(Proof continued from last time...)

Last time, we introduced functions

$$\Gamma : \mathbb{Z} \longrightarrow \pi_1(S^1), \qquad w : \pi_1(S^1) \longrightarrow \mathbb{Z}.$$

Γ was defined by $\Gamma(n) = \gamma_n$, where $\gamma_n(s) = e^{2\pi ins}$. The function w was defined by $w(\alpha) = \tilde{\alpha}(1)$, where $\tilde{\alpha} : I \longrightarrow \mathbb{R}$ was a lift of α starting at 0. We were left with showing that w is the inverse of Γ.

First note that $\delta(s) = ns$ is a path in $\mathbb{R}$ starting at 0, and $p \circ \delta(s) = e^{2\pi i(ns)} = \gamma_n(s)$, so δ is a lift of γ_n starting at 0. By uniqueness of lifts (Lemma 4.3 from last time), δ must be $\tilde{\gamma_n}$. Therefore

$$w \circ \Gamma(n) = w(\gamma_n) = \tilde{\gamma}_n(1) = \delta(1) = n.$$

It remains to check that $\Big[\Gamma(w(\gamma))\Big] = [\gamma]$ for any loop γ. Consider lifts $\widetilde{\Gamma(w(\gamma))}$ and $\tilde{\gamma}$. These are both paths in $\mathbb{R}$ starting at 0 and ending at $\tilde{\gamma}(1) = w(\gamma)$ (this uses that $w \circ \Gamma(n) = n$). But any two such paths are homotopic (use a straight-line homotopy)! Composing that homotopy with the exponential map p will produce a path-homotopy $\Gamma(w(\gamma)) \simeq_p \gamma$ as desired. ∎

Using problem 2 from Homework II, we get the following result.

Corollary 5.1. *Let T^n denote the n-torus $T^n = S^1 \times S^1 \times \cdots \times S^1$ (n times). Then $\pi_1(T^n) \cong \mathbb{Z}^n$.*

We can derive a number of very interesting consequences from our knowledge of the fundamental group of S^1.

Theorem 5.2. *(Brouwer fixed point theorem) For any map $f : D^2 \longrightarrow D^2$, there exists at least one point $x \in D^2$ such that $f(x) = x$. Such an x is called a* **fixed point** *of the map f.*

Proof. Assume for a contradiction that f has *no* fixed points. Then $x - f(x)$ is not the origin, and for each point x there is a unique $t_x \geq 1$ such that $f(x) + t_x(x - f(x))$ lies on S^1. This is where the ray starting at $f(x)$ and passing through x meets the circle. Define $g(x) : D^2 \longrightarrow S^1$ by the formula

$$g(x) = f(x) + t_x(x - f(x)).$$

You should convince yourself that t_x, and therefore $g(x)$, is a continuous function of x.

Now the key point is that if x starts in the boundary S^1 of D^2, then $t_x = 1$ and $g(x) = x$. In other words, the composition

$$S^1 \xrightarrow{i} D^2 \xrightarrow{g} S^1$$

is the identity map of S^1. Consider what happens on the fundamental group. The conclusion would be that the composition

$$\pi_1(S^1) = \mathbb{Z} \xrightarrow{i_*} \pi_1(D^2) = 0 \xrightarrow{g_*} \pi_1(S^1) = \mathbb{Z}$$

is the identity map of $\mathbb{Z}$, which is impossible. ∎

Application: Take a cup of coffee and move it around, so that the coffee gets mixed up. When it comes to rest, there is some particle that ends up where it started. (Okay, this is sort of BS since it assumes every particle stays on the surface, but it is a common description of Brouwer fixed point theorem.)

Theorem 5.3. *(Fundamental theorem of algebra) Every nonconstant polynomial with complex coefficients has a solution in $\mathbb{C}$.*

Proof. Assume that $p(z) \neq 0$ for all $z \in \mathbb{C}$. We will show that p must be constant. Define a function $f : S^1 \longrightarrow S^1$ by $f(z) = p(z)/\|p(z)\|$. We can define a homotopy by

$$h(z,t) = p(zt)/\|p(zt)\|.$$

Thus f is homotopic to a constant map, which means that it has "degree" zero.

