CLASS NOTES
MATH 527 (SPRING 2011)
WEEK 6

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1. Mon, Feb. 21

Note that since we have $C(f) = X \cup_A C_*(A)$ and the inclusion $A \hookrightarrow C_*(A)$ at time 0 is a cofibration, it follows that the pushout map $i : X \longrightarrow C(f)$ is a cofibration. In particular, the quotient $C(f)/X$ is based homotopy equivalent to $C(X \xrightarrow{i} C(f))$. The quotient $C(f)/X$ can also be identified with ΣA. So we get a "cofiber sequence"

$$A \xrightarrow{f} X \xrightarrow{i} C(f) \xrightarrow{\partial} \Sigma A.$$

Proposition 1.1. *The cofiber of ∂ is homotopy equivalent to ΣX, and under this identification, the map $\Sigma A \longrightarrow C(\partial) \simeq \Sigma X$ is identified with the map $-\Sigma f$ defined by $-\Sigma f(x,t) = (x, 1-t)$.*

(Draw pictures)

So the cofiber sequence extends to a long sequence of based maps

$$A \xrightarrow{f} X \xrightarrow{i} C(f) \xrightarrow{\partial} \Sigma A \xrightarrow{-\Sigma f} \Sigma X \xrightarrow{-\Sigma i} \dots.$$

Theorem 1.2. *(Long exact sequence) For any based space Z, mapping the long cofiber sequence into Z produces a long exact sequence*

$$\dots \longrightarrow [\Sigma X, Z]_* \longrightarrow [\Sigma A, Z]_* \longrightarrow [C(f), Z]_* \longrightarrow [X, Z]_* \longrightarrow [A, Z]_*.$$

At the right end of the sequence, these are just pointed sets, but $[\Sigma A, Z]_* \cong \pi_1(\mathrm{Map}_*(A, Z))$ and $[\Sigma^2 A, Z]_* \cong \pi_2(\mathrm{Map}_*(A, Z))$, so we get groups and eventually abelian groups in this long exact sequence. The proof of this result is completely analogous to the proof of the long exact sequence in homotopy.

In fact, this result can also be deduced from the earlier result for the following reason:

Proposition 1.3. *Let $f : A \longrightarrow X$ be a based map and let $\mathrm{Map}_*(f) : \mathrm{Map}_*(X, Z) \longrightarrow \mathrm{Map}_*(A, Z)$ be the induced map. Then the homotopy fiber $F(\mathrm{Map}_*(f))$ is homeomorphic to the mapping space $\mathrm{Map}_*(C(f), Z)$.*

Proof. This follows from the universal property of the pushout $C(f) = X \cup AC_*(A)$:

$$\begin{aligned}\mathrm{Map}_*(C(f), Z) &\cong \mathrm{Map}_*(X, Z) \times_{\mathrm{Map}_*(A,Z)} \mathrm{Map}_*(C_*(A), Z) \\ &\cong \mathrm{Map}_*(X, Z) \times_{\mathrm{Map}_*(A,Z)} P_* \mathrm{Map}_*(A, Z) \cong F(f),\end{aligned}$$

where $P_*(Y)$ is the space of based paths in Y (ending at the basepoint). ∎

Although cofiber sequences do not give rise to a long exact sequence in homotopy, they do give rise to a long exact sequence in homology:

Theorem 1.4. *For any based map $f : A \longrightarrow X$, there is a long exact sequence in reduced homology*

$$\dots \longrightarrow \tilde{\mathrm{H}}_n(A) \longrightarrow \tilde{\mathrm{H}}_n(X) \longrightarrow \tilde{\mathrm{H}}_n(C(f)) \longrightarrow \tilde{\mathrm{H}}_{n-1}(A) \longrightarrow \dots.$$

Proof. Up to homotopy equivalence, we may assume that f is a cofibration and that the cofiber $C(f)$ is the quotient X/A. This is then the long exact sequence in homology for a "good pair". ■

Homotopy pushouts

Last time, we said that if $f : A \longrightarrow X$ is a cofibration, then the cofiber $C(f)$ is homotopy equivalent to X/A. This generalizes as follows:

