CLASS NOTES
MATH 751 (SPRING 2015)

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1. Wed, Jan. 14

Standing assumption: All spaces will be CW complexes, or at least of the homotopy type of a CW complex. For any based space (X, x_0), the basepoint will always be assumed to be a 0-cell for some CW structure.

Some basic questions

(1) When does a space decompose as a product $X \simeq Y \times Z$?
(2) How is this useful?
(3) We will also look at generalized products: bundles and fibrations

We want to understand spaces X up to homotopy. We use algebraic data to do this. The first approximation is the collection $\pi_n(X)$ of homotopy groups of X. In practice, this is too hard!

Example 1.1. For $X = S^2$, we know

$$\pi_1(S^2) = 0, \quad \pi_2(S^2) \cong \mathbb{Z}, \quad \pi_3(S^2) \cong \mathbb{Z} \text{ (Prop 12.2)}, \quad \pi_4(S^2) \cong \mathbb{Z}/2\mathbb{Z} \text{(Remark 39.1)}.$$

But these homotopy groups $\pi_n(S^2)$ are only known up to $n = 64$, although it is known that infinitely many are nonzero.

Similarly, any simpy-connected finite complex that is not contractible must have infinitely many nontrivial homotopy groups (Theorem 43.2).

Homology (or cohomology) groups are, in general, much more computable.

Example 1.2. We know that $\mathrm{H}_0(S^2) \cong \mathbb{Z}$ and $\mathrm{H}_2(S^2) \cong \mathbb{Z}$ are the only nonzero homology groups of S^2.

In some cases, homology informs us about homotopy:

1.1. Preliminaries: The Hurewicz and Whitehead Theorems.

Let $\alpha \in \pi_n(X)$ be represented by a map $a : S^n \longrightarrow X$. Denote by $\iota_n \in \mathrm{H}_n(S^n)$ "the" fundamental class. The **Hurewicz homomorphism**

$$h : \pi_n(X) \longrightarrow \mathrm{H}_n(X)$$

is defined by

$$h(\alpha) = a_*(\iota_n).$$

Date: May 1, 2015.

Theorem 1.3 (Hurewicz)**.** , [H, Theorem 4.32] *Let $n \geq 2$ and suppose that $\pi_k(X) = 0$ for $k < n$ (we say that X is $(n-1)$*-**connected***). Then the Hurewicz homomorphism*

$$h : \pi_n(X) \longrightarrow \mathrm{H}_n(X)$$

is an isomorphism.

Remark 1.4. Let $n \geq 2$. Given the homology computation for S^n and the fact that S^n is simply-connected, it immediately follows that $\pi_k(S^n) = 0$ for $k < n$ and $\pi_n(S^n) \cong \mathbb{Z}$.

Theorem 1.5 (Whitehead)**.** , [H, Corollary 4.33] *If $f : X \longrightarrow Y$ is a homology-isomorphism and both X and Y are simply-connected, then f is a homotopy equivalence.*

You have already seen the version of the Hurewicz theorem in the non-simply-connected case. There is also a more general version of the Whitehead theorem in the presence of fundamental groups. We may discuss this later in the course.

There is another Whitehead theorem. Recall that $f : X \longrightarrow Y$ is called a **weak homotopy equivalence** if $f_* : \pi_n(X) \longrightarrow \pi_n(Y)$ is an isomorphism for all n and all choices of basepoint in X.

Theorem 1.6 (Whitehead II)**.** , [H, Theorem 4.5] *If $f : X \longrightarrow Y$ is a weak homotopy equivalence and both X and Y are CW, then f is a homotopy equivalence.*

Since we are *always* assuming our spaces are CW in this course, the theorem tells us there is no difference between homotopy equivalences and weak homotopy equivalences. The first Whitehead theorem is a computational tool, whereas this second version justifies the restriction to weak homotopy equivalences and the focus on homotopy groups.

One of the central questions of the course will be:

Question 1.7. *Suppose given some decomposition of X. What does this tell us about $\pi_*(X)$ or $\mathrm{H}_*(X)$ or $\mathrm{H}^*(X)$?*

In many cases, this question can be turned on its head. That is, if we can show that the homotopy (or homology or cohomology) groups do not decompose, then there cannot be a space-level decomposition.

The simplest example of a decomposition we have in mind is $X \simeq Y \times Z$. Then we have

Homotopy of a product: $\pi_n(X) \cong \pi_n(Y) \oplus \pi_n(Z)$.

Homology of a product: (Künneth formula)

$$\mathrm{H}_n(X) \cong \bigoplus_{i+j=n} \mathrm{H}_i(Y) \otimes \mathrm{H}_j(Z) \quad \oplus \bigoplus_{k+\ell=n-1} \mathrm{Tor}(\mathrm{H}_k(Y), \mathrm{H}_\ell(Z)).$$

So homology, while typically much more computable for a particular space, has a more complicated product formula. We will see this again in the more general case of a fibration.

There is a similar Künneth formula in cohomology, but it requires a few extra hypotheses (for example, "finite type").

Another type of decomposition is a wedge decomposition $X \simeq Y \vee Z$. In this case, we get

Homotopy of a wedge: Assume that Y is p-connected and that Z is q-connected, where p and q are ≥ 1. Then $\pi_n(X) \cong \pi_n(Y) \oplus \pi_n(Z)$ for $n < p + q$.

This can be shown using the homotopy-excision theorem (see Prop 2.2 in my old course notes). But there is not much control in higher dimensions. For example, $\pi_2(S^1 \vee S^2)$ is not finitely-generated. The sum $S^2 \vee S^2$ has infinite homotopy groups in dimensions 4, 5, and 6 (see [BT], p. 264), whereas the only infinite homotopy groups of S^2 occur in dimensions 2 and 3 (Theorem 42.1).

Homology of a wedge: $\tilde{H}_n(X) \cong \tilde{H}_n(Y) \oplus \tilde{H}_n(Z)$.

Example 1.8. The nontrivial homology groups of $\mathbb{CP}^2$ are $H_n(\mathbb{CP}^2) \cong \mathbb{Z}$ for $n = 0, 2, 4$. These are the same as the homology groups of $S^2 \vee S^4$, so we can ask whether the two spaces are equivalent. The above homotopy formula tells us that

$$\pi_3(S^2 \vee S^4) \cong \pi_3(S^2) \oplus \pi_3(S^4) \cong \mathbb{Z} \oplus 0 \cong \mathbb{Z}.$$

We will see in Prop 12.2 that $\pi_3(\mathbb{CP}^2) = 0$, so $\mathbb{CP}^2 \not\simeq S^2 \vee S^4$

As indicated above, we will primarily be interested in product-decompositions (and generalizations).

Since homotopy interacts well with products, a natural question is: given the homotopy groups $\pi_n(X)$ of a space, is that enough information to recover the space? We will see that the answer is no, but to relate this to products, we make the following definition.

Definition 1.9. Given a group G and $n \geq 0$ an **Eilenberg-Mac Lane space** of type $K(G, n)$ is a CW complex satisfying

$$\pi_k(K(G, n)) \cong \begin{cases} G & k = n \\ 0 & \text{else.} \end{cases}$$

For any space X, let us write $G_n = \pi_n(X)$. Then a more precise version of the above question is

Question 1.10. *Is there a homotopy equivalence* $X \simeq \prod_n K(G_n, n)$*?*

Again, we will see later (in Lecture 39) that the answer to this question is no.

2. Fri, Jan. 16

It may initially seem surprising that spaces of the type $K(G, n)$ exist.

Example 2.1. One simple example is that S^1 is a $K(\mathbb{Z}, 1)$. To see that the higher homotopy groups of S^1 are all trivial, recall that a covering map $E \longrightarrow B$ always induces an isomorphism on higher homotopy groups. Since the universal cover of S^1 is the contractible space $\mathbb{R}$, we conclude that a cover of S^1 has no higher homotopy groups and so the same must be true of S^1 itself.

