41. Mon, Apr. 28

We saw last time that homology interacts nicely with disjoint unions. We list here a few more nice properties of homology, without proof.

Proposition 41.1.

(1) *If X is a k-dimensional CW complex, then $H_n(X) = 0$ for all $n > k$.*
(2) *Let $f : X \longrightarrow Y$ be a* **cellular** *map, meaning that $f(X^k) \subseteq Y^k$ for all k. Then f induces homomorphisms $f_* : H_n(X) \longrightarrow H_n(Y)$ for all n.*
(3) *Let $\iota : X^k \hookrightarrow X$ be the inclusion of the k-skeleton. Then the induced map*

$$\iota_* : H_n(X^k) \longrightarrow H_n(X)$$

is an isomorphism for $n < k$ and a surjection for $n = k$.
(4) *Suppose that $f \simeq g : X \longrightarrow Y$ are both cellular maps. Then $f_* = g_*$.*

Note that (4) implies that homotopy equivalent spaces have isomorphic homology.

On the other hand, this version of homology also has its drawbacks.

- This requires a cell structure on a space, and these are not always easy to come by.
- We only get induced maps on homology coming from cellular maps of spaces. But most maps are not cellular! A famous example of a noncellular map is the diagonal map $X \longrightarrow X \times X$ for any space X (think of the case $X = I$).

I encourage you to read the first five pages of Chapter 2 of Hatcher. He has a nice discussion of how you might be lead to (cellular) homology and why you might turn to other forms (simplicial and singular).

Ok, last week we discussed $H_0(X)$. What about $H_1(X)$? In the examples we've seen, it looks like $H_1(X)$ is close to $\pi_1(X)$.

Theorem 41.2 (Hurewicz)**.** *Assume that X is a connected CW complex. Then*

$$H_1(X) \cong \pi_1(X)_{ab}.$$

Proof. First, note that cells in dimensions 3 or higher affect neither π_1 nor H_1. In other words, if X^2 is the 2-skeleton, then $\pi_1(X^2) \cong \pi_1(X)$ and $H_1(X^2) \cong H_1(X)$.

42. Wed, Apr. 30

By the van Kampen theorem, we know that $\pi_1(X^1) \twoheadrightarrow \pi_1(X^2)$ is surjective. Moreover, if we denote by $\beta_1, \dots, \beta_k$ the 2-cells of X (or really, their attaching maps, thought of as elements of $\pi_1(X^1)$), then the van Kampen theorem tells us that

$$\pi_1(X^2) \cong \pi_1(X^1)/\langle \beta_1, \dots, \beta_k \rangle.$$

Denote by $\tilde{X}^1$ the result of collapsing out a maximal tree in the graph X^1, and recall that the natural map $X^1 \longrightarrow \tilde{X}^1$ is a homotopy equivalence. The space $\tilde{X}^1$ is a wedge of circles $\tilde{X}^1 \cong \bigvee S^1$, each circle corresponding to a generator of $\pi_1(X^1)$. We now have

$$\pi_1(X^2) \cong \pi_1(\tilde{X}^1)/\langle \beta_1, \dots, \beta_k \rangle \cong F(\alpha_1, \dots, \alpha_n)/\langle \beta_1, \dots, \beta_k \rangle.$$

Let's now turn to homology. We know that $H_1(X)$ is computed as a quotient

$$C_2(X) \longrightarrow Z_1(X).$$

Lemma 42.1. *We have $Z_1(X) = Z_1(X^1) = H_1(X^1) \cong H_1(\tilde{X}^1) = Z_1(\tilde{X}^1) = C_1(\tilde{X}_1)$.*

The homology isomorphism follows from the fact that $X \longrightarrow \tilde{X}^1$ is a homotopy equivalence.
The lemma implies that $H_1(X)$ is the quotient

$$H_1(X) \cong \mathbb{Z}(\alpha_1, \dots, \alpha_n)/\langle \beta_1, \dots, \beta_k \rangle.$$

There is now an obvious surjection

$$\pi_1(X) \longrightarrow H_1(X)$$

induced by the abelianization map $F(\alpha_1, \dots, \alpha_n) \twoheadrightarrow \mathbb{Z}[\alpha_1, \dots, \alpha_n]$. It follows that the induced map is also abelianization, as we saw on April 7. ■

There is also a statement in higher dimensions, assuming that all lower homotopy groups vanish. We state it without proof.

