Math 751 - Topics in Topology
Homework 4
Spring 2015

1. (a) (Replacing a map by a fibration) Let $f : X \longrightarrow Y$ be a map. Let $P(Y)$ be the path space on Y. This is the space of paths in Y, topologized as $P(Y) = \mathrm{Map}(I, Y)$. Now define the mapping path space $P(f)$ by the pullback

$$\begin{tikzcd} P(f) \arrow[r] \arrow[d] & P(Y) \arrow[d, "ev_1"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

   Show that we have a commutative triangle

$$\begin{tikzcd} X \arrow[rr, "\alpha"] \arrow[dr, "f"'] & & P(f) \arrow[dl, "\beta_f"] \\ & Y & \end{tikzcd}$$

   in which α is a homotopy equivalence and β_f is a fibration.

   (b) What do you get if you replace $* \longrightarrow Y$ by a fibration?

2. Suppose that $f : X \longrightarrow Y$ is a fibration and Y is path-connected. Show that if y and y' are points of Y, then $f^{-1}(y) \simeq f^{-1}(y')$. This means that it still makes sense to talk about "the" fiber of a fibration that is not necessarily a bundle.

3. Let $p : E \longrightarrow B$ be a fibration and let $\iota : A \longrightarrow B$ be the inclusion of a subspace.

   (a) Show that the restriction $p : p^{-1}(A) \longrightarrow A$ is also a fibration.

   (b) Show that if $\iota : A \longrightarrow B$ induces an isomorphism in homotopy groups π_n for $n < k$ and a surjection on π_k, then the same is true of the inclusion $j : p^{-1}(A) \longrightarrow E$. (Such a map is called an n-equivalence.)