Proposition 1.5. *Let $j : A \longrightarrow X$ be a cofibration and consider a pushout square*

$$\begin{array}{ccc} A & \xrightarrow{j} & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & P. \end{array}$$

Then P is homotopy equivalent to the "double mapping cylinder" $B \cup_A A \times I \cup_A X$. Moreover, suppose given a map of diagrams

$$\left(\begin{array}{ccc} A & \xrightarrow{j} & X \\ \downarrow & & \\ B & & \end{array}\right) \longrightarrow \left(\begin{array}{ccc} A' & \xrightarrow{j'} & X' \\ \downarrow & & \\ B' & & \end{array}\right)$$

in which both j and j' are cofibrations and the component maps $f_A : A \xrightarrow{\sim} A'$, $f_B : B \xrightarrow{\sim} B'$, $f_X :\xrightarrow{\sim} X'$ are homotopy equivalences. Then $P \simeq P'$ (where P' is the pushout of the second diagram).

The space P above is called the **homotopy pushout** of the maps $B \leftarrow A \rightarrow X$.

Proof. A very similar argument to the one from last time shows that P is homotopy equivalent to the pushout of the diagram

$$\begin{array}{ccc} A & \longrightarrow & M(j) \\ \downarrow & & \\ B. & & \end{array}$$

But this pushout is homeomorphic to the double mapping cylinder.

For the second part, it is not difficult to see that replacing X or B up to homotopy equivalence results in a homotopy equivalent pushout. That replacing A does not change the homotopy type follows from the following lemma:

Lemma 1.6. *A pushout of a homotopy equivalence along a cofibration is a homotopy equivalence. That is, if j is a cofibration and f is a homotopy equivalence in the square*

$$\begin{array}{ccc} A & \xrightarrow{j} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \alpha} \\ A' & \longrightarrow & X', \end{array}$$

then α is a homotopy equivalence.

Proof. We may as well assume that X is a mapping cylinder M and that j is the inclusion at time 1. Let g be a homotopy inverse for f. Up to homeomorphism, we may replace X' by the double mapping cylinder $M \cup_A A \times I \cup_A A'$ and the map α by the inclusion (this is homotopic to the original map). We define a map $\beta : X' \longrightarrow M$ in the other direction by the identity map on the subspace M of X' and by using a homotopy $h : \mathrm{id}_A \sim g \circ f$ on the rest. Then $\beta \circ \alpha = \mathrm{id}$ on the nose, and $\alpha \circ \beta \simeq \mathrm{id}$ by first using a homotopy to pull the image of A' into A' and then using a homotopy $f \circ g \simeq \mathrm{id}$. ■

■

Example 1.7. Here's an easy example to see that we need to assume one of the maps is a cofibration:

$$\begin{array}{ccc} S^n & \longrightarrow & D^{n+1} \\ \downarrow & & \downarrow \\ D^{n+1} & \longrightarrow & S^{n+1} \end{array} \quad \longrightarrow \quad \begin{array}{ccc} S^n & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \end{array}$$

Here we have homotopy equivalences that do not produce a homotopy equivalence on the pushout.

2. Wed, Feb. 23

Last time, we showed that as long as one of the maps $B \leftarrow A \rightarrow X$ is a cofibration, then the pushout $B \cup_A X$ is equivalent to the double mapping cylinder (the homotopy pushout). Here is another situation in which one can deduce the that the pushout is a homotopy pushout. First recall that a triple $(X; A, B)$ consisting of a space X and subspaces A and B is called an **excisive triad** if $X = \mathrm{int}A \cup \mathrm{int}B$.

Theorem 2.1. *Let $(X; A, B)$ be an excisive triad and let $C = A \cap B$. Then the square*

$$\begin{array}{ccc} C & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{array}$$

is a homotopy pushout square.

See [May, §10.7] for a proof.