Example 2.2. The infinite real projective space $\mathbb{RP}^\infty$ is a $K(\mathbb{Z}/2\mathbb{Z}, 1)$. Again, this can be seen using covering space theory. The universal cover of $\mathbb{RP}^\infty$ is the infinite sphere S^∞, which is again contractible.

Example 2.3. The Eilenberg-Mac Lane spaces of type $K(\mathbb{Z}/n\mathbb{Z}, 1)$ are given by infinite Lens spaces. Again, these have S^∞ as universal cover.

Example 2.4. The infinite complex projective space $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$. We will show this later in Proposition 12.1. For the moment, we can compute the homotopy groups up to level 2: recall that $\mathbb{CP}^1 = S^2$, so we know that $\pi_1(\mathbb{CP}^1) = 0$ and $\pi_2(\mathbb{CP}^1) \cong \mathbb{Z}$. Now $\mathbb{CP}^2$ can be obtained from $\mathbb{CP}^1$ be attaching a single 4-cell. This does not affect the homotopy groups below dimension 3. Similarly $\mathbb{CP}^\infty$ is obtained from $\mathbb{CP}^1$ by attaching cells in dimensions 4 and higher, so π_1 and π_2 are not affected.

Construction 2.5. One way to see that $K(G, n)$'s always exist (assuming that G is abelian if $n \geq 2$) is to build them via attaching cells as follows. First, express G as a quotient $G \cong F/H$ as a quotient of a free group. We start building a CW complex by taking a single 0-cell and then adding in one n-cell for each generator of the free group F. So far, we have no homotopy below level n, and we first need to fix the homotopy group π_n. For each generator of the subgroup $H \leq F$, attach an $(n+1)$-cell to kill off that element in π_n. In this way, we will force π_n to be correct, but we now have no control over the higher homotopy groups. But we don't want to have any higher homotopy groups. In order to get rid of π_{n+1}, we attach $(n+2)$-cells to wipe out all of π_{n+1}. For example, we could attach one $(n+2)$-cell for each generator of π_{n+1}. As we attach these cells, we do not change the homotopy groups in dimensions n and lower. Now we keep attaching cells in higher and higher dimensions to wipe out the homotopy groups, and this produces a CW complex in the end with the desired homotopy groups.

Unfortunately, the previous construction involves making a lot of choices and is not very convenient for some purposes. For example, suppose we have a homomorphism $\varphi : A \longrightarrow B$. We might then expect to get a (well-behaved) continuous map $K(A, n) \longrightarrow K(B, n)$ inducing φ on the nth homotopy groups. In other words, we might ask that the assignment $A \rightsquigarrow K(A, n)$ is *functorial*. Another point is that $S^1 = K(\mathbb{Z}, 1)$ is not just a space. It is a topological abelian group.

Remark 2.6. The projective spaces $\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$ can similarly be given abelian monoid structures. The standard models don't have inverse maps, but there are ways in general of replacing a topological (connected) monoid by a homotopy-equivalent topological group. We work with $\mathbb{RP}^\infty$, and the story for $\mathbb{CP}^\infty$ is completely analogous. Recall that one model for $\mathbb{RP}^n$ is as $\big(\mathbb{R}^{n+1} - \{0\}\big)/\mathbb{R}^\times$. We can think of an element of $\mathbb{R}^{n+1} - \{0\}$ as a nonzero polynomial of degree n. Multiplication of polynomials gives a map

$$\mathbb{R}^{n+1} - \{0\} \times \mathbb{R}^{k+1} - \{0\} \longrightarrow \mathbb{R}^{nk+1} - \{0\}.$$

According to this model, the space $\mathbb{RP}^n$ consists of the space of nonzero degree n polynomials up to scalar multiplication (meaning we pass to equivalence classes under scalar multiplication). The multiplication is compatible with scalar multiplication, so we get an induced map

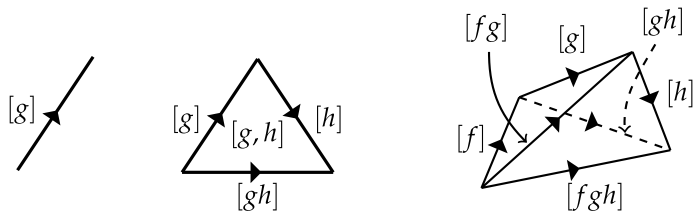
$$\mathbb{RP}^n \times \mathbb{RP}^k \longrightarrow \mathbb{RP}^{n+k}.$$

Passing to the limit gives the multiplication on $\mathbb{RP}^\infty$. It is clear from the construction that this is associative, and the unit is the class of the nonzero constant polynomial.

Here are two constructions of $K(G,n)$'s with some better properties.

Construction 2.7 (The simplicial construction)**.** Given a group G, we will build a Δ-complex BG which will be a $K(G,1)$. The complex BG has a single 0-simplex. We add in one 1-simplex, denoted $[g]$, for each $g \in G$. In general, we have an n-simplex $[g_1, \ldots, g_n]$ for each ordered n-tuple of elements of G.The faces are as depicted below:

BG:



The 3-simplex $[f,g,h]$

The faces of the 3-simplex are:

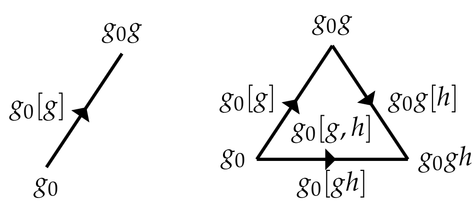
- Left: $[f,g]$
- Bottom: $[f,gh]$
- Right: $[fg,h]$
- Back: $[g,h]$

To see that BG is a $K(G,1)$, it suffices to find a contractible cover. We will do this next time.

3. Wed, Jan. 21

We define a Δ-complex EG. It has one 0-simplex g for each $g \in G$. The n-simplices are of the form $g_0[g_1, \ldots, g_n]$.

EG:



There is a map of Δ-complexes $p : EG \longrightarrow BG$ defined by

$$p(g_0[g_1, \ldots, g_n]) = [g_1, \ldots, g_n].$$

You should convince yourself that this really is a map of Δ-complexes (in other words, check it is compatible with passage to faces). [1] Moreover, there is a left G-action on EG: we define

$$g \cdot (g_0[g_1, \ldots, g_n]) = gg_0[g_1, \ldots, g_n].$$

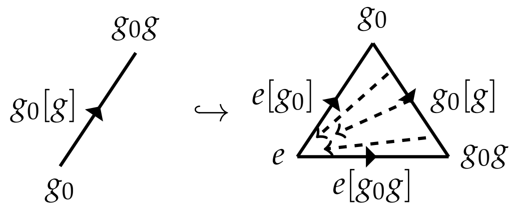
It is clear that the map p identifies BG with the quotient EG/G. It is easy to see that the action of G on EG is free. In order to deduce that p is a covering space, it suffices to check

Lemma 3.1. *Each point $y \in EG$ has a neighborhood U such that all g-translates gU, where $g \neq e$, are disjoint from U.*

[1] This example is discussed in [H, Example 1B.7], but I am using different notation. The 1-simplex in EG that I am writing as $g_0[g_1]$ would be written as $[g_0, g_0g_1]$ in [H].

Sketch. Argue by induction up the skeleta of EG. □

We now know that $p : EG \longrightarrow BG$ is a covering. If we show that EG is contractible, it will follow that EG is the universal cover and that $\pi_1(BG) \cong G$. The main idea is that any simplex $g_0[g_1, \ldots, g_n]$ can be included as a face of the $(n+1)$-simplex $e[g_0, g_1, \ldots, g_n]$, which is the cone of the previous simplex. Inside this cone, there is the straight-line homotopy of the n-simplex to the cone point.



