**Example 38.1.** Take $X = T^2$. The standard cell structure we have used has a single 0, two 1-cells $a$ and $b$, and a single 2-cell $e$ attached via $aba^{-1}b^{-1}$. Since there is a single 0-cell, this means that automatically $d_1 = 0$. To calculate $d_2(e)$, we wish to calculate the coefficient in front of $a$ and $b$. For $a$, we must compose the attaching map $aba^{-1}b^{-1}$ with the projection onto the circle $a$. This means all of the $b$'s are sent to 0, so in the end we have $aa^{-1} = 0$. The same goes for $b$, so $d_2 = 0$. The chain complex $C_*(T^2)$ is

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z}.$$

Before we consider other examples, like the Klein bottle or $\mathbb{RP}^2$, let's discuss what these chain complexes are for.

**Definition 38.2.** Given a chain complex $C_*$, define a subgroup $Z_n \subseteq C_n$ to be the kernel of $d_n$. Elements of $Z_n$ are referred to as $n$-**cycles**. We also define $B_n \subseteq C_n$ to be the image of $d_{n+1}$. Note that since $d_n \circ d_{n+1} = 0$, we have $B_n \subseteq Z_n$. Define the **$n$-th homology group of the chain complex** $C_*$ to be

$$H_n(C_*) = Z_n/B_n.$$

In the case of the complex $C_*(X)$ of cellular chains on a cell complex, we write $H_n(X)$ or $H_n(X;\mathbb{Z})$ for $H_n(C_*(X))$.

Let's compute the homology groups of the above spaces.

**Example 38.3.** ($S^2$, first approach) In the first CW structure on $S^2$, it is clear that we get $H_0 = H_2 = \mathbb{Z}$ and $H_1 = 0$.

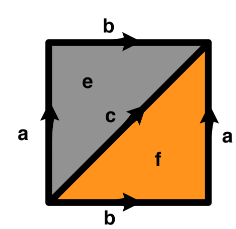
**Example 38.4.** ($S^2$, second approach) In the second CW structure on $S^2$, we again see that $H_0 \cong \mathbb{Z}$ since $d_1 = 0$, so that $B_0 = 0$ and $H_0 = Z_0 = \mathbb{Z}$. Next, the statement $d_1 = 0$ also means that $Z_1 = C_1 = \mathbb{Z}$, and we see that $d_2$ is surjective, so that $B_1 = Z_1 = C_1$. It follows that $H_1 \cong \mathbb{Z}$. Finally, the kernel of $d_2$ is the cyclic subgroup of $\mathbb{Z}^2$ generated by $(1,-1)$, so $H_2 = Z_2 \cong \mathbb{Z}$.

**Example 38.5.** ($S^2$, third approach) In the third CW structure, the differential $d_1$ has image the subgroup generated by $(-1,1)$, so $H_0 \cong \mathbb{Z}^2/(-1,1) \cong \mathbb{Z}$. The kernel of $d_1$ is the subgroup generated by $(1,-1)$, which is the image of $d_2$, so $H_1 = 0$. The kernel of $d_2$ is again the subgroup generated by $(-1,1)$, so that $H_2 \cong \mathbb{Z}$.

**Example 38.6.** (torus, first approach) Since all differentials were zero in $C_*(T^2)$ given above, it is immediate that

$$H_0(T^2) \cong \mathbb{Z}, \qquad H_1(T^2) \cong \mathbb{Z}^2, \qquad H_2(T^2) \cong \mathbb{Z}.$$

**Example 38.7.** (torus, second approach) Consider the CW structure on $T^2$ as given in the picture to the right. The resulting chain complex is




$$\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\quad 0 \quad} \mathbb{Z}$$

We read off right away that $H_0(T^2) \cong \mathbb{Z}$. Then

$$H_1(T^2) = Z^3/\operatorname{im}(d_2) = \mathbb{Z}^3/\mathbb{Z}(1,1,-1) \cong \mathbb{Z}^2.$$

The last isomorphism is induced by the map $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} : \mathbb{Z}^3 \longrightarrow \mathbb{Z}^2$. Finally,

$$H_2(T^2) = \ker(d_2) = \mathbb{Z}(1,1) \cong \mathbb{Z}.$$

We continue with more homology calculation examples.

