CLASS NOTES
MATH 651 (SPRING 2013)

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1. Wed, Jan. 15

Here are a list of main topics for this semester:

(1) homotopy, homotopy equivalence (Hatcher - Ch. 0, Ch. 1.1; Lee - Ch. 7)
(2) the fundamental group (topology $\rightsquigarrow$ algebra) (Hatcher - Ch. 1.1; Lee - Ch. 7, Ch. 8)
(3) the theory of covering spaces (Hatcher - Ch. 1.3; Lee - Ch. 11, Ch. 12)

Example 1.1.
(a) What spaces cover $\mathbb{R}$? Only $\mathbb{R}$ itself. Every covering map $E \longrightarrow \mathbb{R}$ is a homeomorphism.
(b) What spaces cover S^1? There is the n-sheeted cover of S^1 by itself, for any nonzero integer n. (Wrap the circle around itself n times.) There is also the exponential map $\mathbb{R} \longrightarrow S^1$.
(c) What spaces cover S^2? Only S^2 itself. Every covering map $E \longrightarrow S^2$ is a homeomorphism.
(d) What spaces cover $\mathbb{RP}^2$? There is the defining quotient map $S^2 \longrightarrow \mathbb{RP}^2$ and the homeomorphisms.

(4) computation of the fundamental group via the Seifert-van Kampen theorem. (Hatcher - Ch. 1.2, Lee - Ch. 9, Ch. 10)
(5) classification of surfaces (compact, connected) and the Euler characteristic. (Lee - Ch. 6, Ch. 10)
(6) homology of CW complexes (Hatcher - Ch. 2.1, Lee - Ch. 13)

The fundamental group, an algebraic object, will turn out to be crucial for understanding topics in geometric topology (coverings, surfaces).

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One of the main questions in topology is that of distinguishing topological spaces. Often, this is done by finding a topological property that one, but not both, of the spaces in question possesses. For example, $\mathbb{R}$ is not homeomorphic to S^1 since the latter is compact, whereas the former is not. Similarly, $I = [0,1]$ is not homeomorphic to S^1 since the former can be disconnected by removing a point, whereas the latter cannot. But this is a tough row to hoe. Neither of the above criteria suffice to distinguish S^2, $\mathbb{RP}^2$, and $T^2 = S^1 \times S^1$. The notion of homotopy will give us another means of distinguishing spaces.

Date: January 17, 2014.

Definition 1.2. Given maps f and $g : X \longrightarrow Y$, a **homotopy** h between f and g is a map $h : X \times I \longrightarrow Y$ ($I = [0,1]$) such that $f(x) = h(x,0)$ and $g(x) = h(x,1)$. We say f and g are **homotopic** if there exists a homotopy between them (and write $h : f \simeq g$).

Example 1.3. Let $f = \mathrm{id} : \mathbb{R} \longrightarrow \mathbb{R}$ and take $g : \mathbb{R} \longrightarrow \mathbb{R}$ to be the constant map $g(x) = 0$. Then a homotopy $h : f \simeq g$ is given by

$$h(x,t) = x(1-t).$$

Check that $h(x,0) = f(x)$ and $h(x,1) = g(x)$. Since f is homotopic to a <u>constant</u> map, we say that f is **null-homotopic** (and h is a **null-homotopy**).

Example 1.4. Consider $f = \mathrm{id} : S^1 \longrightarrow S^1$ and the map $g : S^1 \longrightarrow S^1$ defined by $g(\cos(\theta), \sin(\theta)) = (\cos(2\theta), \sin(2\theta))$. Thinking of S^1 as the complex numbers of unit norm, the map g can alternatively be described as $g(z) = z^2$. Then the maps f and g are <u>not</u> homotopic. Furthermore, neither is null-homotopic. (Though we won't be able to show this for a couple of weeks.)

Proposition 1.5. *The property of being homotopic defines an equivalence relation on the set of maps $X \longrightarrow Y$.*

Proof. (Reflexive): Need to show $f \simeq f$. Use the **constant homotopy** defined by $h(x,t) = f(x)$ for all t.