On the other hand, write a_i for the coefficients of the degree n polynomial $p(z)$. For convenience, we assume $p(z)$ is monic. Let $k(z,t)$ be the homotopy between z^n and $p(z)$ given by the formula

$$k(z,t) = \sum_{i=0}^{n} a_i z^i t^{n-i} = z^n + a_{n-1}z^{n-1}t + \cdots + a_0 t^n.$$

Note that, for $t \neq 0$ this can be rewritten as $k(z,t) = t^n p(z/t)$. In particular, this is never 0 by hypothesis. It follows that the map $H : S^1 \times I \longrightarrow S^1$ defined by the formula

$$H(z,t) = \frac{k(z,t)}{\|k(z,t)\|}$$

defines a homotopy from z^n to f. This shows that f has degree n.

Combining the two statements gives that $n = 0$, so that p is a constant polynomial. ∎

Application: . . . everything?

6. Wed, Jan. 29

Theorem 6.1. *(Borsuk-Ulam Theorem) For every continuous map $f : S^2 \longrightarrow \mathbb{R}^2$, there is an antipodal pair of points $\{x, -x\} \subset S^2$ such that the $f(x) = f(-x)$.*

Proof. Suppose not. Then we can define a map $g : S^2 \longrightarrow S^1$ by $g(x) = \frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}$. Let $\gamma : S^1 \longrightarrow S^1$ be the restriction to the equator. Note that since γ extends over the northern (or southern) hemisphere, the loop γ is null. We also write δ for the composition $I \longrightarrow S^1 \xrightarrow{\gamma} S^1$.

The equation $g(-z) = -g(z)$ means that $\gamma(-z) = -\gamma(z)$ or $\delta(t + \frac{1}{2}) = -\delta(t)$. Denote by $\tilde{\delta}$ a lift to a path in $\mathbb{R}$. Then $\tilde{\delta}$ must satisfy the equation $\tilde{\delta}(t + \frac{1}{2}) = \tilde{\delta}(t) + \frac{1}{2} + k$ for some integer k. In particular, we find that

$$\tilde{\delta}(1) = \tilde{\delta}\left(\frac{1}{2}\right) + \frac{1}{2} + k = \tilde{\delta}(0) + 1 + 2k.$$

Thus the degree of γ is the odd integer $1 + 2k$. This contradicts that γ is null. ∎

Application: At any point in time, there are two polar opposite points on Earth having the same temperature and same barometric pressure. (Or pick any two continuously varying parameters)

Corollary 6.2. *The sphere S^2 is not homeomorphic to any subspace of $\mathbb{R}^2$.*

Proof. According to the theorem, there is no continuous injection $S^2 \longrightarrow \mathbb{R}^2$. ∎

Dependence on the basepoint

Although we often talk about "the fundamental group" of a space X, this group depends on the choice of basepoint for the loops. One thing at least should be clear: if we want to understand $\pi_1(X, x_0)$, only the path component of x_0 in X is relevant. Any other path component can be ignored. More precisely, if PC_x denotes the path-component of a point x, then for any choice of basepoint x_0, we get an **isomorphism of groups**

$$\pi_1(PC_{x_0}, x_0) \cong \pi_1(X, x_0).$$

For this reason, we will often assume from now on that our spaces are path-connected.

Under this assumption that X is path-connected, how does the fundamental group depend on the choice of base point? Suppose that x_0 and x_1 are points in X. How can we compare loops based at x_0 to loops based at x_1? Since X is path-connected, we may choose some path α in X from x_0 to x_1. Then, if γ is a loop based at x_0, we get a loop $\overline{\alpha} \cdot \gamma \cdot \alpha$. Let us write $\Phi_\alpha(\gamma)$ for this loop.

