26. Mon, Mar. 24

The next application is the computation of the fundamental group of any *graph*. We start by specifying what we mean by a graph. Recall that $S^0 \subseteq \mathbb{R}$ is usually defined to be the set $S^0 = \{-1, 1\}$. For the moment, we take it to mean instead $S^0 = \{0, 1\}$ for convenience.

Definition 26.1. A **graph** is a 1-dimensional CW complex.

Of special importance will be the following type of graph.

Definition 26.2. A **tree** is a connected graph such that it is not possible to start at a vertex v_0, travel along successive edges, and arrive back at v_0 without using the same edge twice.

(Give examples and nonexamples)

Proposition 26.3. *Any tree is contractible. Even better, if v_0 is a vertex of the tree T, then v_0 is a deformation retract of T.*

Proof. We give the proof in the case of a finite tree. Use induction on the number of edges. If T has one edge, it is homeomorphic to I. Assume then that any tree with n edges deformation retracts onto any vertex and let T be a tree with $n+1$ edges. Let $v_0 \in T$. Now let $v_1 \in T$ be a vertex that is maximally far away from v_0 in terms of number of edges traversed. Then v_1 is the endpoint of a unique edge e, which we can deformation retract onto its other endpoint. The result is then a tree with n edges, which deformation retracts onto v_0. ■

Corollary 26.4. *Any tree is simply connected.*

Definition 26.5. If X is a graph and $T \subseteq X$ is a tree, we say that T is a **maximal tree** if it is not contained in any other (larger) tree.

By Zorn's Lemma, any tree is contained in some maximal tree.

Theorem 26.6. *Let X be a connected graph and let $T \subseteq X$ be a maximal tree. The quotient space X/T is a wedge of circles, one for each edge not in the tree. The quotient map $q : X \longrightarrow X/T$ is a homotopy equivalence.*

Proof. Since T contains every vertex, all edges in the quotient become loops, or circles. To see that q is a homotopy equivalence, we first define a map $b : X/T \cong \bigvee S^1 \longrightarrow X$. Recall that to define a continuous map out of a wedge, it suffices to specify the map out of each wedge summand. Fix a vertex $v_0 \in X$. Suppose we have a circle corresponding to the edge e in X from v_1 to v_2. Pick paths α_1 and α_2 in T from v_0 to v_1 and v_2, respectively. We then send our circle to the loop $\alpha_1 e \alpha_2^{-1}$.

The composition $q \circ b$ on a wedge summand S^1 looks like $c * \mathrm{id} * c$ and is therefore homotopic to the identity. For the other composition, first note that the deformation retraction of T onto v_0 gives a homotopy $b \circ q \simeq \mathrm{id}$ on T. We can then extend this to a homotopy on all of X by specifying the homotopy on each edge not in T. This requires specifying a homotopy $\alpha_1 e \overline{\alpha}_2 \simeq e$ which is easily done. ■

Corollary 26.7. *The fundamental group of any graph is a free group.*

Last time, we showed that the fundamental group of any graph is free. We will use this to deduce an algebraic result about free groups. But first, a result about coverings of graphs.

Theorem 26.8. *Let $p : E \longrightarrow B$ be a covering, where B is a connected graph. Then E is also a connected graph.*

Proof. Recall our definition of a graph. It is a space obtained by glueing a set of edges to a set of vertices. Let B_0 be the vertices of B, and let $E_0 \subseteq E$ be $p^{-1}(B_0)$. We define

$$E_1 \subseteq B_1 \times E_0$$

to be the set of pairs $(\alpha : S^0 \longrightarrow B_0, e)$ such that $\alpha(0) = p(e)$. We then have compatible maps $E_0 \hookrightarrow E$ and $\amalg_{E_1} I \longrightarrow E$. The second map is given by the unique path-lifting property. These assemble to give a continuous map from the "pushout" $\hat{E}$ to E. It is clear that this "pushout" is a graph. We have taken the same vertices, so the map on vertices is a bijection. The map on edges is also a bijection by unique path-lifting. There are now several arguments for why this must be a homeomorphism. If B is a finite graph and E is a finite covering, we are done since E is compact and B is Hausdorff. More generally, the map $\hat{E} \longrightarrow E$ is a map of covers which induces a bijection on fibers, so it must be an isomorphism of covers. ∎

Now here is a purely algebraic statement, which we can prove by covering theory.