**Example 39.1.** (Klein bottle, first version) Recall that we have a CW structure on $K$ having a single 0-cell and 2-cell and two 1-cells. The 2-cell is attached according to the relation $aba^{-1}b$. It follows that $C_*(K)$ is the chain complex

$$\mathbb{Z} \xrightarrow{\binom{0}{2}} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z}$$

We read off immediately that $H_0(K) \cong \mathbb{Z}$ and that $H_2(K) = 0$ since $d_2$ is injective. The remaining calculation is

$$H_1(K) = \mathbb{Z}^2/\mathbb{Z}(0,2) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

**Example 39.2.** (Klein bottle, second version) Recall that we discussed a second CW structure on $K$ having a single 0-cell and 2-cell and two 1-cells. The 2-cell is attached according to the relation $c^2d^2$. It follows that $C_*(K)$ is the chain complex

$$\mathbb{Z} \xrightarrow{\binom{2}{2}} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z}$$

We read off immediately that $H_0(K) \cong \mathbb{Z}$ and that $H_2(K) = 0$ since $d_2$ is injective. The remaining calculation is

$$H_1(K) = \mathbb{Z}^2/\mathbb{Z}(2,2) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Here the isomorphism $\mathbb{Z}^2/\mathbb{Z}(2,2) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is induced by the map

$$\mathbb{Z}^2 \twoheadrightarrow \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$
$$(n,k) \mapsto (n-k,k).$$

**Example 39.3.** ($\mathbb{RP}^2$) We have a CW structure with a single cell in dimensions 0, 1, and 2. The attaching map for the 2-cell is $\gamma_2 : S^1 \longrightarrow S^1$. It follows that the chain complex $C_*(\mathbb{RP}^2)$ is

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

Thus $H_0(\mathbb{RP}^2) \cong \mathbb{Z}$, $H_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$, and $H_2(\mathbb{RP}^2) = 0$.

Comparing what we have found in the examples so far suggest what would happen with a general surface.

**Example 39.4.** (Orientable surfaces) We have a CW structure on $M_g$ with a single 0-cell and 2-cell and $2g$ 1-cells. The attaching map for the 2-cell is the product of commutators $[a_1,b_1]\dots[a_g,b_g]$. It follows that $C_*(M_g)$ is the chain complex

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z}.$$

So $H_0(M_g) \cong Z$, $H_1(M_g) \cong \mathbb{Z}^{2g}$, and $H_2(M_g) \cong \mathbb{Z}$.

**Example 39.5.** (Nonorientable surfaces) We have a CW structure on $N_g$ with a single 0-cell and 2-cell and $g$ 1-cells. The attaching map for the 2-cell is the product $a_1^2 \dots a_g^2$. It follows that $C_*(N_g)$ is the chain complex

$$\mathbb{Z} \xrightarrow{\begin{pmatrix} 2 \\ \vdots \\ 2 \end{pmatrix}} \mathbb{Z}^g \xrightarrow{0} \mathbb{Z}$$

So $H_0(N_g) \cong \mathbb{Z}$, $H_1(N_g) \cong \mathbb{Z}^g/\mathbb{Z}(2,\dots,2) \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}$, and $H_2(N_g) = 0$. Again, the isomorphism $\mathbb{Z}^g/\mathbb{Z}(2,\dots,2) \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}$ is induced by

$$\mathbb{Z}^g \twoheadrightarrow \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}$$
$$(n_1,\dots,n_g) \mapsto (n_1-n_g, n_2-n_g, \dots, n_{g-1}-n_g, n_g).$$

**Remark 39.6.** According to the previous examples and our Proposition 34.1, a compact, connected surface $M$ satisfies $H_2(M) \cong \mathbb{Z}$ if $M$ is orientable and satisfies $H_2(M) = 0$ if $M$ is not orientable.

