1. Mon, Mar. 7

Matt Ando lectured today.

2. Wed, Mar. 9

Last time, we used Blakers-Massey to prove the Freudenthal theorem, which says that if X is well-pointed and n-connected, then the map suspension map \( \pi_j(X) \to \pi_{j+1}(\Sigma X) \) is an isomorphism for \( j \leq 2n \) and a surjection for \( j = 2n + 1 \). Of course, important examples come from taking X to be a sphere.

**Example 2.1.** (1) Take \( X = S^2 \). We know \( S^2 \) is 1-connected, so

\[
\pi_j(S^2) \to \pi_{j+1}(S^3)
\]

is an isomorphism for \( j \leq 2 \) and a surjection for \( j = 3 \). Recall that we know \( \pi_2(S^2) \cong \mathbb{Z} \) from the long exact sequence for the Hopf fibration \( \eta \). The Freudenthal theorem then says that \( \pi_3(S^3) \cong \mathbb{Z} \) and that \( \pi_4(S^3) \) is cyclic (it turns out this is \( \mathbb{Z}/2 \)).

(2) Taking \( X = S^3 \), since \( S^3 \) is 2-connected, we get isomorphisms

\[
\pi_3(S^3) \cong \pi_4(S^4) \cong \mathbb{Z}, \quad \pi_4(S^3) \cong \pi_5(S^4) \cong \mathbb{Z}/2
\]

(3) As the connectivity of spheres improves as we suspend, for any fixed \( j \) and \( k \), the sequence

\[
\pi_j(S^k) \to \pi_{j+1}(S^{k+1}) \to \pi_{j+2}(S^{k+2}) \to \ldots
\]

eventually stabilizes to produce the **stable homotopy groups of spheres**. We write

\[
\pi^s_n(S^0) = \operatorname{colim}_k \pi_{n+k}(S^k).
\]

This stabilizes at \( \pi^s_n(S^0) = \pi_{2n+2}(S^{n+2}) \). The first few stable homotopy groups of spheres are

\[
\begin{align*}
\pi_0^s &\cong \pi_2(S^0) \cong \mathbb{Z} & \pi_1^s &\cong \pi_4(S^1) \cong \mathbb{Z}/2 & \pi_2^s &\cong \pi_6(S^3) \cong \mathbb{Z}/2 \\
\pi_3^s &\cong \pi_8(S^3) \cong \mathbb{Z}/24 & \pi_4^s &\cong \pi_{10}(S^6) \cong 0 & \pi_5^s &\cong 0 \\
\pi_6^s &\cong \mathbb{Z}/2 & \pi_7^s &\cong \mathbb{Z}/240 & \pi_8^s &\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2
\end{align*}
\]

(4) One can also talk of stable homotopy groups for arbitrary based spaces. If X is a based space, then we define the nth stable homotopy group of X to be

\[
\pi^s_n(X) = \operatorname{colim}_k \pi_{n+k}(\Sigma^k X).
\]

These stabilize as above.

We will return to stable homotopy groups when we discuss the **stable homotopy category of spectra**.

The Blakers-Massey theorem also allows us to compare wedges with products:
Proposition 2.2. Let $X$ be $p$-connected and $Y$ be $q$-connected well-pointed space ($p, q \geq 2$). Then the inclusion $X \vee Y \hookrightarrow X \times Y$ is $p + q$-connected.

Proof. Since $X$ and $Y$ are well-pointed, the square

\[
\begin{array}{ccc}
* & \rightarrow & Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{i_X} & X \vee Y
\end{array}
\]

is a homotopy pushout square. Homotopy excision tells us that the homotopy pullback $X \times Y$ is $(p + q - 1)$-connected. As we have discussed, this is equivalent to the statement that the map

$F(\ast \rightarrow X) \rightarrow F(Y \xrightarrow{i_Y} X \vee Y)$

is $(p + q - 1)$-connected. Note that the first fiber is just $\Omega(X)$. So the maps

$\pi_i(X) \cong \pi_{i-1}(\Omega(X)) \rightarrow \pi_{i-1}(F(i_Y)) \cong \pi_i(X \vee Y, Y)$

are isomorphisms for $i \leq p + q - 1$ and surjective for $i = p + q$.

Consider now the long exact sequence for the pair $(X \vee Y, Y)$:

\[
\ldots \rightarrow \pi_i(Y) \rightarrow \pi_i(X \vee Y) \rightarrow \pi_i(X \vee Y, Y) \rightarrow \ldots
\]

The map $X \vee Y \rightarrow Y$ which collapses $X$ to a point gives a splitting to $i_Y$, and we conclude that

$\pi_i(X \vee Y) \cong \pi_i(Y) \oplus \pi_i(X \vee Y, Y)$.

