The interesting, new result here concerns the existence of lifts.

**Proposition 14.1.** *(Lifting Criterion)* Let \( p : E \to B \) be a covering and let \( f : Z \to B \), with \( Z \) very connected. Given points \( z_0 \in Z \) and \( e_0 \in E \) with \( f(z_0) = p(e_0) \), there is a lift \( \tilde{f} \) with \( \tilde{f}(z_0) = e_0 \) if and only if \( f_* (\pi_1(Z, z_0)) \subseteq p_* (\pi_1(E, e_0)) \).

*Proof.* (\( \Rightarrow \)) This is clear. Since \( f = p \circ \tilde{f} \), we have \( f_* = p_* \circ \tilde{f}_* \).

(\( \Leftarrow \)) Here is the more interesting direction. Suppose that \( f_* (\pi_1(Z, z_0)) \subseteq p_* (\pi_1(E, e_0)) \). Let \( z \in Z \). We wish to define \( \tilde{f}(z) \). Pick any path \( \alpha \) in \( Z \) from \( z_0 \) to \( z \). Then \( f \circ \alpha \) is a path in \( B \), which therefore lifts uniquely to a path \( \tilde{\alpha} \) in \( E \) starting at, say \( e_0 \). We define \( \tilde{f}(z) = \tilde{\alpha}(1) \). It is clear that \( \tilde{f} \) is a lift of \( f \).

Why is the lift \( \tilde{f} \) well-defined? Suppose \( \beta \) is another path in \( Z \) from \( z_0 \) to \( z \). Then \( f \circ (\alpha \cdot \beta) \) is a loop in \( B \) at \( b_0 = f(z_0) \). By assumption, this means that for some loop \( \delta \) in \( E \), we have

\[
p \circ \delta \simeq_p f \circ (\alpha \cdot \beta) = f(\alpha) \cdot f(\beta)
\]

in \( B \). Since path-composition behaves well with respect to path-homotopy, we have a path-homotopy

\[
h : (p \circ \delta) \cdot f(\beta) \simeq_p f(\alpha)
\]

of paths in \( B \). Note that the path \( (p \circ \delta) \cdot f(\beta) \) lifts to the path \( \delta \cdot \tilde{\beta} \). The homotopy \( h \) then lifts (uniquely) to a path-homotopy in \( E \)

\[
\tilde{h} : \delta \cdot \tilde{\beta} \simeq_p \tilde{\alpha}.
\]

In particular, these have the same endpoints. Of course, the endpoint of \( \delta \cdot \tilde{\beta} \) is simply the endpoint of \( \beta \). It follows that \( \tilde{f} \) is well-defined at \( z \).

Just for emphasis, let’s go through the proof that \( \tilde{f} \) is continuous. Let \( z \in Z \) and let \( U \) be an evenly covered neighborhood \( U \) of \( f(z) \), and let \( V \) be the component of \( p^{-1}(U) \) containing the lift \( \tilde{f}(z) \). Let \( W \subseteq Z \) be the path-component of \( f^{-1}(U) \) containing \( z \). Since \( Z \) is locally path-connected, \( W \) is open. Moreover, since \( W \) is path-connected and \( \tilde{f}(W) \cap V \neq \emptyset \), we must have \( \tilde{f}(W) \subseteq V \). Then on the neighborhood \( W \) of \( z \), the lift \( \tilde{f} \) may be described as the composition \( p|_V^{-1} \circ f \). It follows that \( \tilde{f} \) is continuous on the neighborhood \( W \) of \( z \). Since \( z \) was arbitrary, \( \tilde{f} \) is continuous.

This implies what we already know: \( S^1 \) is not a retract of \( \mathbb{R} \). More generally, and less trivially, we have that the identity map \( S^1 \to S^1 \) does not lift against the \( n \)-fold cover \( p_n : S^1 \to S^1 \). Even more generally, we might ask about lifting some \( p_k : S^1 \to S^1 \) against the cover \( p_n : S^1 \to S^1 \).

By the result above, this happens if and only if \( k \mathbb{Z} \subseteq n \mathbb{Z} \). In other words, this happens if and only if \( n \mid k \).

