The interesting, new result here concerns the existence of lifts.

Proposition 14.1. *(Lifting Criterion) Let $p : E \longrightarrow B$ be a covering and let $f : Z \longrightarrow B$, with Z very connected. Given points $z_0 \in Z$ and $e_0 \in E$ with $f(z_0) = p(e_0)$, there is a lift $\tilde{f}$ with $\tilde{f}(z_0) = e_0$ if and only if $f_*(\pi_1(Z, z_0)) \subseteq p_*(\pi_1(E, e_0))$.*

Proof. ($\Rightarrow$) This is clear. Since $f = p \circ \tilde{f}$, we have $f_* = p_* \circ \tilde{f}_*$.

($\Leftarrow$) Here is the more interesting direction. Suppose that $f_*(\pi_1(Z, z_0)) \subseteq p_*(\pi_1(E, e_0))$. Let $z \in Z$. We wish to define $\tilde{f}(z)$. Pick any path α in Z from z_0 to z. Then $f \circ \alpha$ is a path in B, which therefore lifts uniquely to a path $\tilde{\alpha}$ in E starting at, say e_0. We define $\tilde{f}(z) = \tilde{\alpha}(1)$. It is clear that $\tilde{f}$ is a lift of f.

Why is the lift $\tilde{f}$ well-defined? Suppose β is another path in Z from z_0 to z. Then $f \circ (\alpha \cdot \overline{\beta})$ is a loop in B at $b_0 = f(z_0)$. By assumption, this means that for some loop δ in E, we have

$$p \circ \delta \simeq_p f \circ (\alpha \cdot \overline{\beta}) = f(\alpha) \cdot \overline{f(\beta)}$$

in B. Since path-composition behaves well with respect to path-homotopy, we have a path-homotopy

$$h : (p \circ \delta) \cdot f(\beta) \simeq_p f(\alpha)$$

of paths in B. Note that the path $(p \circ \delta) \cdot f(\beta)$ lifts to the path $\delta \cdot \tilde{\beta}$. The homotopy h then lifts (uniquely) to a path-homotopy in E

$$\tilde{h} : \delta \cdot \tilde{\beta} \simeq_p \tilde{\alpha}.$$

In particular, these have the same endpoints. Of course, the endpoint of $\delta \cdot \tilde{\beta}$ is simply the endpoint of $\tilde{\beta}$. It follows that $\tilde{f}$ is well-defined at z.

Just for emphasis, let's go through the proof that $\tilde{f}$ is continuous. Let $z \in Z$ and let U be an evenly covered neighborhood U of $f(z)$, and let V be the component of $p^{-1}(U)$ containing the lift $\tilde{f}(z)$. Let $W \subseteq Z$ be the path-component of $f^{-1}(U)$ containing z. Since Z is locally path-connected, W is open. Moreover, since W is path-connected and $\tilde{f}(W) \cap V \neq \emptyset$, we must have $\tilde{f}(W) \subseteq V$. Then on the neighborhood W of z, the lift $\tilde{f}$ may be described as the composition $p|_V^{-1} \circ f$. It follows that $\tilde{f}$ is continuous on the neighborhood W of z. Since z was arbitrary, $\tilde{f}$ is continuous. ■

This implies what we already know: S^1 is not a retract of $\mathbb{R}$. More generally, and less trivially, we have that the identity map $S^1 \longrightarrow S^1$ does not lift against the n-fold cover $p_n : S^1 \longrightarrow S^1$. Even more generally, we might ask about lifting some $p_k : S^1 \longrightarrow S^1$ against the cover $p_n : S^1 \longrightarrow S^1$. By the result above, this happens if and only if $k\mathbb{Z} \subseteq n\mathbb{Z}$. In other words, this happens if and only if $n \mid k$.