Proposition 6.3.

(1) *The operation Φ_α gives a well-defined operation on homotopy-classes of loops.*
(2) *The operation Φ_α only depends on the homotopy-class of α.*
(3) *The operation Φ_α induces an isomorphism of groups*

$$\pi_1(X, x_0) \cong \pi_1(X, x_1)$$

with inverse induced by $\Phi_{\overline{\alpha}}$.

Proof. Both (1) and (2) follow immediately from Proposition 3.1. It is also clear that $\Phi_{\overline{\alpha}}$ is inverse to Φ_α, since

$$\Phi_{\overline{\alpha}}(\Phi_\alpha(\gamma)) = \alpha \cdot \overline{\alpha} \cdot \gamma \cdot \alpha \cdot \overline{\alpha} \simeq_p \gamma.$$

Finally, we show Φ_α is a homomorphism:

$$[\Phi_\alpha(\gamma \cdot \delta)] = [\overline{\alpha} \cdot \gamma \cdot \delta \cdot \alpha] = [\overline{\alpha} \cdot \gamma \cdot (\alpha \cdot \overline{\alpha}) \cdot \delta \cdot \alpha] = [\Phi_\alpha(\gamma)] \cdot [\Phi_\alpha(\delta)].$$

■

So, as long as X is path-connected, the isomorphism-type of the fundamental group of X does not depend on the basepoint. For example, once we know that $\pi_1(\mathbb{R}^2, \mathbf{0}) = \langle e \rangle$, it follows that the same would be true with any other choice of basepoint. A (path-connected) space with trivial fundamental group is said to be **simply connected**. Again, we know that any convex subset of $\mathbb{R}^n$ is simply connected.

We saw that S^1 has a nontrivial fundamental group, but in contrast we will see that the higher spheres are all simply connected.

Theorem 6.4. *The n-sphere S^n is simply connected if $n \geq 2$.*

This follows easily from the following theorem.

Theorem 6.5. *Any continuous map $S^1 \longrightarrow S^n$ is path-homotopic to one that is not surjective.*

Let's first use this to deduce the statement about n-spheres. Let γ be a loop in S^n. We know it is path-homotopic to a loop δ that is not surjective. But recall that $S^n - \{P\} \cong \mathbb{R}^n$. Thus we can contract δ using a straight-line homotopy in the complement of any missed point. It remains to prove the latter theorem.

Proof. There are a number of ways to prove this result. For instance, it is an easy consequence of "Sard's Theorem" from differential topology. Here is a proof using once again the Lebesgue number lemma.

Let $\{U, V\}$ be the covering of S^n, where U is the upper (open) hemisphere, and V is the complement of the North pole. Let $\gamma : S^1 \longrightarrow S^n$ be a loop. By Lebesgue, we can subdivide the interval I into finitely many subintervals $[s_i, s_{i+1}]$ such that on each subinterval, γ stays within either U or V. We will deform γ so that it misses the North pole. On the subintervals that are mapped into V, nothing needs to be done.

Suppose $[s_i, s_{i+1}]$ is not mapped into V, so that γ passes through the North pole on this segment. Recall that the open hemisphere U is homeomorphic to $\mathbb{R}^n$. The problem thus reduces to the following: given a path in $\mathbb{R}^n$, show it is path-homotopic to one not passing through the origin. This is simple. First, any path is homotopic to the straight-line path. If that does not pass through the origin, great. If it does, just wiggle it a little, and it won't any more. ■

Corollary 6.6. *The infinite sphere S^∞ is simply connected.*

Proof. Consider a loop α in S^∞. The image of α is then a compact subset of the CW complex S^∞. It follows (see Hatcher, A.1) that the image of α is contained in a finite union of cells. In other words, the image of α is contained in some S^n. By the above, α is null-homotopic in S^n and therefore in S^∞ as well. ■

You showed on your homework that S^∞ is contractible, and this in fact implies simply connected, as the next result shows.

