Ok, so we know that $\pi_n(\mathbb{RP}^2) \cong \pi_n(S^2)$. What are these groups? We will show later that $\pi_2(S^2) \cong \mathbb{Z}$. Just like for S^1, a generator for this group is the identity map $S^2 \longrightarrow S^2$. But the fascinating thing is that, in contrast to S^1, there are plenty of interesting higher homotopy groups! Here is a table of homotopy groups of spheres, taken from Wikipedia.

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	Z	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^4	0	0	0	Z	Z_2	Z_2	$Z\times Z_{12}$	Z_2^2	Z_2^2	$Z_{24}\times Z_3$	Z_{15}	Z_2	Z_2^3	$Z_{120}\times Z_{12}\times Z_2$	$Z_{84}\times Z_2^5$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2	Z_2	Z_{30}	Z_2	Z_2^3	$Z_{72}\times Z_2$
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	Z	Z_2	Z_{60}	$Z_{24}\times Z_2$	Z_2^3
S^7	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	Z_{120}	Z_2^3
S^8	0	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	$Z\times Z_{120}$

There are several things to note in this table.

(1) We have $\pi_n(S^3) = \pi_n(S^2)$ for $n \geq 3$. There is a map $S^3 \longrightarrow S^2$ that induces this isomorphism on homotopy groups. It is the Hopf map η we studied before ($\mathbb{C}^2 - \{0\} \longrightarrow \mathbb{CP}^1$). This map is not a cover, since the fibers are circles. But this is a higher analogue of a covering: it is an S^1-bundle. The analogue of the "evenly covered neighborhoos" here is called "local triviality" of the bundle. This means that each point in $x \in \mathbb{CP}^1$ has a neighborhood U such that $\eta^{-1}(U) \cong S^1 \times U$. Remembering that a point in $\mathbb{CP}^1$ is of the form $x = [z_1 : z_2]$, consider the open sets $U_1 = \{[z_1 : z_2] | z_1 \neq 0\}$ and $U_2 = \{[z_1 : z_2] | z_2 \neq 0\}$. These certainly cover $\mathbb{CP}^1$, and the isomorphism

$$\eta^{-1}(U_1) \cong S^1 \times U_1$$

is

$$(z_1, z_2) \mapsto \left(\frac{z_1}{\|z_1\|}, [z_1 : z_2] \right).$$

A bundle still has a lifting property for paths and homotopies, but the lifts are no longer unique. This means that we can't necessarily lift an arbitrary map $Y \longrightarrow S^2$ up to a map $Y \longrightarrow S^3$, and it need not be true that *all* higher homotopy groups of S^2 are identified with those of S^3. It turns out that what happens here is that we have a "long exact sequence" relating the homotopy groups of S^3, S^2, and S^1 (most of which are trivial).

(2) We have $\pi_n(S^k) = 0$ if $n < k$. The argument is similar to the one that showed the higher spheres are all simply-connected. The main step is to show that any map $S^n \longrightarrow S^k$ is homotopic to a nonsurjective map if $n < k$.

(3) The answers are eventually constant on each diagonal. There is a suspension homomorphism $\pi_n(S^k) \longrightarrow \pi_{n+1}(S^{k+1})$ that induces these isomorphisms. The stable answer for $\pi_{k+n}(S^k)$

is known as the nth stable homotopy group of spheres and is written π_n^s. We have

$$\pi_0^s = \mathbb{Z}, \qquad \pi_1^s = \mathbb{Z}/2, \qquad \pi_2^s = \mathbb{Z}/2, \qquad \pi_3^s = \mathbb{Z}/24.$$

These groups are known out to around $n = 60$.

(4) Most of the unstable groups are finite. The only infinite ones are $\pi_n(S^n) = \mathbb{Z}$ and $\pi_{4k-1}(S^{2k})$. The latter are all $\mathbb{Z}\times$(finite group). This is a theorem of J. P. Serre. This implies that all of the stable groups are finite, except $\pi_0^s = \mathbb{Z}$.