Theorem 42.2 (Hurewicz)**.** *Assume that X is a CW complex satisfying $\pi_k(X) = 0$ for $k < n$ (we say that X is $(n-1)$-**connected**), where $n \geq 2$. Define*

$$h_n : \pi_n(X) \longrightarrow H_n(X)$$

by

$$h_n(\alpha) = \alpha_*(x_n),$$

where $x_n \in H_n(S^n)$ is the class of the unique n-cell (in the minimal CW structure on S^n). Then h_n is an isomorphism of groups, known as the Hurewicz map.

Using induction and the fundamental group Hurewicz theorem, this implies the following result.

Corollary 42.3. *Suppose that X is a CW complex that is $(n-1)$-connected. Then $H_k(X) = 0$ for $0 < k < n$ as well.*

Note that the torus T^2 shows that Theorem 42.2 fails if we drop the connectivity hypothesis.

Remark 42.4. The Hurewicz theorem is often mentioned as the simplest way to establish the isomorphism $\pi_n(S^n) \cong \mathbb{Z}$ for $n \geq 2$. But for us this would be circular, since we needed to know this isomorphism in order to define the cellular differentials. The point is that the Hurewicz theorem also holds in simplicial or singular homology, neither of which relies on the calculation $\pi_n(S^n) \cong \mathbb{Z}$ in the definition.

Recall that we talked about the Euler characteristic for surfaces. For any chain complex C_*, we define the Euler characteristic of C_* by $\chi(C_*) = \sum (-1)^i \operatorname{rank}(C_i)$. Recall that the **rank** of a free abelian group is the maximal number of linearly independent elements. For example, if $C \cong \mathbb{Z}^r \oplus A$, where A is finite, then $\operatorname{rank} C = r$.

43. Fri, May 2

Proposition 43.1. *For any chain complex, we have $\chi(C_*) = \chi(H_*(C_*))$.*

Proof. The key is to note that we have short exact sequences

$$0 \longrightarrow Z_i \longrightarrow C_i \longrightarrow B_{i-1} \longrightarrow 0.$$

and

$$0 \longrightarrow B_i \longrightarrow Z_i \longrightarrow H_i \longrightarrow 0.$$

By a homework problem, these tell us that

$$\operatorname{rank}(C_i) = \operatorname{rank}(Z_i) + \operatorname{rank}(B_{i-1})$$

and

$$\operatorname{rank}(Z_i) = \operatorname{rank}(B_i) + \operatorname{rank}(H_i).$$

So

$$\sum_i (-1)^i \operatorname{rank}(C_i) = \sum_i (-1)^i (\operatorname{rank}(B_i) + \operatorname{rank}(H_i) + \operatorname{rank}(B_{i-1})).$$

This is a telescoping sum, and we end up with $\chi(H_*)$. ■

So this tells us that the Euler characteristic only depends on the homology of the space, not on the particular cellular model.

Example 43.2. We talked about the homology of $\mathbb{RP}^2$ earlier. We saw this was

$$H_0(\mathbb{RP}^2) \cong \mathbb{Z}, \qquad H_1(\mathbb{RP}^2) = \mathbb{Z}/2, \qquad H_2(\mathbb{RP}^2) = 0.$$

Since the standard model for $\mathbb{RP}^2$ has no cells above dimension 2, there is of course no homology in higher dimensions. The Euler characteristic computation according to homology is

$$\chi(\mathbb{RP}^2) = \operatorname{rank}(\mathbb{Z}) - \operatorname{rank}(\mathbb{Z}/2) = 1.$$

Now let's consider $\mathbb{RP}^n$ for $n > 2$. The cellular chain complex is

$$\begin{tikzcd} C_n \arrow[r, "1+(-1)^n"] \arrow[d, equal] & C_{n-1} \arrow[r] \arrow[d, equal] & \dots \arrow[r, "2"] & C_1 \arrow[r, "0"] \arrow[d, equal] & C_0 \arrow[d, equal] \\ \mathbb{Z} & \mathbb{Z} & & \mathbb{Z} & \mathbb{Z} \end{tikzcd}$$