Relation of cofiber sequences to fiber sequences

Let $f : A \longrightarrow X$ be a based map. Then we can form the cofiber sequence

$$A \xrightarrow{f} X \xrightarrow{i} C(f).$$

Consider the diagram

$$\begin{array}{ccccccc} F(f) & \xrightarrow{p} & A & \xrightarrow{f} & X & \xrightarrow{i} & C(f) \\ \varphi\downarrow & & \downarrow & \nearrow & & & \\ \Omega C(f) & \longrightarrow & F(i) & & & & \end{array}$$

The left vertical map (φ) is specified by sending (γ, a) to the loop in $C(f)$ given on the first half of the interval by γ^{-1} and on the second half by the path $t \mapsto (a, 2t-1)$. The second vertical map sends a to $(\delta_a, f(a))$, where δ_a is the path $t \mapsto (a, t)$. The square does not commute on the nose, but it commutes up to homotopy (the pair $(\delta_a \cdot \gamma^{-1}, *)$ can be contracted to $(\delta_a, f(a))$ by moving the initial point along γ^{-1}).

The map φ is usually not an equivalence, but we will see later that if A and X are both n-connected, then φ induces an isomorphism on π_n. We will also see that the analogue of this map in *spectra* is an equivalence.

Example 2.2. Recall the Hopf map $\eta : S^3 \longrightarrow S^2$. In homework 3, you show that $F(\eta) \simeq S^1$. What is $C(\eta)$? Certainly η is not a cofibration, so we should first convert it into a cofibration to compute the cofiber. Since the cofiber is the homotopy pushout of the maps $* \leftarrow S^3 \rightarrow S^2$, by what we have said previously, we may equally well replace the map $S^3 \longrightarrow *$ by a cofibration, namely the cofibration $S^3 \hookrightarrow D^4$. That is, the cofiber is the result of attaching a 4-cell to S^2 via η. This is precisely the CW structure on $\mathbb{CP}^2$ that we mentioned earlier.

Here is a careful description of this CW structure, from Hatcher. We usually write $\mathbb{CP}^2$ as the quotient of S^5 by the action of S^1 (complex multiplication). But every point of $S^5 \subseteq \mathbb{C}^3$ can be identified, under this action of S^1, with a point (x, y, z) such that the third complex coordinate z is in fact real and ≥ 0. If $z \neq 0$, there is a unique such point, whereas if $z = 0$, then the whole S^1 orbit of (x, y, z) is of this form. This subspace of S^5 is a hemisphere of $S^4 \subseteq S^5$, so it is D^4. Furthermore, we can now write $\mathbb{CP}^2$ as a quotient of D^4 by an action of S^1 on the boundary S^3. This identification on the boundary is precisely $\eta : S^3 \longrightarrow S^2$.

There is another way to identify the cofiber of η as $\mathbb{CP}^2$, using that excisive triads produce homotopy pushouts. We may replace the diagram $* \leftarrow S^3 \rightarrow S^2$ up to homotopy by the excisive triad diagram

$$\begin{array}{ccc} V - \{[0:0:1]\} & \longrightarrow & \mathbb{CP}^2 - \{[0:0:1]\} \\ \downarrow & & \\ V, & & \end{array}$$

where $V \subseteq \mathbb{CP}^2$ is the open subset

$$V = \{[x : y : z] \mid z \neq 0\}.$$

The open subsets V and $\mathbb{CP}^2 - \{[0 : 0 : 1]\}$ certainly cover $\mathbb{CP}^2$, so this gives an excisive triad. The subset V is homeomorphic to $\mathbb{C}^2$ (send $[x : y : z]$ to $(x/z, y/z)$) and is therefore contractible. It also follows that $V - \{[0 : 0 : 1]\} \simeq S^3$. Finally, the map $\mathbb{CP}^2 - \{[0 : 0 : 1]\} \longrightarrow \mathbb{CP}^1$ sending $[x : y : z]$ to $[x : y]$ is well-defined since x and y cannot both be zero. The inclusion $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2 - \{[0 : 0 : 1]\}$ defined by $[x : y] \mapsto [x : y : 0]$ is a homotopy inverse.

So we have identified the cofiber of η with the space $\mathbb{CP}^2$. The comparison map $S^1 \simeq F(\eta) \longrightarrow \Omega C(\eta) \simeq \Omega\mathbb{CP}^2$ cannot be an equivalence: the exact sequence

$$\pi_5(S^1) \longrightarrow \pi_5(S^5) \longrightarrow \pi_5(\mathbb{CP}^2)\pi_4(S^1)$$

shows that $\pi_5(\mathbb{CP}^5) \cong \mathbb{Z}$, so that $\pi_4(\Omega\mathbb{CP}^5) \cong \mathbb{Z}$, whereas $\pi_4(S^1) = 0$.