More interestingly, we have

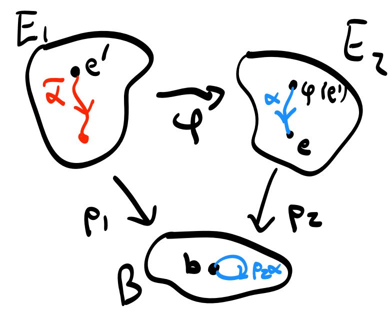
Corollary 14.2. *Suppose that the covering space E is simply-connected. Then a map $f : Z \longrightarrow B$ lifts to some $\tilde{f} : Z \longrightarrow E$ if and only if f induces the trivial map on fundamental groups.*

Corollary 14.3. *Suppose that Z is simply-connected and $p : E \longrightarrow B$ is a covering map. Then any map $f : Z \longrightarrow B$ lifts against p.*

Thus if $X \longrightarrow B$ is a simply connected covering and $E \longrightarrow B$ is any covering, we automatically get a map of covers $X \longrightarrow E$. For this reason, simply connected covers are referred to as **universal covers**.

Proposition 14.4. *Suppose that $\varphi : E_1 \longrightarrow E_2$ is a map of covers. Then φ is a covering map.*

Proof. Let $e \in E_2$. We need to find an evenly-covered neighborhood of e. We know that the point $p_2(e) \in B$ has an evenly covered neighborhood U_2 (with respect to p_2). Let U_1 be an evenly covered neighborhood, with respect to p_1, of $p_2(e)$. Write U for the component of $U_1 \cap U_2$ containing $p_2(e)$. Then $p_2^{-1}(U) \cong \amalg V_i$. Let V_0 be the component containing e. Write $p_1^{-1}(U) \cong \amalg W_j$. Then, since U is connected, each V_i and W_j must be connected. It follows that φ takes each W_j into a single V_i, so that $\varphi^{-1}(V_0) \subseteq p_1^{-1}(U)$ is a disjoint union of some of the W_j's, and it follows that φ restricts to a homeomorphism on each component because both p_1 and p_2 do so.



It only remains to show that $\varphi^{-1}(e)$ is nonempty. Let $b = p_2(e)$, and pick any $e' \in p_1^{-1}(b)$. Since E is very connected, we can find a path $\alpha : \varphi(e') \rightsquigarrow e$ in E. We can push this path α down to a loop $p_2\alpha$ in B and then lift this uniquely to a path $\tilde{\alpha}$ in E_1 starting at e'. Now $\varphi(\tilde{\alpha})$ is a lift of $p_2\alpha$ in E_1 starting at $\varphi(e')$, so by uniqueness of lifts, we must have $\varphi(\tilde{\alpha}) = \alpha$. In particular, $\varphi(\tilde{\alpha}(1)) = e$. ■

15. Wed, Feb. 19

It follows that any universal cover $X \longrightarrow B$ *covers every other covering* $E \longrightarrow B$.

Remark 15.1. Recall that in the proof of Theorem 11.4, we ended up building a map of covers $\varphi : X \longrightarrow X$ corresponding to any point in the fiber F, but we wanted to know it was in fact a homeomorphism. Prop 14.4 now gives us that it is a covering map, so that according to the homework, it suffices to show that the φ we constructed was injective. This can be seen by verifying that it is injective on each fiber.

Our next goal is to completely understand the possible covers of a given space B. There are two avenues of approach. On the one hand, Prop. 13.2 tells us that covering spaces give rise to subgroups of $\pi_1(B)$, so we can try to understand the collection of subgroups. Another approach, which we will look at next, focuses on the fiber $F = p^{-1}(b_0)$.

It will be convenient in what follows to write $G = \pi_1(B, b_0)$ and $F = p^{-1}(b_0) \subset E$. Given a loop γ based at b_0 and a point $f \in F$, we will write $\tilde{\gamma}_f$ for the lift of γ which starts at f.

Theorem 15.2. *Let $p : E \longrightarrow B$ be a covering and let $F = p^{-1}(b)$ be the fiber over the basepoint. Then the function*

$$a : F \times \pi_1(B) \longrightarrow F, \qquad (f, [\gamma]) \mapsto \tilde{\gamma}_f(1)$$

specifies a transitive right action of $\pi_1(B)$ on the fiber F. This is called the **monodromy action**.