It is straightforward to check that this homotopy is compatible with restriction to faces and therefore glues together to define a null-homotopy of id_{EG}. We have proved

Proposition 3.2. *BG is a $K(G,1)$.*

It is easily verified from the construction that both EG and BG are *functorial*. That is, any group homomorphism $\varphi : H \longrightarrow G$ induces simplicial maps

$$E\varphi : EH \longrightarrow EG, \qquad \text{and} \qquad B\varphi : BH \longrightarrow BG.$$

The next property we want (we'll explain why in a minute) is that $B(-)$ preserves products. In other words, $B(H \times G) \cong BH \times BG$. Unfortunately, this is not true with the model we introduced above. Technically, the above space is the geometric realization as a Δ-set, but we really need the realization as a *simplicial* set. All that this means for us is that the Δ complex we have been working with has some "degenerate" simplices that we need to remove. Any simplex $[g_1, \ldots, g_n]$ in which at least one g_i is e is considered degenerate, and we need to collapse these. For instance, we collapse the 1-simplex $[e]$ to the vertex, and we collapse the 2-simplex $[g, e]$ onto the 1-simplex $[g]$. Writing $B^\Delta(X)$ for the Δ-complex above and $B^s(X)$ for the new, collapsed, complex, the natural collapse map $B^\Delta(X) \longrightarrow B^s(X)$ is a homotopy equivalence.

Exercise 3.3. *Let G be a group. Then the following are equivalent:*

(1) *$m : G \times G \longrightarrow G$ is a homomorphism of groups*
(2) *$\mathrm{inv} : G \longrightarrow G$ is a homomorphism*
(3) *G is abelian .*

It follows that if G is abelian, then we can define a multiplication on B^sG by

$$B^sG \times B^sG \cong B^s(G \times G) \longrightarrow B^s(G),$$

where we have used that $B^s(-)$ is functorial in group homomorphisms. We similarly get an inverse map, and it follows that B^sG is again a topological abelian group. From now on, when we refer to the simplicial, or bar, construction of BG, we will always mean B^sG.

Great! We have produced a functorial construction of $K(G,1)$'s from groups, and these are topological abelian groups if G is abelian. In the abelian case, we can then iterate the construction, forming $BBBB \cdots BG$.

Proposition 3.4. *If G is an abelian group, then $\overbrace{B\cdots B}^{n}G$ is a $K(A,n)$.*

This does not follow from what we have done so far. Thinking about the $B(BG)$ case, there are two problems. Write $\mathcal{G}$ for BG. This is a topological group, but when we built $B\mathcal{G}$ before, we were not taking into account any topology on $\mathcal{G}$. So we need to fix that. The bigger problem is that if we consider $E\mathcal{G} \longrightarrow B\mathcal{G}$, the fibers are homeomorphic to $\mathcal{G}$ and so are not discrete! So this is not a covering space. We need a generalization of covering spaces. We will come to this later in the course, in lecture 12.

4. Fri, Jan. 21

We had some discussion last time about delta sets and simplicial sets. See [F] for a good introduction to the subject.

A **Δ-set** is a sequence of sets L_n for $n \geq 0$ together with "face maps" $d_i : L_n \longrightarrow L_{n-1}$ for $0 \leq i \leq n$. You should think of d_i as picking out the $(n-1)$-dimensional face of an n-simplex that is missing the ith vertex. The face maps are required to satisfy the relation $d_i \circ d_j = d_{j-1} \circ d_i$ whenever $i < j$. This relation is easy to see if you visualize the faces of a simplex. It is common to represent the data of a Δ-set graphically by the diagram

$$L_0 \leftleftarrows L_1 \underleftarrow{\leftleftarrows} L_2 \;\substack{\longleftarrow\\\longleftarrow\\\longleftarrow\\\longleftarrow}\; L_3 \;\cdots$$

Example 4.1.

(1) $L_0 = \{0\}$ and $L_n = \emptyset$ for $n > 0$. There are no face maps to specify. This Δ-set is just a vertex.
(2) $L_0 = \{0,1\}$, $L_1 = \{[0,1]\}$, $L_n = \emptyset$ for $n > 1$. Here $d_0([0,1]) = 1$ and $d_1([0,1]) = 0$. This is an interval as a Δ-set.
(3) We can similarly build an n-simplex as a Δ-set. It will have $\binom{n+1}{k+1}$ k-simplices. We write Δ^n_Δ for this Δ-set.
(4) The complex $B_\bullet G$ is a Δ-set, where $B_nG = G^n$. The face maps are described as follows. In B_nG, d_0 drops the first group element and d_n drops the last, while the intermediary d_i's multiply g_i with g_{i+1}.
(5) $E_\bullet G$ is a Δ-set, where $E_nG = G^{n+1}$. Here, d_n still drops the last group element, while d_i multiplies g_i with g_{i+1} for $0 \leq i < n$.

Given a Δ-set $L_\bullet$, there is a corresponding topological space, called the **geometric realization** $|L_\bullet|_\Delta$ of $L_\bullet$. To define it, first let $\mathbf{\Delta}^n \subseteq \mathbb{R}^{n+1}$ be the standard topological n-simplex defined by

$$\mathbf{\Delta}^n = \{(t_0,\ldots,t_n) \mid \sum_i t_i = 1, t_i \geq 0\}.$$

For any $0 \leq i \leq n+1$, we have inclusions $\delta^i : \mathbf{\Delta}^{n-1} \hookrightarrow \mathbf{\Delta}^n$ defined by

$$\delta^i(t_0,\ldots,t_{n-1}) = (t_0,\ldots,t_{i-1},0,t_i,\ldots,t_{n-1}).$$

We now define the geometric realization as

$$|L_\bullet|_\Delta = \coprod_n L_n \times \mathbf{\Delta}^n / \sim$$

where $(d_i(x),t) \sim (x,\delta^i(t))$ for $x \in L_{n+1}$ and $t \in \mathbf{\Delta}^n$.

Example 4.2. The realization of the Δ-set Δ^n_Δ is $|\Delta^n_\Delta|_\Delta \cong \mathbf{\Delta}^n$.

5. Monday, Jan. 26

In contrast, a **simplicial set** is a Δ-set $K_\bullet$ but further equipped with "degeneracy maps" $s_i : K_n \longrightarrow K_{n+1}$ for $0 \le i \le n$ satisfying $s_i s_j = s_{j+1} s_i$ if $i \le j$. These are related to the face maps by the following equations:

$$d_i s_j = s_{j-1} d_i \text{ if } i < j, \qquad d_j s_j = \mathrm{id} = d_{j+1} s_j, \qquad \text{and}$$
$$d_i s_j = s_j d_{i-1} \text{ if } i > j+1.$$

You should think of the degeneracy map s_i as taking an n-simplex and producing an $(n+1)$-simplex by repeating the ith vertex. That should allow you to make sense of the above formulas. The visual representation of a simplicial set is now

$$K_0 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; K_1 \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; K_2 \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; K_3 \;\cdots$$

Example 5.1.

(1) If we want to define the 0-simplex as a simplicial set, we can't just set $K_0 = \{0\}$ and the other K_n's to be empty, since this makes it impossible to define $s_0 : K_0 \longrightarrow K_1$. In fact, as soon as K_0 is nonempty, then *every* K_n must also be nonempty. The minimal choice we can make is to put a single simplex in each dimension, and that defines the simplicial Δ^0_s. Note that all simplices in dimension greater than zero are degenerate simplices.

(2) We can define Δ^1_s by specifying two 0-simplices and one nondegenerate 1-simplex $[0,1]$ as in the Δ-set case. But now we are also required to add in the degenerate simplices $s_0([0])$ and $s_0([1])$. When $n > 1$, there will be $n+2$ n-simplices, all of which are degenerate.

(3) In general, given a Δ-set $L_\bullet$, it is always possible to build a simplicial set by freely adding in degenerate simplices.

(4) The Δ-sets $B_\bullet G$ and $E_\bullet G$ extend to simplicial sets by defining the degeneracy maps s_i by inserting the identity element e in the ith slot.