Theorem 26.9. *Any subgroup H of a free group G is free. If G is free on n generators and the index of H in G is k, then H is free on $1 - k + nk$ generators.*

Proof. Define B to be a wedge of circles, one circle for each generator of G. Then $\pi_1(B) \cong G$. Let $H \leq G$ and let $p : E \longrightarrow B$ be a covering such that $p_*(\pi_1(E)) = H$. By the previous result, E is a graph and so $\pi_1(E)$ is a free group by the result from last time.

27. Wed, Mar. 26

For the second statement, we introduce the **Euler characteristic** of a graph, which is defined as $\chi(B) = \#$ vertices $-\#$ edges. In this case, we have $\chi(B) = 1 - n$. Since H has index k in G, this means that G/H has cardinality k. But this is the fiber of $p : E \longrightarrow B$. So E has k vertices, and each edge of B lifts to k edges in E. So $\chi(E) = k - kn$.

On the other hand, we know from last time that E is homotopy equivalent to E/T, where $T \subseteq E$ is a maximal tree. Note that collapsing any edge in a tree does not change the Euler characteristic. The number of generators, say m of E is then the number of edges in E/T, so we find that $\chi(E) = 1 - m$. Setting these equal gives

$$k - kn = 1 - m, \qquad \text{or} \qquad m = 1 - k + kn.$$

∎

Shifting gears a little, here is an important topological result.

Theorem 27.1. *Let G be any group. Then there exists a space X with $\pi_1(X) \cong G$.*

Proof. Write $G = F/N$, where F is a free group and N is a normal subgroup (for example, you could take F to be the free group on all of the elements of G). Let $B = \bigvee S^1$, such that $\pi_1(B) \cong F$. Let $p : E \longrightarrow B$ be a covering with $p_*(\pi_1(E)) = N$. We want to somehow kill the subgroup $N \leq F$.

Consider the cone

$$C = C(E) = E \times I / E \times \{1\}.$$

Then form the union $X = B \cup_E C(E)$, where $p(e) \sim (e, 0)$. This construction is called the **mapping cone** on the map p. Cover this using $U = B \cup E \times [0, 2/3)$ and $V = E \times (1/3, 1]/E \times \{1\}$. Note that V is contractible and that $U \simeq B$. Furthermore $U \cap V \simeq E$. The van Kampen theorem then implies that

$$\pi_1(X) \cong \pi_1(U) * \pi_1(V)/\pi_1(U \cap V) = G * \langle e \rangle / N \cong G/N \cong F.$$

∎

There is another way to describe what we have done here, through the approach of attaching cells.

Given a space X and a map $\alpha : S^1 \longrightarrow X$, we may attach a disc along the map α to form a new space

$$X' = X \cup_\alpha D^2.$$

Since the inclusion of the boundary $S^1 \hookrightarrow D^2$ is null, it follows that the inclusion

$$\alpha : S^1 \longrightarrow X \longrightarrow X'$$

is also null. If $h : S^1 \times I \longrightarrow D^2$ is a null homotopy for the inclusion, then the composition

$$S^1 \times I \xrightarrow{h} D^2 \xrightarrow{\iota_{D^2}} X \cup_\alpha D^2$$

is a null-homotopy for the composition $S^1 \longrightarrow D^2 \xrightarrow{\iota_{D^2}} X \cup_\alpha D^2$. By definition of the pushout, this is the same as $S^1 \xrightarrow{\alpha} X \xrightarrow{\iota_X} X \cup_\alpha D^2$. So we have effectively killed off the class $[\alpha] \in \pi_1(X)$.