Theorem 6.7. *Let $f : X \longrightarrow Y$ be a homotopy equivalence. Then, for any choice of basepoint $x \in X$, the induced map*

$$f_* : \pi_1(X, x) \xrightarrow{\cong} \pi_1(Y, f(x))$$

is an isomorphism.

At first glance, this might seem obvious, since we have a quasi-inverse $g : Y \longrightarrow X$ to f, and so we would expect g_* to be the inverse of f_*. But note that there is no reason that $g(f(x))$ would be x again, so g_* does not even map to the correct group to be the inverse of f_*. We need to employ some sort of change-of-basepoint to deal with this.

Proposition 6.8. *Let h be a homotopy between maps $f, g : X \rightrightarrows Y$. For a chosen basepoint $x_0 \in X$, define a path α in Y by $\alpha(s) = h(x_0, s)$. Then the diagram to the right commutes.*

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "f_*"] \arrow[dr, "g_*"'] & \pi_1(Y, f(x_0)) \arrow[d, "\Phi_\alpha", "\cong"'] \\ & \pi_1(Y, g(x_0)) \end{tikzcd}$$

Proof. For any loop γ in X based at x_0, we want a path-homotopy $H : \Phi_\alpha(f \circ \gamma) \simeq_p g \circ \gamma$. For each t, let α_t denote the path $\alpha_t(s) = \alpha(1 - (1 - s)(1 - t))$. Note that $\alpha_0 = \alpha$ and α_1 is constant at $\alpha(1) = g(x_0)$.

Such a path-homotopy is given by

$$H_t = \overline{\alpha_t} \cdot (h_t \circ \gamma) \cdot \alpha_t$$

■

Proof of Theorem 6.7. Let $g : Y \longrightarrow X$ be a quasi-inverse to f. Then $g \circ f \simeq \mathrm{id}_X$, so Prop 6.8 gives us a diagram

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "\mathrm{id}_*", "\cong"'] \arrow[dr, "(gf)_*"'] & \pi_1(X, x_0) \arrow[d, "\Phi_\alpha", "\cong"'] \\ & \pi_1(X, gf(x_0)) \end{tikzcd}$$

Now $(gf)_*$ must be an isomorphism since the other two maps in the diagram are isomorphisms. Since $(gf)_* = g_* \circ f_*$, the map f_* must be injective and similarly g_* must be surjective.

But now we can swap the roles of f and g, getting a diagram

$$\begin{tikzcd} \pi_1(Y, f(x_0)) \arrow[r, "\mathrm{id}_*", "\cong"'] \arrow[dr, "(fg)_*"'] & \pi_1(Y, f(x_0)) \arrow[d, "\Psi_\alpha", "\cong"'] \\ & \pi_1(Y, fgf(x_0)) \end{tikzcd}$$

It then follows that $g_* : \pi_1(Y, f(x_0)) \longrightarrow \pi_1(X, gf(x_0))$ is injective. Since we already showed it is surjective, we deduce that it is an isomorphism. Now going back to our first diagram, we get

$$g_* \circ f_* = \Phi_\alpha, \qquad \text{or} \qquad f_* = g_*^{-1} \circ \Phi_\alpha,$$

so that $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$ is an isomorphism. ■

So far, we know a number of simply connected spaces ($\mathbb{R}^n$, S^n for $n \geq 2$), and we know that $\pi_1(T^n) \cong \mathbb{Z}^n$ for any $n \geq 1$. Can there be torsion in the fundamental group? For example, is it possible that for some nontrivial loop γ in X, winding around the loop twice gives a trivial loop? The next example has this property.

Recall that the real projective plane $\mathbb{RP}^2$ is defined as the quotient of S^2 by the equivalence relation $x \sim -x$. The equivalence classes are precisely the sets of pairs of antipodal points. Another way to think about this is that each pair of antipodal points corresponds to a straight line through the origin. We will determine $\pi_1(\mathbb{RP}^2)$.