(5) Given any space X, there is a simplicial set $\mathrm{Sing}_\bullet(X)$, the singular complex of X, defined by $\mathrm{Sing}_n(X) = \mathcal{C}(\mathbf{\Delta}^n, X)$. The face maps d_i are induced by the δ^i. Similarly, the degeneracy maps s_i are induced by maps $\sigma^i : \mathbf{\Delta}^n \longrightarrow \mathbf{\Delta}^{n-1}$ which we describe below.

Dual to the face inclusions $\delta^i : \mathbf{\Delta}^n \hookrightarrow \mathbf{\Delta}^{n+1}$ are the collapse maps $\sigma^i : \mathbf{\Delta}^n \to \mathbf{\Delta}^{n-1}$ defined for $0 \le i \le n-1$ by

$$\sigma^i(t_0, \dots, t_n) = (t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_n).$$

The **geometric realization** of a simplicial set $K_\bullet$ is defined as

$$|K_\bullet| = \coprod_n K_n \times \mathbf{\Delta}^n / \sim$$

where (i) $(d_i(x), t) \sim (x, \delta^i(t))$ for $x \in K_{n+1}$ and $t \in \mathbf{\Delta}^n$ and also (ii) $(s_i(x), t) \sim (x, \sigma^i(t))$ for $x \in K_n$ and $t \in \mathbf{\Delta}^{n+1}$.

Example 5.2.

(1) We have $|\Delta^n_s| \cong \mathbf{\Delta}^n$. In particular, $|\Delta^0_s|$ is just a point, even though Δ^0_s has simplices in all dimensions.

(2) Note that the Δ-set realization $|\Delta^n_s|_\Delta$ is much bigger, since we do not collapse the degenerate simplices.

Set-theoretically, the realization of a simplicial set is the disjoint union of the (open) *nondegenerate* simplices.

Why do we care about simplicial sets rather than the more intuitive Δ-sets? There are many reasons, but one is the already mentioned property that, given simplicial sets $K_\bullet$ and $L_\bullet$, we have a *homeomorphism*

$$|K_\bullet \times L_\bullet| \cong |K_\bullet| \times |L_\bullet|.$$

This is not true for Δ-sets and the Δ-realization.

6. Wed, Jan. 28

We have recently discussed Δ-sets and simplicial sets. We want to generalize the constructions $E\mathcal{G}$ and $B\mathcal{G}$ to allow for a topological group $\mathcal{G}$.

This means that we don't just have a *set* of 1-simplices but rather a *space* of 1-simplices. In practice, this does not add significant complications. Recall that for a general Δ-complex X, if we denote by $\mathcal{F}_n$ the set of n-simplices, then X is defined to be a quotient

$$\coprod_n \coprod_{f \in \mathcal{F}_n} \Delta^n \twoheadrightarrow X.$$

In the more general context in which each $\mathcal{F}_n$ comes with a topology, we simply define X to be an analogous quotient

$$\coprod_n \mathcal{F}_n \times \Delta^n \twoheadrightarrow X.$$

We thus have quotients

$$\coprod_n \mathcal{G}^{n+1} \times \Delta^n \twoheadrightarrow E\mathcal{G} \qquad \text{and} \qquad \coprod_n \mathcal{G}^n \times \Delta^n \twoheadrightarrow B\mathcal{G}.$$

This describes the geometric realization of a Δ-space. The realization of a simplicial space works completely analogously.

There is another, more categorical, approach to Δ-sets and simplicial sets. Let $\mathbf{ord}$ denote the category with objects the linearly-ordered sets $\mathbf{n} = \{0 < 1 < \cdots < n\}$. The morphisms are the order-preserving maps. Let $\mathbf{ord}^{\text{inj}}$ denote the subcategory in which we include only *injective* order-preserving maps.

Proposition 6.1. *A Δ-set is the same as a contravariant functor $(\mathbf{ord}^{\text{inj}})^{op} \longrightarrow \text{Set}$, and a simplicial set is the same as a contravariant functor $(\mathbf{ord})^{op} \longrightarrow \text{Set}$.*

The point is that we have maps $\delta^i : \mathbf{n} \longrightarrow \mathbf{n+1}$ skipping $i \in \mathbf{n+1}$ and $\sigma^i : \mathbf{n+1} \longrightarrow \mathbf{n}$ collapsing i and $i+1$ together. Of course, a simplicial space would similarly just be a contravariant functor $(\mathbf{ord})^{op} \longrightarrow \text{Top}$. We can similarly define a simplicial object in any category $\mathscr{C}$ as a contravariant functor $(\mathbf{ord})^{op} \longrightarrow \mathscr{C}$.

This concludes our brief introduction to Δ-sets and simplicial sets.

7. Fri, Jan. 30

We went on this simplicial digression because it was used in the simplicial bar construction of a $K(G,n)$. Here is yet another construction due to McCord.

Construction 7.1 (The physics-inspired construction)**.** Let A be an abelian group and let X be a space. Let $B(X,A)$ be the set of finitely-supported functions $X \longrightarrow A$. That is, we consider functions which are nonzero at only finitely points. This is an abelian group under pointwise addition. We topologize the space $B(X,A)$ as follows. For any $k \geq 1$, let $B_k(X,A)$ be the set of functions $X \longrightarrow A$ which are nonzero at at most k points. For $k=1$, we have a surjection $X \times A \twoheadrightarrow B_1(X,A)$ which sends a pair (x,a) to the function $a_x : X \longrightarrow A$, where

$$a_x(y) = \begin{cases} a & y = x \\ 0 & y \neq x. \end{cases}$$

We topologize $B_1(X,A)$ as the quotient of $X \times A$. More generally, for any k, we have a surjection $(X \times A)^k \twoheadrightarrow B_k(X,A)$ defined by

$$((x_1,a_1),\dots,(x_k,a_k)) \mapsto (a_1)_{x_1} + \cdots + (a_k)_{x_k}.$$

We topologize $B_k(X,A)$ as the above quotient space. The space $B(X,A)$ is the union of the spaces $B_k(X,A)$, and we give $B(X,A)$ the topology of the union. In mathematical physics-speak, this is the "space of A-charged particles in X".

We claim that $B(X,A)$ becomes a topological abelian group. That is, the addition and inverse maps are continuous. For the addition, it suffices to show that for any n and k, the addition $B_n(X,A) \times B_k(X,A) \longrightarrow B_{n+k}(X,A)$ is continuous. But we have the diagram

$$\begin{tikzcd} B_n(X,A) \times B_k(X,A) \arrow[r] & B_{n+k}(X,A) \\ (X \times A)^n \times (X \times A)^k \arrow[u] \arrow[r, "\cong"] & (X \times A)^{n+k} \arrow[u] \end{tikzcd}$$

Continuity of the top horizontal arrow follows from the following example of the magic of Compactly Generated Weak Hausdorff spaces:

Proposition 7.2 (Point-set topological lemma, ([Str], 2.20))**.** *If $f_1 : X_1 \longrightarrow Y_1$ and $f_2 : X_2 \longrightarrow Y_2$ are quotient maps, then so is the product $f_1 \times f_2$.*

Similarly, to see that the inverse is continuous, it is enough to see that the restriction $B_k(X,A) \longrightarrow B_k(X,A)$ is continuous. Again, we have the diagram

$$\begin{tikzcd} B_k(X,A) \arrow[r] & B_k(X,A) \\ (X \times A)^k \arrow[u] \arrow[r, "\cong"] & (X \times A)^k \arrow[u] \end{tikzcd}$$

with both vertical maps quotient maps.

What does all of this have to do with Eilenberg-Mac Lane spaces? The claim is that $B(S^n,A)$ will be a model for $K(A,n)$. Actually, we modify the above construction slightly: if X is a based space, then we define $\tilde{B}(X,A) \subseteq B(X,A)$ to be the functions taking value 0 at the basepoint.

Proposition 7.3. *The topological abelian group $\tilde{B}(S^n,A)$ is a $K(A,n)$.*

We will not prove this. See Section 4.K of [H] for a related discussion.