Proof. Recall that we have already showed this to be well-defined.

Let c_{b_0} be the constant loop at b_0. Then the constant loop c_f at f in E is a lift of c_{b_0} starting at f, so by uniqueness it must be the only lift. Thus $f \cdot [c_{b_0}] = f$.

Now let α and β be loops at b. We wish to show that $(f \cdot \alpha) \cdot \beta = f \cdot (\alpha \cdot \beta)$. Let $f_2 = \tilde{\alpha}_f(1)$. Then $\tilde{\alpha}_f \cdot \tilde{\beta}_{f_2}$ is a (= the) lift of $\alpha \cdot \beta$ starting at f, so

$$f \cdot (\alpha \cdot \beta) = \tilde{\alpha}_f \cdot \tilde{\beta}_{f_2}(1).$$

On the other hand, $f \cdot \alpha = \tilde{\alpha}_f(1) = f_2$, so

$$(f \cdot \alpha) \cdot \beta = f_2 \cdot \beta = \tilde{\beta}_{f_2}(1)$$

Finally, to see that this action is transitive, let f_1 and f_2 be points in the fiber F. Let γ be a path in E from f_1 to f_2. Then $\alpha = p \circ \gamma$ is a loop at b_0. Furthermore $\tilde{\alpha}_{f_1} = \gamma$, so $f_1 \cdot \alpha = \gamma(1) = f_2$. ■

Note that if we instead wrote path-composition in the "correct" order (i.e. in the same order as function composition), this would give a left action of $\pi_1(B)$ on F.

By the Orbit-Stabilizer theorem, since G acts transitively on F, there is an isomorphism of right G-sets $F \cong G_{e_0}\backslash G$, where $G_{e_0} \leq G$ is the stabilizer of e_0.

Proposition 15.3. *The stabilizer of $e \in F$ under the monodromy action is the subgroup $p_*(\pi_1(E,e)) \leq \pi_1(B,b_0)$.*

Proof. Let $[\gamma] \in \pi_1(E,e)$. Then γ is a lift of $p \circ \gamma$ starting at e, so $e \cdot p_*(\gamma) = \gamma(1) = e$. Thus $p_*(\gamma)$ stabilizes e.

On the other hand, let $[\alpha] \in \pi_1(B,b_0)$ and suppose that $e \cdot [\alpha] = e$. This means that α lifts to a loop $\tilde{\alpha}$ in E. Thus $\alpha = p \circ \tilde{\alpha}$ and $[\alpha] \in p_*(\pi_1(E,e))$. ∎

Corollary 15.4. *Let $p : E \longrightarrow B$ be a covering. Then there is an identification of right $\pi_1(B)$-sets*

$$F \cong p_*(\pi_1(E,e))\backslash\pi_1(B,b).$$

16. Fri, Feb. 21

We have seen that any covering gives rise to a transitive G-set. We would also like to understand maps of coverings.

Definition 16.1. Let X and Y be (right) G-sets. A function $f : X \longrightarrow Y$ is said to be G-**equivariant** (or a map of G-sets) if $f(xg) = f(x) \cdot g$ for all x.

Proposition 16.2. *Let $\varphi : E_1 \longrightarrow E_2$ be a map of covers of B. The induced map on fibers $F_1 \longrightarrow F_2$ is $\pi_1(B)$-equivariant.*

Proof. Let $[\gamma] \in \pi_1(B)$ and $f \in F_1$. We have $f \cdot [\gamma] = \tilde{\gamma}_f(1)$, where $\tilde{\gamma}_f$ is the lift of γ starting at f. Similarly, we have $\varphi(f) \cdot [\gamma] = \tilde{\gamma}_{\varphi(f)}(1)$. But $\varphi(\tilde{\gamma})$ is a lift of γ starting at $\varphi(\gamma(0)) = \varphi(f)$, so $\tilde{\gamma}_{\varphi(f)} = \varphi(\tilde{\gamma}_f)$. Thus

$$\varphi(f) \cdot [\gamma] = \tilde{\gamma}_{\varphi(f)}(1) = \varphi(\tilde{\gamma}_f)(1) = \varphi(\tilde{\gamma}_f(1)) = \varphi(f \cdot [\gamma]).$$

∎

We showed last time that F_1 and F_2 are G-orbits, so we pause to analyze maps between orbits in general.