(Symmetric): If $h : f \simeq g$, we need a homotopy from g to f. Define $H(x,t) = h(x, 1-t)$ (reverse time).

(Transitive): If $h_1 : f_1 \simeq f_2$ and $h_2 : f_2 \simeq f_3$, we define a new homotopy h from f_1 to f_3 by the formula

$$h(x,t) = \begin{cases} h_1(x,2t) & 0 \leq t \leq 1/2 \\ h_2(x, 2t-1) & 1/2 \leq t \leq 2. \end{cases}$$

■

We write $[X,Y]$ for the set of homotopy classes of maps $X \longrightarrow Y$.

Proposition 1.6. *(Interaction of composition and homotopy) Suppose given maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $X \xrightarrow{f'} Y \xrightarrow{g'} Z$. If $f \simeq f'$ and $g \simeq g'$ then $g \circ f \simeq g' \circ f'$.*

Proof. We will show that $g \circ f \simeq g' \circ f$. The required homotopy is given by

$$H(x,t) = h'(f(x), t).$$

It is easily verified that $H(x,0) = g \circ f(x)$ and $H(x,1) = g' \circ f(x)$. Why is the map $H : X \times I \longrightarrow Z$ continuous? It is the composition of the continuous maps

$$X \times I \xrightarrow{f \times \mathrm{id}} Y \times I \xrightarrow{h'} Z.$$

That the map $f \times \mathrm{id}$ is continuous can be easily verified using the universal property. If you are not familiar with this, consult Theorem 10.2 from the fall semester course notes. ■

Definition 1.7. A map $f : X \longrightarrow Y$ is a **homotopy equivalence** if there is a map $g : Y \longrightarrow X$ such that both composites $f \circ g$ and $g \circ f$ are homotopic to the identity maps. We say that spaces X and Y are **homotopy equivalent** if there exists some homotopy equivalence between them, and we write $X \simeq Y$.

Remark 1.8. It is clear that any homeomorphism is a homotopy equivalence, since then both composites are <u>equal</u> to the idenitity maps.

2. Fri, Jan. 17

Last time, we introduced the related notions of homotopy and homotopy equivalence. We ended by saying that every homeomorphism is a homotopy equivalence. The following example shows that the converse is not true.

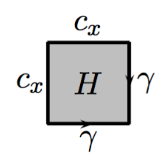
Example 2.1. The (unique) map $f : \mathbb{R} \longrightarrow *$, where $*$ is the one-point space, is a homotopy equivalence. Pick any map $g : * \longrightarrow \mathbb{R}$ (for example, the inclusion of the origin). Then $f \circ g = \mathrm{id}$. The other composition $g \circ f : \mathbb{R} \longrightarrow \mathbb{R}$ is contant, but we have already seen last time that the identity map of $\mathbb{R}$ is null-homotopic. So $\mathbb{R} \simeq *$. The same argument works equally well to show that $\mathbb{R}^n \simeq *$ for any n. Even more generally, if X is a convex subset of $\mathbb{R}^n$, then $X \simeq *$.

Here's some more terminology: any space that is homotopy-equivalent to the one-point space is said to be **contractible**. As we have just seen in the example above, this is equivalent to the statement that the identity map is null-homotopic.

We will see later that the spaces S^2, $\mathbb{RP}^2$, and T^2 are not homotopy-equivalent (and therefore not homeomorphic).

Homotopy of paths

Recall that a **path** in a space X is simply a continuous map $\gamma : I \longrightarrow X$. It will turn out to be fruitful to study homotopy-classes of paths in a space X. But this is not very interesting if we don't impose additional restrictions: every path is null! A contracting homotopy for the path γ is given by

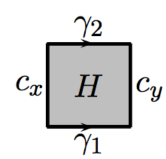


$$H(s,t) = \gamma(s(1-t)).$$

We need to modify our notion of homotopy to get an interesting relation for paths.