More interestingly, we have

**Corollary 14.2.** Suppose that the covering space \( E \) is simply-connected. Then a map \( f : Z \to B \) lifts to some \( \tilde{f} : Z \to E \) if and only if \( f \) induces the trivial map on fundamental groups.

**Corollary 14.3.** Suppose that \( Z \) is simply-connected and \( p : E \to B \) is a covering map. Then any map \( f : Z \to B \) lifts against \( p \).

Thus if \( X \to B \) is a simply connected covering and \( E \to B \) is any covering, we automatically get a map of covers \( X \to E \). For this reason, simply connected covers are referred to as universal covers.

**Proposition 14.4.** Suppose that \( \varphi : E_1 \to E_2 \) is a map of covers. Then \( \varphi \) is a covering map.
Proof. Let \( e \in E_2 \). We need to find an evenly-covered neighborhood of \( e \). We know that the point \( p_2(e) \in B \) has an evenly covered neighborhood \( U_2 \) (with respect to \( p_2 \)). Let \( U_1 \) be an evenly covered neighborhood, with respect to \( p_1 \), of \( p_2(e) \). Write \( U \) for the component of \( U_1 \cap U_2 \) containing \( p_2(e) \). Then \( p_2^{-1}(U) \cong \Pi V_i \). Let \( V_0 \) be the component containing \( e \). Write \( p_1^{-1}(U) \cong \Pi W_j \). Then, since \( U \) is connected, each \( V_i \) and \( W_j \) must be connected. It follows that \( \varphi \) takes each \( W_j \) into a single \( V_i \), so that \( \varphi^{-1}(V_0) \subseteq p_1^{-1}(U) \) is a disjoint union of some of the \( W_j \)'s, and it follows that \( \varphi \) restricts to a homeomorphism on each component because both \( p_1 \) and \( p_2 \) do so.

It only remains to show that \( \varphi^{-1}(e) \) is nonempty. Let \( b = p_2(e) \), and pick any \( e' \in p_1^{-1}(b) \). Since \( E \) is very connected, we can find a path \( \alpha : \varphi(e') \sim e \) in \( E \). We can push this path \( \alpha \) down to a loop \( p_2 \alpha \) in \( B \) and then lift this uniquely to a path \( \tilde{\alpha} \) in \( E_1 \) starting at \( e' \). Now \( \varphi(\tilde{\alpha}) \) is a lift of \( p_2 \alpha \) in \( E_1 \) starting at \( \varphi(e') \), so by uniqueness of lifts, we must have \( \varphi(\tilde{\alpha}) = \alpha \). In particular, \( \varphi(\tilde{\alpha}(1)) = e \).

\[ \blacksquare \]

15. Wed, Feb. 19

It follows that any universal cover \( X \to B \) covers every other covering \( E \to B \).

Remark 15.1. Recall that in the proof of Theorem 11.4, we ended up building a map of covers \( \varphi : X \to X \) corresponding to any point in the fiber \( F \), but we wanted to know it was in fact a homeomorphism. Prop 14.4 now gives us that it is a covering map, so that according to the homework, it suffices to show that the \( \varphi \) we constructed was injective. This can be seen by verifying that it is injective on each fiber.

Our next goal is to completely understand the possible covers of a given space \( B \). There are two avenues of approach. On the one hand, Prop. 13.2 tells us that covering spaces give rise to subgroups of \( \pi_1(B) \), so we can try to understand the collection of subgroups. Another approach, which we will look at next, focuses on the fiber \( F = p_1^{-1}(b_0) \).

It will be convenient in what follows to write \( G = \pi_1(B, b_0) \) and \( F = p_1^{-1}(b_0) \subset E \). Given a loop \( \gamma \) based at \( b_0 \) and a point \( f \in F \), we will write \( \tilde{\gamma}_f \) for the lift of \( \gamma \) which starts at \( f \).

Theorem 15.2. Let \( p : E \to B \) be a covering and let \( F = p_1^{-1}(b) \) be the fiber over the basepoint. Then the function

\[ a : F \times \pi_1(B) \to F, \quad (f, [\gamma]) \mapsto \tilde{\gamma}_f(1) \]

specifies a transitive right action of \( \pi_1(B) \) on the fiber \( F \). This is called the monodromy action.

