$\begin{tikzcd} \Gamma(X) \arrow[r, "\Gamma(f)"] \arrow[d, "\gamma_X"'] & \Gamma(Y) \arrow[d, "\gamma_Y"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Furthermore, one can check that $\Gamma(\mathrm{id}) = \mathrm{id}$ and $\Gamma(f \circ g) = \Gamma(f) \circ \Gamma(g)$, so that Γ is a functor. ■

Proposition 2.2. *The functor Γ descends to a functor $\mathbf{Ho}(\Gamma) : \mathbf{Ho}(\mathbf{Top}) \longrightarrow \mathbf{Ho}(CW)$.*

Proof. This follows from the (generalization of the) Whitehead theorem. Suppose that $f \simeq g : X \longrightarrow Y$. The map $\gamma_Y : \Gamma(Y) \xrightarrow{\sim} Y$ induces a bijection

$$[\Gamma X, \Gamma Y] \cong [\Gamma X, Y].$$

In particular, the classes of $\Gamma(f)$ and $\Gamma(g)$ are sent to the classes of $f \circ \gamma_X$ and $g \circ \gamma_X$, respectively. Since these classes coincide, we conclude that the classes of $\Gamma(f)$ and $\Gamma(g)$ similarly coincide. ■

Corollary 2.3. *The category $\mathbf{Ho}(CW)$ "is" the homotopy category of spaces, that is, the localization of $\mathbf{Top}$ with respect to the weak homotopy equivalences. The composition $\mathbf{Top} \xrightarrow{\Gamma} CW \longrightarrow \mathbf{Ho}(CW)$ is the universal functor from $\mathbf{Top}$ that takes weak equivalences to isomorphisms.*

Proof. Given a functor $F : \mathbf{Top} \longrightarrow \mathscr{C}$ that converts weak homotopy equivalences to isomorphisms, we define $\tilde{F} : \mathbf{Ho}(CW) \longrightarrow \mathscr{C}$ by $\tilde{F}(X) = F(X)$. Note that since $\tilde{F}$ converts homotopy equivalences to isomorphisms, it follows that F factors through $\mathbf{Ho}(\mathbf{Top})$, so that $\tilde{F}$ is well-defined on morphisms. The required natural isomorphism $\tilde{F} \circ \Gamma \cong F$ is given on a space X by the map $F(\gamma_X) : \tilde{F} \circ \Gamma(X) = F(\Gamma X) \longrightarrow F(X)$. ■

Proposition 2.4. *The above CW approximation theorem generalizes to the following statement: let $f : W \longrightarrow Y$ be a map of spaces. Then f can be factored, in a functorial way, as a composition $W \xrightarrow{\lambda} \Gamma(Y, W) \xrightarrow{\gamma} Y$ in which λ is a relative CW complex and γ is a weak equivalence.*

3. Fri, Feb. 18

More on the homotopy category. Another (equivalent) formulation: the category $w\mathbf{Ho}(\mathbf{Top})$ has the same objects as $\mathbf{Top}$, but we define

$$w\mathbf{Ho}(\mathbf{Top})(X, Y) := [\Gamma X, \Gamma Y].$$

The universal functor $\mathbf{Top} \xrightarrow{\iota} w\mathbf{Ho}(\mathbf{Top})$ then looks like the identity on objects and like Γ on morphisms.

Given a functor $F : \mathbf{Top} \longrightarrow \mathscr{C}$, a **left derived functor** for F is a functor $\tilde{F} : w\mathbf{Ho}(\mathbf{Top}) \longrightarrow \mathscr{C}$ and a natural transformation $\tilde{F} \circ \iota \Rightarrow$ which is terminal among such data. That is, if we have $G : w\mathbf{Ho}(\mathbf{Top}) \longrightarrow \mathscr{C}$ and a transformation $G \circ \iota \Rightarrow F$, then we get a natural transformation $G \Rightarrow \tilde{F}$ making the appropriate diagram commute. The left derived functor is the closest approximation to F (from the left) by a functor that takes weak equivalences to isomorphisms.

Proposition 3.1. *Let $F : \mathbf{Top} \longrightarrow \mathscr{C}$ be a functor that factors through $\mathbf{Ho}(\mathbf{Top})$. Then F has a left derived functor $\mathbb{L}F : w\mathbf{Ho}(\mathbf{Top}) = \mathbf{Ho}(CW) \longrightarrow \mathscr{C}$.*

Proof. We define $\mathbb{L}F(X) = F(\Gamma X)$, and the transformation $\mathbb{L}F \circ \iota(X) \longrightarrow F(X)$ is given by the CW approximation map γ_X.