Fibrations & fiber sequences

We have discussed both (homotopy) fibers and cofibers, and we have seen that the cofiber of a cofibration is just the poin-set quotient. There is an analogue of this statement for the fiber of a *fibration*. The notion of a fibration is completely dual to that of a cofibration:

Definition 2.3. A map $p : E \longrightarrow B$ is a (Hurewicz) **fibration** if for every space X, map $f : X \longrightarrow E$ and homotopy $h : X \times I \longrightarrow B$ with $h_0 = p \circ f$, there is an extension $\tilde{h} : X \times I \longrightarrow E$. In other words, we get a lift in the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & E \\ {\scriptstyle i_0}\downarrow & \overset{\tilde{h}}{\nearrow} & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[h]{} & B. \end{array}$$

We also say that the map p satisfies the **homotopy lifting property** (or **covering homotopy property**).

Remark 2.4. There is an important generalization of fibration, called a **Serre fibration**, in which the map $p : E \longrightarrow B$ is only asked to satisfy the homotopy lifting property with respect to spaces $X = I^n$. We will come back to this later.

As for cofibrations, there is a "universal" diagram to check. The pair of maps $X \xrightarrow{f} E$ and $X \xrightarrow{h} B^I$ that make the diagram commute correspond to a single map $X \longrightarrow E \times_B B^I$. This is the unbased path space construction on the map p, and we will write $P(p)$ or $F(p)$ for this as well. We can see that any lifting diagram factors as

$$\begin{array}{ccccc} X & \longrightarrow & P(p) & \longrightarrow & E \\ {\scriptstyle i_0}\downarrow & & {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle p} \\ X \times I & \longrightarrow & P(p) \times I & \longrightarrow & B, \end{array}$$

and finding a lift in the square on the right will give a lift in the big rectangle.

3. Fri, Feb. 25

Proposition 3.1. *The map $p : E \longrightarrow B$ is a fibration if and only if the natural map $k : E^I \longrightarrow P(p)$ has a splitting $s : P(p) \longrightarrow E^I$ so that $k \circ s = \mathrm{id}_{P(p)}$.*

We think of the section s as a "path-lifting function" for the fibration p.

Proposition 3.2. *The class of fibrations is closed under*

(1) *composition,*
(2) *pullbacks,*
(3) *products,*
(4) *retracts, and*
(5) *sequential inverse limits.*

If $f : E \longrightarrow B$ is *any* map, we may factor this as $E \xrightarrow{j} P(f) \xrightarrow{q} B$. The map q evaluates the path in B at time 1, and $j(e) = (e, c_{f(e)})$.

Proposition 3.3. *For any map f, the map $E \xrightarrow{j} P(f)$ is a homotopy equivalence and $P(f) \xrightarrow{q} B$ is a fibration.*

Proof. Projection p_1 onto the E factor gives a map $p_1 : P(f) \longrightarrow E$ so that $p_1 \circ j = \mathrm{id}$. The other composition $j \circ p_1 \simeq \mathrm{id}$ via the homotopy that contracts paths starting at $f(e)$ to the constant path at $f(e)$.

To see that q is a fibration, consider a test diagram

$$\begin{array}{ccc} X & \xrightarrow{(g_1, g_2)} & P(f) \\ {\scriptstyle i_0}\downarrow & \nearrow & \downarrow{\scriptstyle q} \\ X \times I & \xrightarrow[h]{} & B. \end{array}$$

