MATH 651
HOMEWORK VIII
PARTIAL SOLUTIONS
SPRING 2013

Problem 2. Suppose that two finite CW complexes X_1 and X_2 are homeomorphic. In other words, we have two cell complex structures on the same space, each having finitely many cells. Suppose furthermore that X_{12} is a common finite refinement of the two. In other words, every n-cell of X_1 is a union of n-cells of X_{12}, and similarly every k-cell of X_2 is a union of k-cells of X_{12}. (Here an n-cell means the image of D^n in the pushout.)

(*a*) If X_1 and X_2 are both graphs (i.e. 1-dimensional), show that $\chi(X_1) = \chi(X_2)$.
(*b*) Show that this formula still holds when the dimension (of both) is n for any $n \geq 1$.

Solution. In both parts (a) and (b), since X_{12} is a common refinement of the two CW structures, it is enough to show that $\chi(X_i) = \chi(X_{12})$. In other words, without loss of generality it suffices to show that $\chi(X_1) = \chi(X_12)$.

(*a*) The graph X_{12} is a refinement of the graph X_1. That means that each edge e in X_1 is a union of finitely many edges $f_1, \ldots, f_k$ in X_{12}. That is, we can think of the CW structure $f_1 \cup \cdots \cup f_k$ on e as being obtained by subdividing the edge e $n-1$ times into smaller edges. Let's think about what happens each time we do a subdivision. Start with the edge e with vertices x and y. If we subdivide it to get edges f_0 and f_1, then f_0 will have vertiecs x and z, say, and f_1 will have vertices z and y. We have added one edge and one vertex, so the Euler characteristic is unchanged. The same will be true if we subdivide $n-1$ times. Note that we could not have attached any "extra" 0-cells in X_{12} in any way other than in an edge subdivision. In order for an extra 0-cell to be included without changing the homeomorphism type of the space, this new vertex would need to be in an edge, which is precisely what happens in the edge subdivisions.

(*b*) To avoid overly cumbersome notation, I will write Y and Z rather than X_1 and X_{12}. The 1-skeleton Z^1 of Z can have many edges that are not in Y^1, but $Z^1 \cap Y^1$ is a refinement of Y^1, and by part (a), we have $\chi(Y^1) = \chi(Y^1 \cap Z^1)$.

Let's consider the 2-cells of Y one at a time. Let e be a 2-cell of Y. This 2-cell may be subdivided in Z, and we want to show that this does not change the Euler characteristic. If we can do this for each 2-cell of Y, we will be able to conclude that $\chi(Y^2) = \chi(Y^2 \cap Z^2)$. A similar argument will then work in higher dimensions, and we will inductively get

$$\chi(Y) = \chi(Y^n) = \chi(Y^n \cap Z^n) = \chi(Y \cap Z)$$

(here $n = \dim Y = \dim Z$).

So we now suppose $Y = D^2$, with a CW structure having a single 2-cell. We want to show that refining this CW structure to Z does not change the Euler characteristic. Again, Z may have many 0 and 1-cells not contained in Y^1. Rather than working from Z back to Y, we will start with the CW structure on Y and refine it in steps to the one in Z, not changing χ at any step.

First, the CW structure on ∂D^2 in Z refines that in Y, but by part (a) this does not change χ since ∂D^2 is a graph.

Date: April 11, 2013.

Next, suppose Z has some additional edges not in $\partial D^2 = Y^1$. Pick such an edge e_1 with one endpoint in ∂D^2. If the other endpoint is not in ∂D^2, there must be a sequence of edges e_2, $\ldots e_k$ connecting that endpoint to the boundary. Write $e = e_1 \cup \cdots \cup e_k$. If we refine Y by adding this edge, that divides the 2-cell. But we have added one edge and one 2-cell, so we have not changed χ. Now this edge e may need to be refined (to $e_1 \cup \cdots \cup e_k$) in Z, but again by part (a) that does not change χ. Now we have two 2-cells in Y. Use the same argument on each! Rinse and repeat. After finitely many steps, we are done.