Putting this together, we learn that

$\pi_i(X \vee Y) \cong \pi_i(Y) \oplus \pi_i(X)$

for $i \leq p + q - 1$. To see that $\pi_{p+q}(X \vee Y) \rightarrow \pi_{p+q}(X \times Y)$, note that the inclusions $i_X$ and $i_Y$ induce a splitting for this map, so that in fact $\pi_i(X \vee Y) \rightarrow \pi_i(X \times Y)$ is onto for every $i$. ■

Corollary 2.3. Let $n \geq 2$. Then $\pi_n(S^n \vee S^n) \cong \mathbb{Z} \oplus \mathbb{Z}$.

We will use this to prove the Hurewicz theorem. Let $\alpha : S^n \rightarrow X$ be any map. Passing to homology gives a map

$\mathbb{Z} \cong H_n(S^n) \xrightarrow{H(\alpha)} H_n(X),$

and we define $h(\alpha)$ to be the image of $1$ under this map.

Lemma 2.4. The resulting map $h : \pi_n(S^n) \rightarrow H_n(X)$ is a well-defined homomorphism.

Proof. Well-definedness is the statement that homotopic maps induce the same map on homology. To see this is a homomorphism, recall that if $\alpha, \beta \in \pi_n(S^n)$, then their sum is defined as the class of

$S^n \xrightarrow{p} S^n \vee S^n \xrightarrow{\alpha\vee\beta} X \vee X \xrightarrow{\nabla} X.$

The conclusion now follows, using that $\tilde{H}_n(Y \vee Z) \cong \tilde{H}_n(Y) \oplus \tilde{H}_n(Z)$. ■

3. Fri, Mar. 11

Theorem 3.1. Let $X$ be $n-1$-connected. Then the Hurewicz map is an isomorphism

$\pi_n(X) \xrightarrow{\cong} H_n(X)$

if $n \geq 2$. If $n = 1$, this is the abelianization map.
Proof. The \( n = 1 \) statement should be familiar from 525, so we assume \( n \geq 2 \).

As homology takes weak equivalences to isomorphisms, we may assume \( X \) is CW. Since it is \( n - 1 \)-connected, we may assume it has a single 0-cell and no cells in dimensions between 0 and \( n \). Furthermore, the (cellular) homology group \( H_n \) and the homotopy group \( \pi_n \) of \( X \) only depends on the \( n + 1 \)-skeleton \( X_{n+1} \). So we assume that \( X \) is a cofiber of a map

\[
A = \bigvee S^n \rightarrow B = \bigvee S^n \rightarrow X.
\]

We then have a diagram

\[
\begin{array}{cccccc}
\pi_n(A) & \rightarrow & \pi_n(B) & \rightarrow & \pi_n(X) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H_n(A) & \rightarrow & H_n(B) & \rightarrow & H_n(X) & \rightarrow & 0
\end{array}
\]

The bottom row is exact because \( A \xrightarrow{\alpha} B \xrightarrow{\beta} X \) is a cofiber sequence. The Blakers-Massey theorem now tells us that since the maps \( A \rightarrow * \) and \( A \xrightarrow{\alpha} B \) are \( n \)-connected and \( n - 1 \)-connected, respectively, the map \( A \rightarrow F(\beta) \) is \( 2n - 2 \)-connected. As \( n \geq 2 \), \( n \leq 2n - 2 \) and we get \( \pi_n(A) \rightarrow \pi_n(F) \). It follows that the top row above is exact.

**Lemma 3.2.** The Hurewicz map is an isomorphism for any wedge \( \bigvee S^n \).

*Proof.* We showed last time that \( \pi_n(S^n \vee S^n) \cong \mathbb{Z} \), where generators are given by the two natural inclusions \( S^n \hookrightarrow S^n \vee S^n \), and it follows that the Hurewicz map is an isomorphism. By induction, we get the same result for any finite wedge. For an infinite wedge, we have

\[
H_n(\bigvee S^n) \cong \bigoplus H_n(S^n) \quad (n \geq 1)
\]

Since \( S^n \) is compact, any map \( S^n \rightarrow \bigvee S^n \) must have image contained in a finite wedge. ■

We discuss a few more consequences of Homotopy Excision. Recall that, given a map \( f : A \rightarrow X \), we previously defined a map \( \phi : F(f) \rightarrow \Omega C(f) \). This induces a map

\[
\pi_i(X, A) \cong \pi_{i-1}(F(f)) \rightarrow \pi_i(C(f)),
\]

and the latter may be reinterpreted as \( \pi_i(X/A) \) if \( f \) is a cofibration.