So $H_2$ tells us about orientability. What about $H_0$?

**Proposition 39.7.** *A CW complex is path-connected (and nonempty) if and only if $H_0(X) \cong \mathbb{Z}$. In general, we have*

$$H_0(X) \cong \mathbb{Z}[\pi_0(X)].$$

*Proof.* Suppose that $X$ is nonempty and path-connected. Define a homomorphism $\epsilon : C_0(X) \longrightarrow \mathbb{Z}$, which is equal to 1 on every 0-cell. This is clearly surjective (this uses that $X$ is nonempty and thus has at least one 0-cell). We claim that $\ker(\epsilon) = \operatorname{im}(d_1)$. The First Isomorphism Theorem will then imply that $H_0(X) \cong \mathbb{Z}$.

The subgroup $\operatorname{im}(d_1)$ of $C_0(X)$ is generated by the elements $d_1(e) = e(1) - e(0)$. Since each of these lies in $\ker(\epsilon)$, it follows that the entire image is in $\ker(\epsilon)$. For the other containment, suppose that $z = \sum_i n_i x_i \in \ker(\epsilon)$. We then have

$$0 = \epsilon\Big(\sum_i n_i x_i\Big) = \sum_i n_i.$$

The argument is by induction on $N = \sum_i |n_i|$. There is nothing to prove if $N = 0$. Note that the 1-skeleton $X^1$ must be path-connected since $X$ is path-connected. Suppose that some coefficient $n_i > 0$. Then there must be another coefficient $n_j < 0$. By assumption, there is a path in $X^1$ from $x_j$ to $x_i$. By the topology axioms on a CW complex, this path meets finitely many 1-cells. In other words, we can connect $x_j$ to $x_i$ via a finite sequence of edges. If the edges are $e_1, \dots, e_k$, then by construction we have

$$d_1(e_1 + \cdots + e_k) = x_i - x_j.$$

We have thus reduced to the $N - 2$ case.

## 40. Fri, Apr. 25

Now suppose that $H_0(X) \cong \mathbb{Z}$. It follows that $C_0(X) \neq 0$, so that $X$ is nonempty. The argument from above shows that $\epsilon : C_0(X) \longrightarrow \mathbb{Z}$ is surjective and vanishes on $\operatorname{im}(d_1)$, so that we get an induced surjection $H_0(X) \longrightarrow \mathbb{Z}$. Since we have assumed that $H_0(X) \cong \mathbb{Z}$, it follows that $\epsilon$ is an isomorphism.

We can run the above argument backwards to deduce that the 1-skeleton must be path-connected. That is, suppose $x$ and $y$ are 0-cells. Then $[x] = [y]$, so there must be a 1-chain $w$ such that $d(w) = x - y$. Suppose $w = \sum n_i e_i$. Then one edge $e_1$ must end at $x$. Let $a = e_1(0)$. Then, if $a \neq y$, there must be another edge ending at $a$ to cancel it out. Repeat this until we get an edge starting at either $x$ or $y$. If it is $x$, then we may remove all the previously considered 1-cells, and the rest still give a 1-cycle. Repeat the argument. If we get $y$, we are done.

This argument shows that $X^1$ is path-connected. Attaching higher cells does not break the connectivity, so that $X$ is path-connected.

For the general statement, the point is that since $X$ is CW, we can write it as the disjoint union of its path-components. The result follows from the next proposition.

∎

**Proposition 40.1.** *Let $X \cong \coprod_i X_i$. Then*

$$H_n(X) \cong \bigoplus_i H_n(X_i)$$

*for all $n$.*

*Proof.* The point is that there is already a direct sum decomposition

$$C_n(X) \cong \bigoplus_i C_n(X_i)$$

since a cell of $X$ must lie in a single component. Moreover, the differentials $d_n$ are compatible with this direct sum decomposition, in the sense that the restriction of $d_n$ to $C_n(X_i)$ lands in $C_{n-1}(X_i)$. ■