Math 751 - Topics in Topology
Homework 3
Spring 2015

1. A **section** of a bundle $p : E \longrightarrow B$ is a map $s : B \longrightarrow E$ such that $ps = \mathrm{id}_B$. Show that a principal bundle is isomorphic to a trivial bundle if and only if there exists a section of the bundle.

2. For any n, the transitive $O(n)$-action on S^{n-1} defines a principal $O(n-1)$-bundle

$$p : O(n) \longrightarrow S^{n-1}$$
$$A \mapsto A\mathbf{e}_1,$$

   where $\mathbf{e}_1 \in \mathbb{R}^n$ is the first standard basis vector $\mathbf{e}_1 = (1, 0 \dots, 0)$. Show that this bundle has a section when $n = 2, 4$ but not when $n = 3$. (In fact, the only other value for which there is a section is $n = 8$.)

   Hint: for the $n = 4$ case, it may help to think of S^3 as the unit quaternions. For the $n = 3$ case, use the long exact sequence in homotopy of a bundle, together with the facts that (*i*) $O(n)$ has two homeomorphic components (matrices of determinant 1 or -1) and (*ii*) $SO(3)$ is homeomorphic to $\mathbb{RP}^3$.

3. ($\star$) Let SX denote the (unreduced) suspension of X, namely the quotient $SX = (X \times I)/\sim$, where $(x, 0) \sim (y, 0)$ and $(x, 1) \sim (y, 1)$ for any two points $x, y \in X$. Then

$$SX = C_+X \cup C_-X$$

   is the union of two cones on X, intersecting at $X \times \{\frac{1}{2}\} \cong X$. Since each cone is contractible, any bundle $p : E \longrightarrow SX$ restricts to a trivial bundle on each cone. Thus any bundle is completely determined by the transition function $\varphi_{\frac{1}{2}} : X_{\frac{1}{2}} \longrightarrow G$, which is known as a **clutching function** for the bundle. (Really, we should replace the cones by open cones each containing $X_{\frac{1}{2}}$.)

   Show that the above discussion leads to a bijection

$$[X, G] \cong \mathrm{Prin}_G(SX)$$

   assuming that G is connected.

4. (a) What is the clutching function for the Möbius bundle?
   (b) What is the clutching function for the Hopf bundle $S^3 \longrightarrow S^2$?
   (c) ($\star$) What is the clutching function for the bundle from Problem 2, for $n \leq 4$?