Math 651 - Topology II
Homework VI
Spring 2014

1. Show that any two simply-connected covers of B are homeomorphic. (Hint: Use the lifting lemmas!)

2. Let $p : E \longrightarrow B$ be a covering. Use Theorem 16.5 from class to determine the group $\mathrm{Aut}(E)$ of deck transformations of E. (Hint: Use Proposition 16.3 from class to determine the group of G-equivariant automorphisms of any G-set.)

3. Let $p : E \longrightarrow B$ be a covering. Pick $e_0 \in E$ and let $b_0 = p(e_0)$. Show that $p_*(\pi_1(E, e_0)) \leq \pi_1(B, b_0)$ is normal if and only if for every point $f \in F$, there is a deck transformation $\varphi : E \longrightarrow E$ such that $\varphi(e_0) = f$.

4. Let $q : X \longrightarrow B$ be a simply-connected covering. We showed in class that $\mathrm{Aut}(X) \cong G = \pi_1(B)$. This gives an action of G on X, and this action restricts to an action on any fiber. But also discussed a G-action on the fiber for any covering.
   (a) Let $q : \mathbb{R}^2 \longrightarrow S^1 \times S^1$ be the universal covering of the torus. Show the two above actions are the same.
   (b) Let $q : X \longrightarrow S^1 \vee S^1$ be the (fractal) simply-connected covering discussed in class. Show that in this case the two actions *do not* coincide! (Hint: Denote by α and β the loops around the two circles in $S^1 \vee S^1$. Determine (carefully) the action of $\alpha\beta$ on a point in the fiber under the two described actions.)

5. ($\star$) Find a free action of the cyclic group C_6 on the sphere S^3, and let $B = S^3/C_6$ be the quotient. Find all covers of B and determine all maps of coverings between them.

6. ($\star$) Think of $\mathbb{R}^4$ as the ring $\mathbb{H}$ of quaternions, so that S^3 corresponds to the unit quaternions. Then the standard unit vectors $\{\pm 1, \pm i, \pm j, \pm k\}$ form the quaterion group Q_8 of order 8. Let $B = S^3/Q^8$, and find all covers and maps between them as in the previous problem.