Ok, so homotopy groups are hard! But there are a few more examples of spaces whose homotopy groups are all known, so let's mention those before we abandon all hope and despair.

Example 35.1. Remember that we have a double cover $S^n \longrightarrow \mathbb{RP}^n$ inducing an isomorphism on all higher homotopy groups. But S^n does not have any homotopy groups until π_n, so this means that $\pi_k(\mathbb{RP}^n) = 0$ if $1 < k < n$. The inclusion $S^n \hookrightarrow S^{n+1}$, $(x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_n, 0)$ induces an inclusion $\mathbb{RP}^n \hookrightarrow \mathbb{RP}^{n+1}$. As n gets higher, we lose more and more homotopy groups. In the limit, $S^\infty = \bigcup_n S^n$ has no homotopy groups (and in fact it is contractible). Similarly, $\mathbb{RP}^\infty$ has only a fundamental group of $\mathbb{Z}/2$ but no higher homotopy groups.

Example 35.2. There is an analogous story for $\mathbb{CP}^n$. Here, we have for every n, an S^1-bundle $S^{2n-1} \simeq \mathbb{C}^n - \{0\} \longrightarrow \mathbb{CP}^n$. This map induces an isomorphism on π_k for $k \geq 3$ and gives $\pi_2(\mathbb{CP}^n) \cong \pi_1(S^1) \cong \mathbb{Z}$. So the only nontrivial homotopy group of $\mathbb{CP}^\infty$ is $\pi_2(\mathbb{CP}^\infty) \cong \mathbb{Z}$.

36. Wed, Apr. 16

Exam 2

37. Fri, Apr. 18

Last time, we discussed higher homotopy groups of some familiar spaces. We saw that most of the M_g and N_g have no higher homotopy groups. On the other hand, basic spaces like S^2 and $\mathbb{RP}^2$ have very complicated (and unknown) higher homotopy groups. The other examples in which we had complete understanding of the higher homotopy groups were the infinite-dimensional complexes $\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$. It turns out that this is quite typical: a finite cell complex almost always has infinitely many nontrivial homotopy groups!

This is rather disheartening. We think of a cell complex as an essentially finite amount of information. It would be nice if we only got finitely many algebraic objects out of it. There is such a construction: homology. As we will see, this will combine a number of the ideas we have recently encountered: the fundamental group and Euler characteristics. A good way to think about homology is that it is a more sophisticated version of the Euler characteristic.

We will deal with one of the (computationally) simplest forms of homology, known as cellular homology. Suppose that X is a CW complex, and define the **group of n-chains** on X, denoted $C_n(X)$, to be the free abelian group on the set of n-cells. .

$$\sum_{t_i \in T} n_i t_i,$$

where only finitely many n_i are nonzero. Recall the universal property of the free abelian group construction:

Proposition 37.1. *Let $\mathbb{Z}[T]$ be the free abelian group on a set T. Denote by $i : T \longrightarrow \mathbb{Z}[T]$ the function defined by $i(t) = 1 \cdot t$. If A is any abelian group and $f : T \longrightarrow A$ is any function, then there exists a unique homomorphism $\tilde{f} : \mathbb{Z}[T] \longrightarrow A$ such that $\tilde{f} \circ i = f$.*

Proof. The point is that every element of $\mathbb{Z}[T]$ is a finite linear combination of elements from T. Since $\tilde{f}$ is supposed to be a homomorphism, this forces

$$\tilde{f}(\sum n_i t_i) = \sum n_i f(t_i).$$

■

We are going to form a **chain complex** out of the various $C_n(X)$. In general, a chain complex is a collection of abelian groups C_n, one for each $n \in \mathbb{Z}$, and **differentials** (homomorphisms) $d_n : C_n \longrightarrow C_{n-1}$ such that $d_{n-1} \circ d_n = 0$.