Definition 2.2. Let γ_1 and γ_2 be paths in X with the same initial and end points. A **path-homotopy** between γ_1 and γ_2 is simply a homotopy h such that at each time t, the resulting path $h(-,t)$ also has the same initial and end points as γ_1 and γ_2.

Another way to think about this is that a path homotopy is a map from the square $I \times I$ that is constant on the left vertical edge and also on the right vertical edge.



Example 2.3. The two paths $\gamma_1(s) = e^{i\pi s}$ and $\gamma_2(s) = e^{-i\pi s}$ are path-homotopic in $\mathbb{R}^2$. A homotopy is given by $h(s,t) = (1-t)\gamma_1(s) + t\gamma_2(s)$. This is the **straight-line homotopy**. For example, when we restrict to $s = 1/2$, the homotopy gives the vertical diameter of the circle.

On the other hand, we could also consider these as paths in $\mathbb{R}^2 - \{(0,0)\}$ or as paths in S^1. We will see later that these are **not** path-homotopic in either of these spaces.

Proposition 2.4. *Given two points a and b in X, path-homotopy defines an equivalence relation on the set of paths from a to b.*

A path in X that begins and ends at the same point is called a **loop** in X. We call the starting/end point the **basepoint** of the loop (and often of X as well). By the above proposition, path-homotopy defines an equivalence relation on the set of loops in X with basepoint x_0. The set of equivalence classes is denoted $\pi_1(X, x_0)$ and is called the **fundamental group of** X (with basepoint x_0). Of course, so far we have no reason to call this a group, we only know this as a set.

Example 2.5. Use of straight-line homotopies show that $\pi_1(\mathbb{R}^n, x) = \{c_x\}$ for any n and x. More generally, $\pi_1(X, x) = \{c_x\}$ for any convex subset of $\mathbb{R}^n$. This holds even more generally for any

star-shaped region in $\mathbb{R}^n$. A subset $X \subset \mathbb{R}^n$ is said to be star-shaped around x if for any $y \in X$, the straight-line segment $\overline{xy}$ is contained in X.

Here is a slightly different perspective on loops. Since a loop is a map $\gamma : I \longrightarrow X$ that is constant on the subspace $\partial I = \{0, 1\} \subseteq I$, there is an induced map from the quotient space $\overline{\gamma} : I/\partial I \longrightarrow X$. Recall that $I/\partial I$ is homeomorphic to the circle S^1. So a loop in X is the same as a map $\overline{\gamma} : S^1 \longrightarrow X$.

A **based map** between two spaces with chosen basepoints is simply a map that takes the basepoint of one space to the basepoint of the other. By a **based homotopy**, we mean a homotopy through based maps (so the homotopy is constant on the basepoint). Based homotopy defines an equivalence relation on the set of based maps, and the set of based homotopy classes is denoted

$$[(X, x_0), (Y, y_0)]_*.$$

It is customary to take $(1, 0)$ as the basepoint for S^1, and path-homotopy classes of loops in X, based at x_0, correspond to based homotopy classes of maps $S^1 \longrightarrow X$. So

$$\pi_1(X, x_0) \cong [(S^1, (1, 0)), (X, x_0)]_*.$$

Where does the <u>group</u> structure on homotopy classes of loops come from? Well, you can concatenate paths, by traveling first along one and then along the other.

Definition 2.6. Let γ and λ be paths in X. We say the two paths are **composable** in X if $\gamma(1) = \lambda(0)$. When this is the case, we define the **concatenation** of γ and λ to be the path

$$\gamma \cdot \lambda(s) = \begin{cases} \gamma(2s) & s \in [0, 1/2] \\ \lambda(2s - 1) & s \in [1/2, 1]. \end{cases}$$

This formula looks familiar, right? This was the one used in Proposition 1.5 to glue two homotopies together. This is no accident: a path is precisely a homotopy between two constant maps!

Concatenation will provide the group structure on $\pi_1(X)$, as we will investigate next time.