Proof. Recall that we have already shown this to be well-defined.

Let \( c_0 \) be the constant loop at \( b_0 \). Then the constant loop \( c_f \) at \( f \in E \) is a lift of \( c_0 \) starting at \( f \), so by uniqueness it must be the only lift. Thus \( f \cdot [c_0] = f \).

Now let \( \alpha \) and \( \beta \) be loops at \( b \). We wish to show that \( (f \cdot \alpha) \cdot \beta = f \cdot (\alpha \cdot \beta) \). Let \( f_2 \) be \( \tilde{\alpha}_f(1) \). Then \( \tilde{\alpha}_f \cdot \tilde{\beta}_{f_2} \) is a (= the) lift of \( \alpha \cdot \beta \) starting at \( f \), so

\[ f \cdot (\alpha \cdot \beta) = \tilde{\alpha}_f \cdot \tilde{\beta}_{f_2}(1). \]

On the other hand, \( f \cdot \alpha = \tilde{\alpha}_f(1) = f_2 \), so

\[ (f \cdot \alpha) \cdot \beta = f_2 \cdot \beta = \tilde{\beta}_{f_2}(1) \]

Finally, to see that this action is transitive, let \( f_1 \) and \( f_2 \) be points in the fiber \( F \). Let \( \gamma \) be a path in \( E \) from \( f_1 \) to \( f_2 \). Then \( \alpha = p \circ \gamma \) is a loop at \( b_0 \). Furthermore \( \tilde{\alpha}_{f_1} = \gamma \), so \( f_1 \cdot \alpha = \gamma(1) = f_2 \). \[ \blacksquare \]
Note that if we instead wrote path-composition in the “correct” order (i.e. in the same order as function composition), this would give a left action of $\pi_1(B)$ on $F$.

By the Orbit-Stabilizer theorem, since $G$ acts transitively on $F$, there is an isomorphism of right $G$-sets $F \cong G_{e_0} \backslash G$, where $G_{e_0} \leq G$ is the stabilizer of $e_0$.

**Proposition 15.3.** The stabilizer of $e \in F$ under the monodromy action is the subgroup $p_*(\pi_1(E, e)) \leq \pi_1(B, b_0)$.

*Proof.* Let $[\gamma] \in \pi_1(E, e)$. Then $\gamma$ is a lift of $p \circ \gamma$ starting at $e$, so $e \cdot p_*(\gamma) = \gamma(1) = e$. Thus $p_*(\gamma)$ stabilizes $e$.

On the other hand, let $[\alpha] \in \pi_1(B, b_0)$ and suppose that $e \cdot [\alpha] = e$. This means that $\alpha$ lifts to a loop $\tilde{\alpha}$ in $E$. Thus $\alpha = p \circ \tilde{\alpha}$ and $[\alpha] \in p_*(\pi_1(E, e))$. □

**Corollary 15.4.** Let $p : E \to B$ be a covering. Then there is an identification of right $\pi_1(B)$-sets

$$F \cong p_*(\pi_1(E, e)) \backslash \pi_1(B, b).$$

16. Fri, Feb. 21

We have seen that any covering gives rise to a transitive $G$-set. We would also like to understand maps of coverings.

**Definition 16.1.** Let $X$ and $Y$ be (right) $G$-sets. A function $f : X \to Y$ is said to be $G$-equivariant (or a map of $G$-sets) if $f(xg) = f(x) \cdot g$ for all $x$.

**Proposition 16.2.** Let $\varphi : E_1 \to E_2$ be a map of covers of $B$. The induced map on fibers $F_1 \to F_2$ is $\pi_1(B)$-equivariant.