Remark 7.4. The more general construction is not connected, and in fact we get

$$B(S^n, A) \simeq K(A, n) \times A.$$

This ends our mini-unit on Eilenberg-Mac Lane spaces, for now.

Fiber bundles

Definition 7.5. Let G be a topological group and F be a left G-space. A ***G*-bundle with fiber** F is a map $p : E \longrightarrow B$ such that each point $b \in B$ has a neighborhood $b \in V \subseteq B$ for which there exists a homeomorphism $\varphi_V : p^{-1}(V) \longrightarrow V \times F$ making the diagram

$$\begin{tikzcd} p^{-1}(V) \arrow[rr, "\varphi_V"] \arrow[dr, "p"'] & & F \times V \arrow[dl] \\ & V & \end{tikzcd}$$

commute. Given a pair U and V of such neighborhoods of x, let $\varphi_{U,x}$ and $\varphi_{V,x}$ be the functions $p^{-1}(x) \longrightarrow F$ defined by restricting φ_U and φ_V to the fiber $p^{-1}(x)$ and projecting to F. We *require that* the composition $\varphi_{V,x} \circ \varphi_{U,x}^{-1} : F \longrightarrow F$ is given by action by an element $g_{U,V}(x) \in G$. Furthermore, we require that the resulting **transition function** $g_{U,V} : U \cap V \longrightarrow G$ be continuous.

A G-bundle with fiber G is referred to as a **principal** G-bundle.

Example 7.6. Given any G and F and base B, we can form the **trivial** bundle $B \times F$.

In an arbitrary bundle E, the isomorphisms $\varphi_V : p^{-1}(V) \cong F \times V$ are referred to as **local trivializations**. The point is that, in a sufficiently small neighborhood of a point $p \in B$, the bundle looks trivial, though this may not be true globally. A fiber bundle is therefore a generalization of a product.

Example 7.7. A covering space is a bundle[2] with discrete fibers and $G = \pi_1(B)$.

Example 7.8. When the fibers are $\mathbb{R}^n$ (equipped with the vector space structure) and the group is $G = \mathrm{Gl}_n(\mathbb{R})$, a G-bundle is called a **real vector bundle**. Similarly, a **complex vector bundle** has fibers $\mathbb{C}^n$ and structure group $G = \mathrm{Gl}_n(\mathbb{C})$. When $n = 1$, we call this a **line bundle**.

Example 7.9. Let $E \subseteq \mathbb{RP}^n \times \mathbb{R}^{n+1}$ consist of the set of pairs (ℓ, v), where $v \in \ell$. Then the assignment $(\ell, v) \mapsto \ell$ defines a line bundle over $\mathbb{RP}^n$. To see the local trivialization, let $\ell \in \mathbb{RP}^n$. Let $V \subseteq \mathbb{RP}^n$ consist of all lines not orthogonal to ℓ. Pick any linear isomorphism $\mu : \ell \cong \mathbb{R}$, and we define

$$\varphi : p^{-1}(V) \longrightarrow V \times \mathbb{R} \quad \text{by} \quad \varphi(\ell', w) = (\ell', \mu(\mathbf{proj}_\ell(w))).$$

If $\ell' \in V$, then $\mathbf{proj}_\ell : \ell' \longrightarrow \ell$ is an isomorphism, so φ is invertible. This bundle is known as the **tautological line bundle**.

[2] Beware, however, that in MA651 we always discussed the $\pi_1(B)$-action as a *right* action.

8. Mon, Feb. 2

Example 8.1. The last example generalizes to Grassmannians. Recall that $\mathrm{Gr}_k(\mathbb{R}^n)$ denotes the set of k-dimensional subspaces of $\mathbb{R}^n$, suitably topologized. We have seen that $\mathrm{Gr}_k(\mathbb{R}^n) \cong O(n)/\Big(O(k) \times O(n-k)\Big)$. There is again a **tautological bundle** over the Grassmannian. Let $E \subseteq \mathrm{Gr}_k(\mathbb{R}^n) \times \mathbb{R}^n$ be $E = \{(K, v) \mid v \in K\}$. Then projection onto the first coordinate $p_1 : E \longrightarrow \mathrm{Gr}_k(\mathbb{R}^n)$ defines a k-dimensional vector bundle. The trivialization has a similar description. Let $K \in \mathrm{Gr}_k(\mathbb{R}^n)$, and let V consist of all k-planes L such that $K^\perp \cap L = 0$. We can pick a linear isomorphism $\mu : K \cong \mathbb{R}^k$ and define

$$\varphi : p^{-1}(V) \longrightarrow V \times \mathbb{R}^k \quad \text{by} \quad \varphi(L, w) = (L, \mu(\mathbf{proj}_K(w))).$$

Again, for $L \in V$, the map $\mathbf{proj}_K : L \longrightarrow K$ is an isomorphism, so that φ is invertible.

Example 8.2. Consider the defining quotient $S^{2n+1} \longrightarrow \mathbb{CP}^n$. This is a principal S^1-bundle. The desired neighborhoods can be taken to be the standard covering of $\mathbb{CP}^n$ in which one (complex) coordinate is specified to be nonzero. For instance, let $V_0 \subseteq \mathbb{CP}^n$ be the subset of points $[z_0 : \cdots : z_n]$ for which $z_0 \neq 0$. We define $\varphi_0 : p^{-1}(V_0) \cong S^1 \times V_0$ by

$$\varphi_0(z_0, \ldots, z_n) = \left(\frac{z_0}{\|z_0\|}, [z_0 : \cdots : z_n] \right).$$

This is a homeomorphism, as

$$(\lambda, [z_0 : \cdots : z_n]) \mapsto \lambda \frac{\|z_0\|}{z_0}(z_0, \ldots, z_n)$$

gives an inverse. This also works in the infinite-dimensional case, so that we get a principal S^1-bundle $S^\infty \longrightarrow \mathbb{CP}^\infty$. The case $n = 1$ is a famous case, known as the Hopf bundle $S^3 \longrightarrow S^2$.

Example 8.3. Let $V_k(\mathbb{R}^n)$ denote the **Stiefel manifold** of orthonormal subsets of $\mathbb{R}^n$ of cardinality k. A point of $V_k(\mathbb{R}^n)$ is referred to as an **orthonormal** ***k*-frame**. We can topologize this in the same way that we topologies the Grassmannian: there is a transitive action of $O(n)$ on $V_k(\mathbb{R}^n)$, and the stabilizer of $(\mathbf{e}_1, \ldots, \mathbf{e}_k)$ is $1 \times O(n-k) \subseteq O(n)$. Thus $V_k(\mathbb{R}^n) \cong O(n)/O(n-k)$.

An orthonormal k-frame in $\mathbb{R}^n$ corresponds to an isometry $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$. Since $O(k)$ is the group of isometries of $\mathbb{R}^k$ with itself, we get a right action of $O(k)$ on $V_k(\mathbb{R}^n)$. The map $V_k(\mathbb{R}^n) \longrightarrow \mathrm{Gr}_k(\mathbb{R}^n)$ taking a frame to its span then defines a principal $O(k)$-bundle. We leave as an exercise the construction of the local trivialization. Note that the $k = 1$ case is just the 2-sheeted cover $S^{n-1} \longrightarrow \mathbb{RP}^{n-1}$.

9. Wed, Feb. 4

See the notes from last Friday for a corrected definition of bundle. For a general G-bundle $p : E \longrightarrow B$, we do not expect a left action of G on E. We have an action on each fiber, but these actions do not glue together to give a globally defined action in general. In some sense, the transition functions measure the failure of the fiberwise group actions to glue together to form a globally defined action.

To help clarify how bundles work, we start by discussing the transition functions for the principal S^1-bundle $S^{2n+1} \longrightarrow \mathbb{CP}^n$.

