Math 651 - Topology I
Homework 1
Spring 2014

1. Show that if X, X', Y, and Y' are spaces such that $X \simeq Y$ and $X' \simeq Y'$ then $X \times X' \simeq Y \times Y'$.

2. Show that homotopy equivalence is an equivalence relation on the collection of topological spaces.

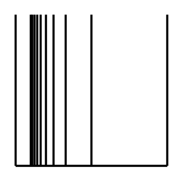
3. (a) Show that if Y is contractible, then any two maps $X \longrightarrow Y$ are homotopic. Conclude that a contractible space is path-connected.

   (b) Show that if X is contractible and Y is path-connected, then any two maps $X \longrightarrow Y$ are homotopic.

4. Recall that a subspace $A \subseteq X$ is said to be a **retract** of X if there exists a map (called a retraction) $r : X \longrightarrow A$ such that $r(a) = a$ for every $a \in A$. Show that if X is contractible and A is a retract of X, then A must also be contractible.

5. Let $A \subseteq X$. A **deformation retraction** of X onto A is a homotopy h starting at id_X such that (i) $h(x, 1) \in A$ for all x and (ii) $h(a, t) = a$ for all t. The map $h(-, 1)$ then defines a retraction $X \longrightarrow A$.

   Consider the space $X = I \times \{0\} \cup \bigcup_{x \in \{0\} \cup \{1/n\}} \{x\} \times I$ depicted to the right.

   (a) Show that X is contractible.

   (b) Show that if a space Y deformation retracts to a point $y_0 \in Y$, then for every neighborhood U of y, there is a neighborhood $V \subset U$ of y such that the inclusion $V \hookrightarrow U$ is nullhomotopic.

   (c) Use part (b) to show that X does not deformation retract onto the point $(0, 1)$.

6. For each n, there is an equatorial inclusion $S^n \hookrightarrow S^{n+1}$. Let $S^\infty = \bigcup_n S^n$, topologized using the topology of the union. Recall that this means that $A \subseteq S^\infty$ is open or closed if and only if each $A \cap S^n$ is open or closed in S^n. Show that S^∞ is contractible.

   (Hint: Start by showing that the identity map of S^∞ is homotopic to the map

   $$(x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, x_3, \dots).$$

   Then show that the latter is nullhomotopic.)