*Proof.* Let $[\gamma] \in \pi_1(B)$ and $f \in F_1$. We have $f \cdot [\gamma] = \tilde{\gamma}_f(1)$, where $\tilde{\gamma}_f$ is the lift of $\gamma$ starting at $f$. Similarly, we have $\varphi(f) \cdot [\gamma] = \tilde{\varphi}(\tilde{\gamma}_f)(1)$. But $\varphi(\tilde{\gamma})$ is a lift of $\gamma$ starting at $\varphi(\gamma(0)) = \varphi(f)$, so $\tilde{\varphi}(\tilde{\gamma}_f) = \varphi(\tilde{\gamma}_f)$. Thus

$$\varphi(f) \cdot [\gamma] = \tilde{\varphi}(\tilde{\gamma}_f)(1) = \varphi(\tilde{\varphi}(\tilde{\gamma}_f))(1) = \varphi(\tilde{\varphi}(\tilde{\gamma}_f)) = \varphi(f \cdot [\gamma]).$$

□

We showed last time that $F_1$ and $F_2$ are $G$-orbits, so we pause to analyze maps between orbits in general.

**Proposition 16.3.** Let $H, K \leq G$. Then every $G$-equivariant map $\varphi : H \backslash G \to K \backslash G$ is of the form $Hg \mapsto K\gamma g$ for some $\gamma \in G$ satisfying $\gamma H \gamma^{-1} \leq K$.

*Proof.* Since $H \backslash G$ is a transitive $G$-set, an equivariant map out of it is determined by the value at any point. Suppose we stipulate

$$He \mapsto K\gamma.$$ 

Then equivariance would force

$$Hg \mapsto K\gamma g.$$ 

Is this well-defined? Since $Hg = Hhg$ for any $h \in H$, we would need $K\gamma g = K\gamma hg$. Multiplying by $g^{-1}\gamma^{-1}$ gives $K = K\gamma h \gamma^{-1}$. Since $h \in H$ is arbitrary, this says that $\gamma H \gamma^{-1} \leq K$. □

**Corollary 16.4.** A $G$-equivariant map $\varphi : H \backslash G \to K \backslash G$ exists if and only if $H$ is conjugate in $G$ to a subgroup of $K$. The two orbits are isomorphic (as right $G$-sets) if and only if $H$ is conjugate to $K$.

**Notation.** Given covers $(E_1, p_1)$ and $(E_2, p_2)$ of $B$, we denote by $\text{Map}_B(E_1, E_2)$ the set of covering homomorphisms $\varphi : E_1 \to E_2$. Given two right $G$-sets $X$ and $Y$, we denote by $\text{Hom}_G(X, Y)$ the set of $G$-equivariant maps $X \to Y$.

The following theorem classifies covering homomorphisms.
Theorem 16.5. Let $E_1$ and $E_2$ be coverings of $B$. Then Proposition 16.2 induces a bijection

$$\text{Map}_B(E_1, E_2) \cong \text{Hom}_G(F_1, F_2).$$

Proof. The key is that a covering homomorphism is a lift in the diagram to the right. Uniqueness of lifts gives injectivity in the theorem. For surjectivity, we use the lifting criterion Prop 14.1. Thus suppose given a $G$-equivariant map $\lambda: F_1 \to F_2$ and fix a point $e_1 \in F_1$. Let $e_2 = \lambda(e_1) \in F_2$. The lifting criterion will provide a lift if we can verify that

$$(p_1)_*(\pi_1(E_1, e_1)) \leq (p_2)_*(\pi_1(E_2, e_2)).$$

But remember that according to Prop 15.3, these are precisely the stabilizers of $e_1$ and $e_2$, respectively. Writing $H_1$ and $H_2$ for these groups, the map $\lambda: F_1 \to F_2$ corresponds to a map $\tilde{\lambda}: H_1 \\cap \ G \to H_2 \\cap \ G$.

According to Prop 16.3, this means that $\gamma H_1 \gamma^{-1} \leq H_2$, where $H_1 e = H_2 e$. The fact that $\lambda(e_1) = e_2$ means that $\gamma = e$. So $H_1 \leq H_2$ as desired. 

We have almost shown that working with covers of $B$ is the same as working with transitive right $G$-sets (technically, we are heading to an “equivalence of categories”). All that is left is to show that for every $G$-orbit $F$, there is a cover $p: E \to B$ whose fiber is $F$ as a $G$-set.