To understand the differential d_k, it suffices to understand what it does to the k-cell e_k. The attaching map for this k-cell is the double cover $S^{k-1} \longrightarrow \mathbb{RP}^{k-1}$. Then $d_k(e_k) = n_k e_{k-1}$, where n_k is the degree of the map

$$S^{k-1} \longrightarrow \mathbb{RP}^{k-1} \longrightarrow \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \cong S^{k-1}.$$

To visualize this, think of $\mathbb{RP}^{k-1}$ as the quotient of the northern hemisphere of S^{k-1} by a relation on the boundary. Then $\mathbb{RP}^{k-2}$ is the quotient of the boundary, so the quotient $\mathbb{RP}^{k-1}/\mathbb{RP}^{k-2}$ is the northern hemisphere with the equator collapsed. The map $S^{k-1} \longrightarrow \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2}$ factors through $S^{k-1}/S^{k-2} \cong S^{k-1} \vee S^{k-1}$. The map on the nothern hemisphere $S^{k-1} \longrightarrow \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \cong S^{k-1}$ is the identity. On the other hand, the map on the southern hemisphere can be identified with the map $(x_1, \dots, x_k) \mapsto (-x_1, \dots, -x_k)$. This is a homeomorphism, so the question is whether it is homotopic to the identity, in which case the map on this hemisphere corresponds to 1, or it is not, in which case the maps corresponds to -1. But this map is a sequence of k reflections, each of which has determinant -1. So the map has determinant $(-1)^k$. This number then agrees with the degree of the map, and we find that $n_k = 1 + (-1)^k$.

It follows that in degrees less than n we have

$$H_{2i}(\mathbb{RP}^n) = 0, i > 0, \qquad H_0(\mathbb{RP}^n) = \mathbb{Z}, \qquad H_{2i+1}(\mathbb{RP}^n) = \mathbb{Z}/2.$$

To determine $H_n(\mathbb{RP}^n)$, we consider $d_n : C_n \longrightarrow C_{n-1}$. If n is even, then d_n is injective, so $H_n(\mathbb{RP}^n) = 0$. On the other hand, if n is odd, then $d_n = 0$, so that $H_n(\mathbb{RP}^n) \cong \mathbb{Z}$.

The Euler characteristic computation according to homology is now

$$\chi(\mathbb{RP}^{2k}) = 0 + 0 + \cdots + 0 + 1 = 1, \qquad \chi(\mathbb{RP}^{2k+1}) = 1 + 0 + 0 + \cdots + 0 + 1 = 2.$$

Recall that we mentioned that for an n-manifold, the top homology group $H_n(M)$ is either $\mathbb{Z}$ or 0, depending on whether the manifold is orientable or not. The above shows that $\mathbb{RP}^n$ is orientable if and only if n is odd ($n \geq 1$).

Recall that one of the themes of this course was to reduce a problem in topology to one in algebra. For instance, we try to understand whether spaces are homotopy equivalent by comparing their homotopy groups.

Definition 43.3. A map $f : X \longrightarrow Y$ is a **weak homotopy equivalence** if it induces a bijection $\pi_0(X) \cong \pi_0(Y)$ and if, for every choice of basepoint $x \in X$, it induces an isomorphism $\pi_n(X, x) \cong \pi_n(Y, f(x))$.

Example 43.4. Consider the set $X = \{L, R, N, S\}$, topologized as follows: the singletons L and R are open, as are the complements of the singletons N and S. Of course this forces $\{L, R\}$ to be open. Then define $f : S^1 \longrightarrow X$ by

$$f((x, y)) = \begin{cases} L & x < 0 \\ R & x > 0 \\ N & (x, y) = (0, 1) \\ S & (x, y) = (0, -1) \end{cases}$$

Then it is quickly verified that this is continuous.

In fact, it is also a weak homotopy equivalence. Again, it is quickly checked that X is path-connected. For a larger theory of which this is merely one example, see Springer Lecture Notes 2032, *Algebraic Topology of Finite Topological Spaces and Applications*.