**Proposition 3.3.** Suppose that \( A \) is \( m \)-connected and that \( f : A \rightarrow X \) is an \( n \)-equivalence. Then the map

\[
F(f) \rightarrow \Omega C(f)
\]

is an \( m + n \)-equivalence.

*Proof.* Consider the homotopy pushout square

\[
\begin{array}{ccc}
A & \rightarrow & CA \\
\downarrow f & & \downarrow \\
X & \rightarrow & C(f)
\end{array}
\]

The inclusion \( A \hookrightarrow CA \) is an \( m + 1 \)-equivalence, and \( f \) is an \( n \)-equivalence. By Blakers-Massey, it follows that induced map

\[
F(f) \rightarrow F(CA \rightarrow C(f)) \simeq \Omega C(f)
\]

is an \( m + n \)-equivalence. ■
Remark 3.4. Another way to interpret the above result in the case that \( f \) is a cofibration is that
\[
\pi_i(X, A) \longrightarrow \pi_i(C(f)) \cong \pi_i(X/A)
\]
is an isomorphism for \( i < m + n + 1 \) and a surjection for \( i = m + n + 1 \).

As we have said previously, the analogue of the map \( F(f) \longrightarrow \Omega C(f) \) in spectra is always an equivalence.

We use the above result to prove a result of Whitehead.

**Theorem 3.5 (Whitehead).** Let \( f : X \longrightarrow Y \) be a map between simply connected spaces such that \( H_*(f) \) is an isomorphism. Then \( f \) is a weak homotopy equivalence.

**Proof.** The cofiber sequence \( X \stackrel{f}{\longrightarrow} Y \longrightarrow C(f) \) induces a long exact sequence in homology, and we deduce that the homology groups of \( C(f) \) vanish. Note that since \( A \) and \( X \) are 1-connected, it follows that \( F(f) \) is 0-connected. By the previous result, \( \pi_0(F(f)) \cong \pi_1(C(f)) \), so \( C(f) \) is simply connected. Since it also has no homology, the Hurewicz theorem implies that \( C(f) \) is weakly contractible \( (C(f) \longrightarrow \ast \) is a weak equivalence). Since \( f \) is a 1-equivalence and \( X \) is 1-connected, the previous result tells us that
\[
\pi_1(F(f)) \cong \pi_2(C(f)) \cong 0,
\]
so that \( f \) is in fact a 2-equivalence. By induction, the same argument will show that \( f \) is an \( n \)-equivalence for every \( n \), i.e. a weak equivalence. 

**Remark 3.6.** There is a more general form of this result in which one assumes only that the spaces \( X \) and \( Y \) are simple. If \( \pi_1(X) \) is nonabelian, this result can easily fail. For instance, there are spaces with nontrivial fundamental group but trivial homology. In this case, the fundamental group must be a nontrivial group with trivial abelianization. In other words, the group must be generated by commutators. Any finite simple group, like \( A_5 \), gives such an example.

**Remark 3.7.** It is not enough to simply know that two spaces are simply connected and have the same homology. For instance, consider \( X = S^2 \vee S^4 \) and \( Y = \mathbb{C}P^2 \). The homology of both spaces is \( \mathbb{Z} \) in degrees 0, 2, and 4. But there is no map between them inducing a homology isomorphism.

There is a natural map \( S^2 \cong \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2 \) given an isomorphism on \( H_2 \), and the quotient map \( \mathbb{C}P^2 \longrightarrow \mathbb{C}P^2/\mathbb{C}P^1 \cong S^4 \) gives an \( H_4 \)-isomorphism. But there is no map \( S^4 \longrightarrow \mathbb{C}P^2 \) inducing the \( H_4 \)-iso. Indeed, the fiber sequence \( S^1 \longrightarrow S^3 \longrightarrow \mathbb{C}P^2 \) shows that \( \pi_4(\mathbb{C}P^2) \cong 0 \) and \( \pi_3(\mathbb{C}P^2) \cong 0 \), so that no such map can exist by Whitehead. Note also that if we consider cohomology, then it is clear that there can be no map inducing a \( \mathbb{H}^* \)-isomorphism since \( Y \) has nontrivial cup products, whereas \( X \) does not.