Math 751 - Topics in Topology  
Homework 4  
Spring 2015

1. (a) (Replacing a map by a fibration) Let \( f : X \longrightarrow Y \) be a map. Let \( P(Y) \) be the path space on \( Y \). This is the space of paths in \( Y \), topologized as \( P(Y) = \text{Map}(I, Y) \). Now define the mapping path space \( P(f) \) by the pullback

\[
\begin{array}{ccc}
P(f) & \longrightarrow & P(Y) \\
\downarrow & & \downarrow \text{ev}_1 \\
X & \xrightarrow{f} & Y.
\end{array}
\]

Show that we have a commutative triangle

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & P(f) \\
\downarrow f & & \downarrow \beta_f \\
Y & & 
\end{array}
\]

in which \( \alpha \) is a homotopy equivalence and \( \beta_f \) is a fibration.

(b) What do you get if you replace \( * \longrightarrow Y \) by a fibration?

2. Suppose that \( f : X \longrightarrow Y \) is a fibration and \( Y \) is path-connected. Show that if \( y \) and \( y' \) are points of \( Y \), then \( f^{-1}(y) \simeq f^{-1}(y') \). This means that it still makes sense to talk about “the” fiber of a fibration that is not necessarily a bundle.

3. Let \( p : E \longrightarrow B \) be a fibration and let \( i : A \longrightarrow B \) be the inclusion of a subspace.

   (a) Show that the restriction \( p : p^{-1}(A) \longrightarrow A \) is also a fibration.

   (b) Show that if \( i : A \longrightarrow B \) induces an isomorphism in homotopy groups \( \pi_n \) for \( n < k \) and a surjection on \( \pi_k \), then the same is true of the inclusion \( j : p^{-1}(A) \longrightarrow E \). (Such a map is called an \( n \)-equivalence.)