Proposition 16.3. *Let $H, K \leq G$. Then every G-equivariant map $\varphi : H\backslash G \longrightarrow K\backslash G$ is of the form $Hg \mapsto K\gamma g$ for some $\gamma \in G$ satisfying $\gamma H \gamma^{-1} \leq K$.*

Proof. Since $H\backslash G$ is a transitive G-set, an equivariant map out of it is determined by the value at any point. Suppose we stipulate

$$He \mapsto K\gamma.$$

Then equivariance would force

$$Hg \mapsto K\gamma g.$$

Is this well-defined? Since $Hg = Hhg$ for any $h \in H$, we would need $K\gamma g = K\gamma hg$. Multiplying by $g^{-1}\gamma^{-1}$ gives $K = K\gamma h\gamma^{-1}$. Since $h \in H$ is arbitrary, this says that $\gamma H\gamma^{-1} \leq K$. ∎

Corollary 16.4. *A G-equivariant map $\varphi : H\backslash G \longrightarrow K\backslash G$ exists if and only if H is conjugate in G to a subgroup of K. The two orbits are isomorphic (as right G-sets) if and only if H is conjugate to K.*

Notation. Given covers (E_1, p_1) and (E_2, p_2) of B, we denote by $\mathrm{Map}_B(E_1, E_2)$ the set of covering homomorphisms $\varphi : E_1 \longrightarrow E_2$. Given two right G-sets X and Y, we denote by $\mathrm{Hom}_G(X, Y)$ the set of G-equivariant maps $X \longrightarrow Y$.

The following theorem classifies covering homomorphisms.

Theorem 16.5. *Let E_1 and E_2 be coverings of B. Then Proposition 16.2 induces a bijection*

$$\mathrm{Map}_B(E_1, E_2) \xrightarrow{\cong} \mathrm{Hom}_G(F_1, F_2).$$

Proof. The key is that a covering homomorphism is a lift in the diagram to the right. Uniqueness of lifts gives injectivity in the theorem. For surjectivity, we use the lifting criterion Prop 14.1.Thus suppose given a G-equivariant map $\lambda : F_1 \longrightarrow F_2$ and fix a point $e_1 \in F_1$. Let $e_2 = \lambda(e_1) \in F_2$. The lifting criterion will provide a lift if we can verify that

$$(p_1)_*(\pi_1(E_1, e_1)) \le (p_2)_*(\pi_1(E_2, e_2)).$$

$$\begin{tikzcd} & E_2 \arrow[d, "p_2"] \\ E_1 \arrow[ur, dashed, "\varphi"] \arrow[r, "p_1"'] & B \end{tikzcd}$$

But remember that according to Prop 15.3, these are precisely the stabilizers of e_1 and e_2, respectively. Writing H_1 and H_2 for these groups, the map $\lambda : F_1 \longrightarrow F_2$ corresponds to a map

$$\widehat{\lambda} : H_1 \backslash G \longrightarrow H_2 \backslash.$$

According to Prop 16.3, this means that $\gamma H_1 \gamma^{-1} \le H_2$, where $\widehat{H_1 e} = H_2 \gamma$. The fact that $\lambda(e_1) = e_2$ means that $\gamma = e$. So $H_1 \le H_2$ as desired. ■

We have almost shown that working with covers of B is the same as working with transitive right G-sets (technically, we are heading to an "equivalence of categories"). All that is left is to show that for every G-orbit F, there is a cover $p : E \longrightarrow B$ whose fiber is F as a G-set.