If G is any other factorization equipped with a transformation $\alpha : G \circ \iota \Rightarrow F$, we define the transformation $G \Rightarrow \mathbb{L}F$ to be the composition

$$G(X) \xleftarrow{\cong} G(\Gamma X) \xrightarrow{\alpha} F(\Gamma X) = \mathbb{L}F(X).$$

■

Example 3.2. (Topological *Ext*) Fix a space Z and consider the (contravariant) functor $\mathrm{Map}(-, Z) : \mathbf{Top}^{op} \longrightarrow \mathbf{Top}$. This does <u>not</u> preserve all weak equivalences. For example, consider the weak equivalence $f : S^0 \longrightarrow X$ from homework 2, where X is the topologist's sine curve. Take $Z = S^0$. Then $\mathrm{Map}(X, S^0) = S^0$ since the sine curve is connected, but $\mathrm{Map}(S^0, S^0)$ has four points. So

$$\mathrm{Map}(f, S^0) : \mathrm{Map}(X, S^0) \longrightarrow \mathrm{Map}(S^0, S^0)$$

is not a weak equivalence. The derive functor is $\mathrm{Map}(\Gamma(-), Z)$, and this is now a functor that preserves weak equivalences. Because of contravariance, this is actually a <u>right</u> derived functor (the map goes $\mathrm{Map}(W, Z) \longrightarrow \mathrm{Map}(\Gamma W, Z)$). So the "correct" (or derived) value of $\mathrm{Map}(X, S^0)$ is $\mathbb{R}\,\mathrm{Map}(X, S^0) \simeq \mathrm{Map}(S^0, S^0)$, the four point space.

Cofiber sequences

Recall that we defined a homotopy fiber of a (based) map $f : A \longrightarrow X$. There is also a notion of homotopy "cofiber", defined as follows. The **(homotopy) cofiber** of a based map $f : A \longrightarrow X$ is the mapping cone

$$C(f) = X \cup_A A \wedge I_+/(A \times \{1\}.$$

The space $X \cup_A A \wedge I_+$ is the **reduced mapping cylinder** $M_*(f)$, and we have $C(f) \cong M_*(f)/A$. The space $A \wedge I_+/(A \times \{1\}$ is the **reduced cone** on A, denoted $C_*(A)$, so we have $C(f) = X \cup_A C_*(A)$. Yet another way to think of this construction: we replace the given map $A \xrightarrow{f} X$ by the composition of based maps

$$A \xrightarrow{j} M_*(f) \xrightarrow{p} X,$$

where j is the inclusion at time 1 and is a cofibration, as we have discussed, and p is a (based) homotopy equivalence. We then take the point-set cofiber (quotient) of j, the replacement for f. The map p induces a map $\overline{p} : C(f) = M_*(f)/A \longrightarrow X/A$.

Proposition 3.3. *Suppose $f : A \longrightarrow X$ is a (based) cofibration. Then the induced map $\overline{p} : C(f) \longrightarrow X/A$ is a based homotopy equivalence.*

Proof. Since f is a based cofibration, we know that $M_*(f)$ is a retract of the cylinder $X \wedge I_+$. Let $r : X \wedge I_+ \longrightarrow M_*(f)$ be a retraction. If we include X in the cylinder $X \wedge I_+$ at time 1, then the image of A under the composition to $M_*(f)$ is a point, so we get an induced map $\overline{r} : X/A \longrightarrow C(f)$. Moreover, the retraction $r : X \wedge I_+ \longrightarrow M_*(f)$ is the identity on the subspace $A \wedge I_+$, so collapsing

$A \wedge I_+$ in the domain and codomain of r gives a map $X/A \wedge I_+ \longrightarrow X/A$. At time 0, this is the identity, since r was a retraction, and at time 1 this is the map $\overline{p} \circ \overline{r}$.

It remains to show that $\overline{r} \circ \overline{p} \simeq \mathrm{id}_{M_*(f)}$. We define a homotopy $M_*(f) \wedge I_+ \longrightarrow M_*(f)$ as follows. On $X \wedge I_+$, we use the retraction r. On $C_*(A) \wedge I_+$, we use the homotopy $C_*(A) \wedge I_+ \longrightarrow C_*(A)$ which at time t uses the linear isomorphism of $[0, 1]$ with $[t, 1]$. ■

This result says that if $A \longrightarrow X$ is a cofibration, then the point-set quotient X/A has the "correct", or derived homotopy type.