As we discussed in class last time, the local trivialization on the subset $V_i \subseteq \mathbb{CP}^n$ in which $x_i \neq 0$ is given by

$$\varphi_i : p^{-1}(V_i) \longrightarrow S^1 \times V_i,$$
$$\varphi_i((x_0, \dots, x_n)) = \Big(\frac{x_i}{\|x_i\|}, [x_0 : \cdots : x_n] \Big).$$

The inverse is

$$\varphi_i^{-1}\Big(\lambda, [x_0 : \cdots : x_n]\Big) = \frac{\lambda |x_i|}{x_i \|\mathbf{x}\|}(x_0, \dots, x_n)$$

Let us describe the transition map $g_{ij} : V_i \cap V_j \longrightarrow S^1$. Recall that this should satisfy the property that

$$S^1 \times (V_i \cap V_j) \xrightarrow{\varphi_i^{-1}} p^{-1}(V_i \cap V_j) \xrightarrow{\varphi_j} S^1 \times (V_i \cap V_j)$$

takes the form

$$(\lambda, \mathbf{x}) \mapsto (g_{ij}(\mathbf{x})\lambda, \mathbf{x}).$$

Composing our formulas for φ_i^{-1} and φ_j gives

$$\varphi_j \varphi_i^{-1}(\lambda, \mathbf{x}) = \varphi_j\Big(\frac{\lambda |x_i|}{x_i \|\mathbf{x}\|}(x_0, \dots, x_n) \Big) = \Big(\frac{x_j |x_i|}{x_i |x_j|}\lambda, \mathbf{x} \Big)$$

It follows that $g_{ij}(\mathbf{x}) = \frac{x_j |x_i|}{x_i |x_j|}$. From this formula, it is easy to verify that

(1) $g_{ii} = e$, the identity element of the group,
(2) $g_{ji} = g_{ij}^{-1}$, and
(3) $g_{jk} g_{ij} = g_{ik}$.

These formulas hold for any G-bundle. The third condition is known as the **cocyle condition**, and it implies the other two.

Proposition 9.1. *If $\mathcal{G}$ is a topological abelian group, then $p : E\mathcal{G} \longrightarrow B\mathcal{G}$ is a principal G-bundle.*

Proof. The following argument is from [Ste, Theorem 8.3]. Recall that we need to cover $B\mathcal{G}$ by neighborhoods on which we have trivializations. We start by constructing a single such neighborhood. By [DK, Theorem 6.23], the inclusion $\mathcal{G} \hookrightarrow E\mathcal{G}$ is an NDR-pair, meaning that there are functions

$$u : E\mathcal{G} \longrightarrow I, \qquad h : E\mathcal{G} \times I \longrightarrow E\mathcal{G}$$

such that

(1) $\mathcal{G} = u^{-1}(0)$,
(2) $h_0 = \mathrm{id}$,
(3) $h(g, t) = g$ for all $g \in \mathcal{G}$ and $t \in I$ and
(4) $h(x, 1) \in \mathcal{G}$ for all x such that $u(x) < 1$.

In fact, since $E\mathcal{G}$ is a $\mathcal{G}$-equivariant complex, we have functions u and h that are $\mathcal{G}$-equivariant. Let $W = u^{-1}[0, 1)$. Note that W is open and saturated (since u is $\mathcal{G}$-equivariant). It follows that $V = p(W) \subseteq B\mathcal{G}$ is open. Define $r : W \longrightarrow \mathcal{G}$ by $r(w) = h(w, 1)$. We note that r is $\mathcal{G}$-equivariant.

We claim that $\varphi_W = (r, p) : W \longrightarrow \mathcal{G} \times V$ is a homeomorphism. To see this, define an "untwisting" map $F : W \longrightarrow W$ by $F(w) = r(w)^{-1}w$. Then it is easily checked that F is constant on $\mathcal{G}$-orbits and so descends to a map $f : V \longrightarrow W$. We can then define an inverse to φ_W by the formula $\varphi_W^{-1}(g, v) = gf(v)$. It is straightforward to check that this is indeed inverse to φ_W.

10. Fri, Feb. 6

Now we need to know that we can cover $B\mathcal{G}$ by such V's, or equivalently that we can cover $E\mathcal{G}$ by W's. To see this, recall that we have a homeomorphism $B\mathcal{G} \times B\mathcal{G} \cong B(\mathcal{G} \times \mathcal{G})$, which allowed us to put a topological group structure on $B\mathcal{G}$. The same goes for $E\mathcal{G}$, in fact. It follows that we can translate $W \subseteq E\mathcal{G}$ by elements of $E\mathcal{G}$. Since $E\mathcal{G}$ is abelian, these translates will still be invariant under the subgroup $\mathcal{G}$, and we can repeat the above argument for the translates of W to get our local trivialization of $E\mathcal{G}$.

In more detail, let $x \in E\mathcal{G}$ and consider the translate xW. Let V_x denote $p(xW)$. Denote by $r_x : xW \longrightarrow G$ the translated function $r_x(y) = r(x^{-1}y)$. Then

$$\varphi_{xW} = (r_x, p) : xW \longrightarrow \mathcal{G} \times V_x$$

defines a local trivialization. The inverse is given as above: one first defines the function $F_x(w) = r_x(w)^{-1}w$. This descends to a map $f_x : V_x \longrightarrow xW$, and $\varphi_{xW}^{-1}(g, v) = gf_x(v)$ is the inverse. It is then straight forward to check that the transition functions are given by the formula

$$g_{x_1,x_2}(v) = r_{x_2}(f_{x_1}(v)).$$

□

Fiber bundles are a generalization of covering spaces, and we know that covering spaces have a path-lifting property. For fiber bundles, we have have the following analogue:

Definition 10.1. A map $p : E \longrightarrow B$ has the **homotopy lifting property** (aka covering homotopy property) with respect to X if, given a map $f : X \longrightarrow E$ and a homotopy $h : X \times I \longrightarrow B$ such that $h_0 = pf$, then there is a lift $\tilde{h} : X \times I \longrightarrow E$ such that $p\tilde{h} = h$ and $\tilde{h}_0 = f$ as in the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\iota_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "h"'] \arrow[ur, dashed, "\tilde{h}"] & B. \end{tikzcd}$$

Note that, in contrast to the situation for covering spaces, we do not assert that the lift $\tilde{h}$ is unique.

Proposition 10.2. *Fiber bundles have the homotopy lifting property with respect to cubes I^n.*

Proof. Let $p : E \longrightarrow B$ be a fiber bundle, and assume given $f : I^n \longrightarrow E$ and $h : I^{n+1} \longrightarrow B$ as in the setup of the homotopy lifting property. Let $\{(U, \varphi_U)\}$ be a local trivialization of

p. We first assume for simplicity that $h(I^{n+1})$ is contained in a single U. Then our lifting problem becomes

$$\begin{tikzcd} I^n \arrow[r, "f"] \arrow[d, "\iota_0"'] & p^{-1}(U) \arrow[r, "\varphi_U", "\cong"'] & U \times F \arrow[d, "p_1"] \\ I^{n+1} \arrow[rru, dashed, "\tilde{h}"] \arrow[rr, "h"'] & & U \end{tikzcd}$$

It is clear that we must define $\tilde{h}$ in the U-coordinate to simply be h. It only remains to specify the F-coordinate. We could simply define the F-coordinate of $\tilde{h}$ to be

$$I^n \times I \xrightarrow{p} I^n \xrightarrow{f} p^{-1}(U) \xrightarrow{\varphi_U} U \times F \xrightarrow{p_2} F,$$

but we will need a little more flexibility. We will need to be able to produce a lift $\tilde{h}$ after already having chosen lifts on $\partial I^n \times I$ (or maybe only part of the boundary). So assume given a lift $\tilde{h}_{\partial I^n} : (\partial I^n) \times I \longrightarrow U \times F$. Let

$$r : I^{n+1} \longrightarrow \Big(I^n \times \{0\}\Big) \cup \Big(\partial I^n \times I\Big)$$

be any retraction. We then define $\tilde{h}_F : I^{n+1} \longrightarrow F$ as the composition

$$I^{n+1} \xrightarrow{r} \Big(I^n \times \{0\}\Big) \cup \Big(\partial I^n \times I\Big) \xrightarrow{f \cup \tilde{h}_{\partial I^n}} F,$$

where we have used a slight abuse of notation in the last line (we really mean the F components of f and $\tilde{h}_{\partial I^n}$).