We can use the van Kampen theorem to show that this is all that we have done.

Proposition 27.2. *Let X be path-connected and let $\alpha : S^1 \longrightarrow X$ be a loop in X, based at x_0. Write $X' = X \cup_\alpha D^2$. Then*

$$\pi_1(X', \iota(x_0)) \cong \pi_1(X)/[\alpha].$$

Of course, we really mean the normal subgroup generated by α.

Proof. Consider the open subsets U and V of D^2, where $U = D^2 - \overline{B_{1/3}}$ and $V = B_{2/3}$. The map $\iota_{D^2} : D^2 \longrightarrow X'$ restricts to a homeomorphism on the interior of D^2, so the image of V in X' is open and path-connected. Now let $U' = X \cup U$. Since this is the image under the quotient map $X \amalg D^2 \longrightarrow X'$ of the saturated open set $X \amalg U$, U' is open in X'. It is easy to see that U' is also path-connected.

Now U' and V cover X'. Since U deformation retracts onto the boundary, it follows that U' deformation retracts onto X. The open set V is contractible. Finally, the path-connected subset $U' \cap V$ deformation retracts onto the circle of radius $1/2$. Moreover, the map

$$\mathbb{Z} \cong \pi_1(U' \cap V) \longrightarrow \pi_1(U') \cong \pi_1(X)$$

sends the generator to $[\alpha]$. The van Kampen theorem then implies that

$$\pi_1(X') \cong \pi_1(X)/\langle \alpha \rangle.$$

■

Actually, we cheated a little bit in this proof, since in order to apply the van Kampen theorem, we needed to work with a basepoint in $U' \cap V$. A more careful proof would include the necessary change-of-basepoint discussion.

What about attaching higher-dimensional cells?

Proposition 27.3. *Let X be path-connected and let $\alpha : S^{n-1} \longrightarrow X$ be an attaching map for an n-cell in X, based at x_0. Write $X' = X \cup_\alpha D^n$. Then, if $n \geq 3$,*

$$\pi_1(X', \iota(x_0)) \cong \pi_1(X).$$

Proof. The proof strategy is the same as for a 2-cell, so we don't reproduce it. The only change is that now $U' \cap V \simeq S^{n-1}$ is simply-connected. ■

Example 27.4. If we attach a 2-cell to S^1 along the identity map $\mathrm{id} : S^1 \longrightarrow S^1$, we obtain D^2. We have killed all of the fundamental group. If we attach another 2-cell, we get S^2. Attaching a 3-cell to S^2 via $\mathrm{id} : S^2 \longrightarrow S^2$ gives D^3. Attaching a second 3-cell gives S^3. The previous results tells us that all spaces obtained in this way (D^n and S^n) will be simply connected.

Example 28.1. ($\mathbb{RP}^2$) A more interesting example is attaching a 2-cell to S^1 along the double covering $\gamma_2 : S^1 \longrightarrow S^1$. Since this loop in S^1 corresponds to the element 2 in $\pi_1(S^1) \cong \mathbb{Z}$, the resulting space X' has $\pi_1(X') \cong \mathbb{Z}/2$. We have previously seen (last semester) that this is just the space $\mathbb{RP}^2$, since $\mathbb{RP}^2$ can be realized as the quotient of D^2 by the relation $x \sim -x$ on the boundary. This presents $\mathbb{RP}^2$ as a cell complex with a single 0-cell (vertex), a single 1-cell, and a single 2-cell.

We can next attach a 3-cell to $\mathbb{RP}^2$ along the double cover $S^2 \longrightarrow \mathbb{RP}^2$. The result is homeomorphic to $\mathbb{RP}^3$ by an analogous argument. In general, we have $\mathbb{RP}^n$ given as a cell complex with a single cell in each dimension. We have $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2$ for all $n \geq 2$.

