MATH 651
HOMEWORK VI
SPRING 2013

Problem 1. We showed in class that if B is connected, locally path connected, and **semilocally simply connected**, then it has a universal cover $q : X \longrightarrow B$.

(a) Let B_n be the circle of radius $1/n$, centered at the point $(1/n, 0)$ in $\mathbb{R}^2$. Let $B = \bigcup_n B_n$. Show that B is not semilocally simply connected by showing that the point $(0, 0)$ has no relatively simply connected neighborhood.

(b) Given any space Z, the **cone** on Z is the space $C(Z) = Z \times I/Z \times \{1\}$. Show that $C(Z)$ is semilocally simply connected.

Problem 2. The space B from problem 1(a) looks as if it might be the infinite wedge $\bigvee_{\mathbb{N}} S^1$. The universal property of the wedge gives a continuous bijection

$$\bigvee_{\mathbb{N}} S^1 \longrightarrow B$$

which sends the circle labelled by n to the circle B_n. Use the following argument to show this is not a homeomorphism.

Consider, for each n, the open subset $U_n \subseteq S^1$ which is an open interval of radian length $1/n$ centered at $(0, 0)$. We have not discussed infinite wedge sums, so you may assume that $\bigvee_n U_n \subseteq \bigvee_n S^1$ is open. Let $W_n \subseteq B_n$ be the image of U_n under the natural homeomorphism $S^1 \cong B_n$. Show that $\bigcup_n W_n \subseteq B$ is not open.

Problem 3. A space X is locally simply connected if, given a neighborhood U of some point x, then there exists a simply connected neighborhood V of x contained in U.

(a) Show that if X is locally simply connected, then it is semilocally simply connected.

(b) Show that the converse is not true by showing that the space $C(Z)$ from problem 1(b) is not locally simply connected.

Problem 4.

(a) Show that in the space B from problem 1, the point $(0, 0)$ is **not** a nondegenerate basepoint. In other words, show that no neighborhood of $(0, 0)$ deformation retracts onto $(0, 0)$.

(b) If no basepoint of X is nondegenerate, there is a way of adding in a good basepoint without changing the homotopy type. The process, known as "attaching a whisker to X", is to consider the space $X \vee I$. If we glue I to X along the point $0 \in I$, show that $1 \in I$ is a nondegenerate baspoint for X.

(c) Show that $X \vee I$ is homotopy equivalent to X.

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