The rest of the argument is just like the one for path-lifting in covering spaces. By the Lebesgue Number Lemma, we may subdivide the cube I^n into subcubes C and the interval I into subintervals J such that each $C \times J$ is taken by h into a single U. We then work our way across $I^n \times I$, lifting in one $C \times J$ at a time, taking care to use the previously constructed lift on any boundaries. □

11. Monday, Feb. 9

Often, we need a relative version of the HLP. This would say: given a map $f : I^n \longrightarrow E$, a homotopy $h : I^n \times I \longrightarrow B$, and a lift $\hat{h} : \partial I^n \times I \longrightarrow E$ of h on the boundary of I^n, then there is a lift $\tilde{h} : I^n \times I \longrightarrow E$ of h which restricts to f and $\hat{h}$. This is in fact equivalent to the version stated above, because of the following lemma.

Lemma 11.1. *There is a homeomorphism $\psi : I^n \times I \cong I^n \times I$ with*

$$\psi(I^n \times \{0\}) = (I^n \times \{0\}) \cup (\partial I^n \times I).$$

We are now ready to state the relation of fiber bundles to homotopy groups.

Proposition 11.2. *Fiber bundles give rise to a long exact sequence in homotopy groups:*

$$\cdots \longrightarrow \pi_{n+1}(B) \xrightarrow{\partial} \pi_n(F) \xrightarrow{j_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \longrightarrow \cdots$$

Proof. Let $b_0 \in B$ be a choice of basepoint, and let $j : F \cong p^{-1}(b_0) \subseteq E$ be the inclusion.

$\boxed{\text{Exactness at } E:}$ The composition $F \xrightarrow{j} E \xrightarrow{p} B$ is constant, so $p_* j_* = 0$. Now suppose $p_*(\beta) = 0$ for $\beta \in \pi_n(E)$. We can represent β as a map $\beta : I^n \longrightarrow E$ which is constant on the boundary. Since $p \circ \beta$ is null, we have a homotopy $h : I^n \times I \longrightarrow B$ with $h_0 = p \circ \beta$ and $h_1 = c_{b_0}$. We thus have a lifting problem

$$\begin{tikzcd} I^n \arrow[r, "\beta"] \arrow[d, "\iota_0"'] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "h"'] \arrow[ur, dashed, "\tilde{h}"] & B, \end{tikzcd}$$

which we can solve by Prop 10.2. According to the discussion before Lemma 11.1, we can choose the lift $\tilde{h}$ to be constant on ∂I^n. Since h_1 is constant at the basepoint b_0, it follows that $\tilde{h}_1$ defines a map $\alpha : I^n/\partial I^n \longrightarrow F$, and the homotopy $\tilde{h}$ shows that $j_*(\alpha) = \beta$

$\boxed{\text{The connecting homomorphism } \partial:}$ Let $\gamma : I^n/\partial I^n \longrightarrow B$ be an element of $\pi_n(B)$. We can regard I^n as $I^{n-1} \times I$, so that γ may be viewed as a homotopy h from c_{b_0} to itself. We can lift h_0 to the constant map at e_0, and we can lift on $\partial I^{n-1} \times I$ to again be constant at e_0.

$$\begin{tikzcd} (I^{n-1} \times \{0\}) \cup (\partial I^{n-1} \times I) \arrow[r, "e_0"] \arrow[d] & E \arrow[d, "p"] \\ I^{n-1} \times I \arrow[r, "\gamma"'] \arrow[ur, dashed] & B \end{tikzcd}$$

By Prop 10.2, we can solve this lifting problem to produce $\tilde{h} : I^{n-1} \times I \longrightarrow E$, and $\alpha = \tilde{h}_1$ defines a map $\alpha : I^{n-1}/\partial I^{n-1} \longrightarrow E$ which lands in the fiber $F = p^{-1}(b_0)$. We then define $\partial(\gamma) = \alpha \in \pi_{n-1}(F)$.

It is not immediate from the above discussion that $\partial : \pi_n(B) \longrightarrow \pi_{n-1}(F)$ is well-defined, since it involves the choice of a lift. We also need to know that the homotopy class of $\partial(\gamma)$ only depends on the homotopy class of γ. Thus let $H : \gamma \simeq \gamma'$ be a based homotopy. As usual, we can regard this as a map $H : I^n \times I \longrightarrow B$ which is constant on $\partial I^n \times I$. Now consider the lifting problem

$$\begin{tikzcd} (I^n \times \{0,1\}) \cup (J^{n-1} \times I) \arrow[r, "\tilde{h}_1 \cup \tilde{h}'_1 \cup e_0"] \arrow[d] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & B, \end{tikzcd}$$

where $J^{n-1} \subseteq I^n$ is the union of all faces *except* $I^{n-1} \times \{1\}$. This has a solution, $\tilde{H}$, and the resriction of $\tilde{H}$ to $I^{n-1} \times \{1\} \times I$ gives a based homotopy $\tilde{h}_1 \simeq \tilde{h}'_1$. This shows that $\partial : \pi_n(B) \longrightarrow \pi_{n-1}(F)$ is well-defined.

$\boxed{\text{Exactness at } B:}$ Let $\beta \in \pi_n(E)$. We want first to see that $\partial(p_*(\beta)) = 0$. But in the construction of $\partial(p_*(\beta))$, we can of course take β itself as the lift of $p_*(\beta)$, and the restriction of β to one face of the cube is certainly constant.

Now suppose that $\partial(\gamma) = 0$. We want to show that γ is in the image of p_*. By construction, $\partial(\gamma) = \tilde{h}_1$ for some lift $\tilde{h} : I^{n-1} \times I \longrightarrow E$ of γ. By assumption, we have a null-homotopy $g : \tilde{h}_1 \simeq c_{e_0}$ in F. For convenience, we think of g as a map $g : I^{n-1} \times [1,2] \longrightarrow E$. We may then glue g to $\tilde{h}$ to get $\beta = g \cup \tilde{h} : I^{n-1} \times [0,2] \longrightarrow E$. Note that β is constant on $\partial(I^{n-1} \times [0,2])$, and since $p \circ g$ is constant (since g is a homotopy in the fiber), it follows that $p \circ \beta \simeq \gamma$.

12. Wednesday, Feb. 11

$\boxed{\text{Exactness at } F\text{:}}$ Let $\gamma \in \pi_n(B)$. We first want to see that $j_*\partial(\gamma) = 0$. Again, in the construction of $\partial(\gamma)$, we produced a map $\tilde{h} : I^n \longrightarrow E$, which gives a null-homotopy in E, $c_{e_0} \simeq \tilde{h}_1$. Since $\partial(\gamma) = \tilde{h}_1$, we are done.

Now let $\alpha \in \pi_{n-1}(F)$ such that $j_*(\alpha) = 0 \in \pi_{n-1}(E)$. We wish to show that α lies in the image of ∂. Let $\tilde{h} : I^{n-1} \times I$ be a null-homotopy for α in E. Then $p \circ \tilde{h} : I^n \longrightarrow B$ is constant on the boundary and so defines an element of $\pi_n(B)$, and it is clear that $\partial(p \circ \tilde{h}) = \tilde{h}_1 = \alpha$. $\square$

Proof of Prop 3.4. We show by induction that if G is a (discrete) abelian group then

$$\overbrace{B \cdots B}^{n} G \sim K(G, n).$$

We already know the case $n = 1$. Let us write $\mathcal{G}$ for the n-fold bar construction $B^n G$. By Prop 11.2 and Prop 9.1, we have a long exact sequence in homotopy

$$\cdots \pi_{j+1}(E\mathcal{G}) \longrightarrow \pi_{j+1}(B\mathcal{G}) \longrightarrow \pi_j(\mathcal{G}) \longrightarrow \pi_j(E\mathcal{G}) \longrightarrow \cdots$$

We showed previously in the discrete case that $EG \simeq *$, and we can carry that proof over to the topological group case. In fact, the simplicial contraction you found on your homework applies in this more general setting.