We define the lift $\tilde{h}$ by the formula

$$\tilde{h}(x,t) = (g_1(x), h(x) \mid_{[0,t]} \cdot g_2(x)),$$

where we renormalize the path $h(x) \mid_{[0,t]} \cdot g_2(x)$ so that we travel along $g_2(x)$ for time 0 to $\frac{1}{1+t}$ and along the $h(x) \mid_{[0,t]}$ for time $\frac{1}{1+t}$ to 1. Since the initial square commutes, we know that $h(x,0) = g_2(x)(1)$, so that it makes sense to compose the paths $h(x) \mid_{[0,t]}$ and $g_2(x)$). Our formula for $\tilde{h}(x,t)$ is a point of $P(f)$ since the path $h(x) \mid_{[0,t]} \cdot g_2(x)$ begins at $f \circ g_2(x)$. The upper triangle then commutes because concatenating $g_2(x)$ with the constant path, renormalized to have length zero, just gives $g_2(x)$ again. The lower triangle commutes because the endpoint of the path $h(x) \mid_{[0,t]} \cdot g_2(x)$ is $h(x,t)$. ■

So any map may be replaced by a fibration by changing the domain up to homotopy equivalence.

Examples.

(1) For any space X, the map $X \longrightarrow *$ is a fibration (we say that the space X is *fibrant*)

(2) Any covering space $E \longrightarrow B$ is a fibration. In fact, for a covering space the lift $\tilde{h}$ is uniquely determined.

(3) Any vector bundle $E \longrightarrow B$ is a fibration if the base B is paracompact (essentially means can find partitions of unity for any cover; any CW complex is paracompact).

Recall that a rank n (real) vector bundle over B is a map $p : E \longrightarrow B$ such that there exists a cover $\mathcal{U}$ of B and a homeomorphism $\varphi_U : p^{-1}(U) \cong U \times \mathbb{R}^n$ compatible with the projections to U. This is subject to the compatibility condition that if U and V are elements of the cover, then the transition function $g_{U,V}$ defined as the composition

$$(U \cap V) \times \mathbb{R}^n \xrightarrow{\varphi_U^{-1}} p^{-1}(U \cap V) \xrightarrow{\varphi_V} (U \cap V) \times \mathbb{R}^n$$

restricts, for each $x \in U \cap V$ to a linear isomorphism $\mathbb{R}^n \cong \mathbb{R}^n$ of fibers.

Without the assumption that the base is paracompact, we can still deduce that a vector bundle is a Serre fibration.

(4) More generally, any fiber bundle in which the base space is paracompact is a fibration.

A map $p : E \longrightarrow B$ is a **fiber bundle with fiber** F if B has a cover $\mathcal{U}$ and for each $U \in \mathcal{U}$ a homeomorphism $\varphi_U : p^{-1}(U) \cong U \times F$ over U.

Without the paracompact hypothesis on the base, we can again say that any bundle is at least a Serre fibration.

There is also an intermediate notion of G-bundle, where a group G acts on the fiber F, and the transition functions are assumed to arise from the action of G on F.

Proposition 3.4. *A fiber bundle $p : E \longrightarrow B$ is a Serre fibration.*

Proof. Let $p : E \longrightarrow B$ be a fiber bundle, and suppose given a a lifting diagram

$$\begin{array}{ccc} I^n & \xrightarrow{f} & E \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle p} \\ I^n \times I & \xrightarrow[h]{} & B. \end{array}$$

Let $\mathcal{U}$ be a covering for B on which we have trivializations for the bundle. Since $I^n \times I$ is compact, we may divide I^n into subcubes C and I into subintervals J, such that each $C \times J$ is contained in a single $h^{-1}(U)$.

For each C, we build a lift on each $C \times J$, starting with the J containting 0, and working our way up in the I coordinate. So by assumption, we have a lift along the initial point of our interval, which we may take to be 0 for simplicity. Moreover, this cube C borders other cubes, and we may have already constructed lifts on the other cubes. So we suppose that for some union D of faces of ∂C, we already have a lift along $D \times I$.

For our fixed C and J, the fiber bundle becomes the trivial fiber bundle $U \times F \longrightarrow U$. A lift in the diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & U \times F \\ {\scriptstyle i_0}\downarrow & \nearrow & \downarrow \\ C \times J & \xrightarrow{h} & U \end{array}$$

must be given in the first coordinate by the map h, so it remains to describe the second coordinate of the lift. By assumption, we already have a lift $C \cup_D D \times J \longrightarrow U \times F \longrightarrow F$. But the space $C \cup_D D \times J$ is a retract of $C \times J$, so we may compose the lift we already have with a choice of retraction $C \times J \xrightarrow{r} C \cup_D D \times J$. ∎