Example 28.2. ($\mathbb{CP}^2$) Recall that $\mathbb{CP}^1 \cong S^2$ is simply connected. Last semester (Dec. 6), we showed that $\mathbb{CP}^n$ has a CW structure with a single cell in every even dimension. For example, $\mathbb{CP}^2$ is obtained from $\mathbb{CP}^1$ by attaching a 4-cell. It follows that every $\mathbb{CP}^n$ is simply-connected.

Why have we been careful to say how many cells we have in each dimension? Recall that a graph G is (for us) just a 1-dimensional CW complex and that we defined the Euler characteristic as

$$\begin{aligned}\chi(G) &= \text{number of vertices} - \text{number of edges}\\ &= \text{number of 0-cells} - \text{number of 1-cells.}\end{aligned}$$

We can make a similar definition for any CW complex.

$$\chi(X) = \text{number of 0-cells} - \text{number of 1-cells} + \text{number of 2-cells} - \text{number of 3-cells} + \dots.$$

Let's compute this for S^n and $\mathbb{RP}^n$. We have the following, using *any* of the cell structures we mentioned previously:

$$\chi(S^0) = 2, \qquad \chi(S^1) = 0, \qquad \chi(S^2) = 2, \qquad \chi(S^3) = 0,$$

$$\chi(\mathbb{RP}^0) = 1, \qquad \chi(\mathbb{RP}^1) = 0, \qquad \chi(\mathbb{RP}^2) = 1, \qquad \chi(\mathbb{RP}^3) = 0,$$

and

$$\chi(\mathbb{CP}^0) = 1, \qquad \chi(\mathbb{CP}^1) = 2, \qquad \chi(\mathbb{CP}^2) = 3, \qquad \chi(\mathbb{CP}^3) = 4,$$

In general, we see that

$$\chi(S^{2n}) = 2, \qquad \chi(S^{2n+1}) = 0, \qquad \chi(\mathbb{RP}^{2n}) = 1, \qquad \chi(\mathbb{RP}^{2n+1}) = 0, \qquad \chi(\mathbb{CP}^n) = n + 1.$$

Let's look at a few more examples of CW complexes.

Example 28.3. (Torus) Attach a 2-cell to $S^1 \vee S^1$ along the map $S^1 \longrightarrow S^1 \vee S^1$ given by $aba^{-1}b^{-1}$, where a and b are the standard inclusions $S^1 \hookrightarrow S^1 \vee S^1$. Let $X = (S^1 \vee S^1) \cup_{aba^{-1}b^{-1}} D^2$. We claim that

$$\pi_1(X) \cong F_2/aba^{-1}b^{-1} \cong \mathbb{Z}^2.$$

We have a surjective homomorphism

$$F_2 \longrightarrow \mathbb{Z}^2$$

sending any word $a^{n_1}b^{m_1}a^{n_2}b^{m_2}\dots a^{n_k}b^{m_k}$ to (n, m), where $n = \sum n_i$ and $m = \sum m_i$. The kernel consists of words where $\sum n_i = \sum m_j = 0$. This is the normal subgroup generated by $aba^{-1}b^{-1}$. For instance,

$$a^2b^{-5}ab^5a^{-3} = a^2b^{-5}(cb)^5a^{-2},$$

where $c = aba^{-1}b^{-1}$. It is easy to see, by expanding it out, how to parenthesize in order to consider $b^{-5}(cb)^5$ as an element in the normal subgroup generated by c. So this space has the same fundamental group as the torus.

But in fact this space is homeomorphic to the torus! Since the attaching map $S^1 \longrightarrow S^1 \vee S^1$ is surjective, so is $\iota_{D^2} : D^2 \longrightarrow X$. Even better, it is a quotient map. On the other hand, we also have a quotient map $I^2 \longrightarrow T^2$, and using the homeomorphism $I^2 \cong D^2$ from before, we can see that the quotient relation in the two cases agrees. The homeomorphism $T^2 \cong X$ puts a cell structure on the torus. There is a single 0-cell (a vertex), two 1-cells (the two circles in $S^1 \vee S^1$), and a single 2-cell, so that

$$\chi(T^2) = 1 - 2 + 1 = 0.$$