Let's start with the cellular $d_1 : C_1(X) \longrightarrow C_0(X)$. By the universal property of the free abelian group $C_1(X)$, it suffices to specify the value of d_1 on the generators, namely the 1-cells of X. Recall that a 1-cell is a map $e : [0,1] \longrightarrow X$ attached via a map $\{0,1\} \longrightarrow X^0$. We define $d_1(e) = e(1) - e(0)$.

Ok, what about $d_2 : C_2(X) \longrightarrow C_1(X)$? Again, it suffices to specify d_2 on each 2-cell. If f is a 2-cell in X, we need to define $d_2(f) = \sum n_i e_i$ for some appropriate coefficients n_i. Pick a 1-cell e_i. The coefficient n_i is determined as follows: consider the map $S^1 \longrightarrow S^1$ defined by

$$S^1 \xrightarrow{f} X^1 \twoheadrightarrow X^1/X^0 \cong \bigvee S^1 \xrightarrow{e_i} S^1.$$

The map labelled f here is the attaching map, and the last map labelled e_i is the map which collapses all circles not corresponding to the edge e_i. The coefficient n_i needed for $d_2(f)$ is the class of this map in $\pi_1(S^1) \cong \mathbb{Z}$.

Now you know how to define any $d_n : C_n(X) \longrightarrow C_{n-1}(X)$: The coefficients in the expansion are the homotopy class of the composite

$$S^{n-1} \xrightarrow{f} X^{n-1} \twoheadrightarrow X^{n-1}/X^{n-2} \cong \bigvee S^{n-1} \xrightarrow{e_i} S^{n-1}$$

in the group $\pi_{n-1}(S^{n-1}) \cong \mathbb{Z}$.

It remains to check that $d_{n-1} \circ d_n = 0$. We will not give the proof in general, but here is the idea in the case $n = 2$. It suffices to show that $d_1(d_2(f)) = 0$ for each 2-cell f. The 1-chain $d_2(f)$ is defined using the attaching map $S^1 \xrightarrow{f} X^1$ for f. Essentially, $d_2(f)$ is telling us how this attaching map traverses the various edges in X^1. Since we are mapping a circle in, whenever we traverse an edge leading to a vertex v, we must also traverse some edge leaving that vertex. So when we go to compute $d_1(d_2(f))$, each time the vertex v arises with a sign $+1$, it will also necessarily appear again with a sign -1. So, in the end, the 0-cells will all cancel out in $d_1(d_2(f))$.

Let's look at some examples.

Example 37.2. Take $X = S^2$. Pick the CW structure having a single vertex and a single 2-cell. Then $C_1(X) = 0$, so both d_2 and d_1 must be the zero map. The chain complex $C_*(S^2)$ is

$$\mathbb{Z} \xrightarrow{d_2} 0 \xrightarrow{d_1} \mathbb{Z}.$$

Example 37.3. Take $X = S^2$. Pick the CW structure having a single vertex, a single edge, and two 2-cells attached via the identity map $S^1 \cong S^1$. Then $C_0(S^2) = C_1(S^2) = \mathbb{Z}$ and $C_2(S^2) = \mathbb{Z}^2$. The map

$$d_1 : C_1 = \mathbb{Z} \longrightarrow C_0 = \mathbb{Z}$$

is $d_1(e) = 0$ since the edge e is a loop. If we write f_1 and f_2 for the 2-cells, we see that $d_2(f_1) = d_2(f_2) = e$. Thus the resulting chain complex is

$$\mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

Example 37.4. Take $X = S^2$. Pick the CW structure having two cells in each degree ≤ 2. Here each attaching map $S^{n-1} \longrightarrow X^{n-1}$ is an identity map. Write x_1 and x_2 for the vertices and e_1 and e_2 for the edges. We have $d_1(e_i) = x_2 - x_1$. Similarly, we have $d_2(f_i) = e_1 - e_2$. The resulting chain complex is

$$\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}} \mathbb{Z}^2$$