1. Let $F \to E \overset{p}{\to} B$ be a fibration. Suppose that $B$ is path-connected and that $\pi_1(B)$ acts trivially on $H_*(F)$. Let $F$ be a field, and assume that $H_*(B; F)$ and $H_*(F; F)$ are both finite-dimensional. Define the $F$-Euler characteristic of a space $X$ by the formula

$$
\chi_F(X) := \sum_i (-1)^i \dim_F H_i(X; F).
$$

Show that $H_*(E; F)$ is finite-dimensional and that

$$
\chi_F(E) = \chi_F(B) \cdot \chi_F(F).
$$

2. Let $n \geq 1$.

(a) We have, as discussed previously, an action of $S^1$ on $S^{2n+1}$. The subgroup inclusion $\mathbb{Z}/2\mathbb{Z} \cong S^0 \subseteq S^1$ gives a fiber sequence

$$
S^1/\mathbb{Z}/2\mathbb{Z} \to S^{2n+1}/\mathbb{Z}/2\mathbb{Z} \to S^{2n+1}/S^1,
$$

which we can rewrite as a fiber sequence

$$
S^1 \to \mathbb{RP}^{2n+1} \to \mathbb{CP}^n.
$$

Use the Serre spectral sequence to determine the cohomology ring $H^*(\mathbb{RP}^{2n+1}; \mathbb{Z})$.

(b) Use part (a) and the inclusion $\mathbb{RP}^{2n} \hookrightarrow \mathbb{RP}^{2n+1}$ to determine the cohomology ring $H^*(\mathbb{RP}^{2n}; \mathbb{Z})$.

(c) Use parts (a) and (b) to determine $H^*(\mathbb{RP}^\infty; \mathbb{Z})$.

3. (a) Use the path-loop fibration $\Omega(S^n) \to PS^n \to S^n$ to determine the cohomology ring $H^*(\Omega S^n; Q)$. (Hint: Your answer should depend on the parity of $n$).

(b) $(\star)$ Determine the integral cohomology ring $H^*(\Omega S^n; \mathbb{Z})$.

4. For any $n$, denote by $\lambda : \mathbb{R}^n \to \mathbb{R}^{n+1}$ the map $\lambda(x_1, \ldots, x_n) = (0, x_1, \ldots, x_n)$. For any $k \leq n$, define a map $\Lambda : V_k(\mathbb{R}^n) \to V_{k+1}(\mathbb{R}^{n+1})$ by

$$(v_1, \ldots, v_k) \mapsto (e_1, \lambda(v_1), \ldots, \lambda(v_k)).$$
Also denote by $p_k : V_{k+1}(\mathbb{R}^n) \longrightarrow V_k(\mathbb{R}^n)$ the fibration

$$p_k(\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}) = (\mathbf{v}_1, \ldots, \mathbf{v}_k).$$

The maps $\Lambda$ assemble together to yields maps of fiber sequences

$$\begin{CD}
S^{n-k} @>>> V_j(\mathbb{R}^{n-k+j}) @>{p_j}>> V_{j-1}(\mathbb{R}^{n-k+j}) \\
@. @VV\Lambda V @. @VV\Lambda V \\
S^{n-k} @>>> V_{j+1}(\mathbb{R}^{n-k+j+1}) @>{p_{j+1}}>> V_j(\mathbb{R}^{n-k+j+1})
\end{CD}$$

Use induction on $j$ (with base case $j = 2$) to show that $H^*(V_k(\mathbb{R}^n); F_2)$ has a simple system of generators $\{x_{n-k}, \ldots, x_{n-1}\}$, where $\text{deg}(x_i) = i$. 