Since $E\mathcal{G} \simeq *$, we conclude that $\pi_{j+1}(B\mathcal{G}) \cong \pi_j(\mathcal{G})$. By induction, we are done. $\square$

We are now in a position to prove a few statements from the first week of class.

Proposition 12.1. *$\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$.*

Proof. Recall that we have a principal S^1-bundle $S^\infty \longrightarrow \mathbb{CP}^\infty$. Since $S^\infty \simeq *$, the long exact sequence gives isomorphisms $\pi_n(\mathbb{CP}^\infty) \cong \pi_{n-1}(S^1)$. $\square$

Proposition 12.2. *We have $\pi_3(S^2) \cong \mathbb{Z}$ and $\pi_3(\mathbb{CP}^2) = 0$.*

Proof. For the first, the principal S^1-bundle $S^3 \longrightarrow \mathbb{CP}^1 \cong S^2$ gives a long exact sequence

$$\cdots \longrightarrow \pi_n(S^1) \longrightarrow \pi_n(S^3) \longrightarrow \pi_n(S^2) \longrightarrow \pi_{n-1}(S^1) \longrightarrow \cdots$$

Since all higher homotopy groups of S^1 vanish, it follows that $\pi_n(S^3) \cong \pi_n(S^2)$ for $n \geq 3$. Thus $\pi_3(S^2) \cong \pi_3(S^3) \cong \mathbb{Z}$ by the Hurewicz theorem.

For the second statement, we use the principal S^1-bundle $S^5 \longrightarrow \mathbb{CP}^2$. The long exact sequence shows that $\pi_n(\mathbb{CP}^2) \cong \pi_n(S^5)$ when $n \geq 3$, so $\pi_3(\mathbb{CP}^2) \cong \pi_3(S^5) = 0$. $\square$

13. Fri, Feb. 13

Last time, the issue of generalizing the HLP came up. We saw that the HLP with respect to cubes I^n was equivalent to the relative HLP with respect to the pair $(I^n, \partial I^n)$. In fact, the HLP with respect to cubes I^n is also equivalent to the relative HLP with respect to pairs (X, A), where $A \subseteq X$ is a subcomplex. To clarify, let $\iota : A \subseteq X$ be the inclusion of a subcomplex. Recall that the **mapping cylinder** $M(\iota)$ is defined as

$$M(\iota) = (X \times \{0\}) \cup (A \times I) \subseteq X \times I.$$

Recall further that the inclusion $M(\iota) \hookrightarrow X \times I$ is a homotopy equivalence. Then the HLP with respect to cubes is equivalent to the existence of a solution to every lifting problem of the form

$$\begin{tikzcd} M(\iota) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \times I \arrow[r] \arrow[ur, dashed] & B. \end{tikzcd}$$

A map $E \longrightarrow B$ satisfying either of these (equivalent) lifting conditions is known as a **Serre fibration**. So Prop 10.2 states that any bundle is a Serre fibration. There is the stronger notion of a **Hurewicz fibration**, which is a map satisfying the HLP with respect to *all* spaces. It is a nontrivial theorem ([Sp, Cor 2.7.14]) that a bundle over a paracompact base space is a Hurewicz fibration.

Classification of bundles

We will show that every bundle comes from a universal example. First, we discuss a way of producing new bundles.

Definition 13.1. Let $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ be maps. The **pullback** of f and g is a space P with maps

$$\begin{tikzcd} P \arrow[r, "\hat{f}"] \arrow[d, "\hat{g}"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z, \end{tikzcd}$$

which is universal among spaces with compatible maps to X and Y. That is, if W is another such example, we get a unique map as in the diagram

$$\begin{tikzcd} W \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & P \arrow[r, "\hat{f}"] \arrow[d, "\hat{g}"'] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z. \end{tikzcd}$$

The pullback is often written $P = X \times_Z Y$, and it can be defined as the subset of $X \times Y$ defined by

$$X \times_Z Y = \{(x, y) \mid f(x) = g(y)\}.$$

14. Monday, Feb. 16

Snow Day!!

15. Wednesday, Feb. 18

Pseudo-Snow Day!!

16. Friday, Feb. 20

Snow Day. . . again. . .

17. Monday, Feb. 23

Not-a-snow day!!

Proposition 17.1. *Let $p : E \longrightarrow B$ be a G-bundle with fiber F. Then, given a map $f : X \longrightarrow B$, the pullback $p_X : E \times_B X \longrightarrow X$ is a G-bundle with fiber F.*

$$\begin{tikzcd} E \times_B X \arrow[r, "\hat{f}"] \arrow[d, "p_X"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$

Proof. Given a point $x \in X$, let $V \subseteq B$ be a trivialized neighborhood of $f(x)$, and let $\varphi_V : p^{-1}(V) \cong F \times V$ be the trivialization. Then $f^{-1}(V)$ is a neighborhood of x, and we wish to define the trivialization

$$\varphi^X_{f^{-1}(V)} : p_X^{-1} f^{-1}(V) \cong F \times f^{-1}(V).$$

Note that we can identify $p_X^{-1} f^{-1}(V)$ with $p^{-1}(V) \times_V f^{-1}(V)$. The trivialization φ_V then gives us an isomorphism

$$p_X^{-1} f^{-1}(V) = p^{-1}(V) \times_V f^{-1}(V) \xrightarrow{\varphi_V \times \mathrm{id}} (F \times V) \times_V f^{-1}(V) \cong F \times f^{-1}(V).$$

We leave it as an exercise to verify that the composition

$$f^{-1}(U) \cap f^{-1}(V) \xrightarrow{f} U \cap V \xrightarrow{g_{U,V}} G$$

define transition functions for the bundle. $\square$

Example 17.2. Let $p : E \longrightarrow B$ be a G-bundle, and let $\iota : A \hookrightarrow B$ be the inclusion of a subspace. Then $\iota^*(E)$ is the restriction of the bundle E to the subspace A, often written as $E_{|A}$.

In fact, as we will show in Prop 18.1 below, the isomorphism class of a pullback bundle only depends on the homotopy class of f, but we first must say what we mean by isomorphism class of bundle.

Definition 17.3. Let $p : E \longrightarrow B$ and $p' : E' \longrightarrow B'$ be G-bundles with fiber F. A **map of bundles** is a pair of maps $f : B \longrightarrow B'$ and $\hat{f} : E \longrightarrow E'$ such that $p' \circ \hat{f} = f \circ p$. We further require that, if $V \subseteq B$ and $V' \subseteq B'$ are trivializable neighborhoods such that $V \cap f^{-1}(V') \neq \emptyset$, then the composition

$$F \times (V \cap f^{-1}(V')) \xrightarrow{\varphi_V^{-1}} p^{-1}(V \cap f^{-1}(V')) \xrightarrow{\hat{f}} (p')^{-1}(V') \xrightarrow{\varphi'_{V'}} F \times V'$$

is of the form

$$(x, v) \mapsto (f_{V,V'}(v) \cdot x, f(v))$$

for some continuous $f_{V,V'} : V \cap f^{-1}(V') \longrightarrow G$.

Note that the definition implies that a map of bundles induces a homeomorphism on fibers.

Remark 17.4. Consider the case $(E', B', p') = (E, B, p)$ and $f = \mathrm{id}_B$, $\hat{f} = \mathrm{id}_E$. Then, given two overlapping open sets U and V, the required maps $f_{U,V} \longrightarrow G$ for a bundle map are precisely the transition functions.

In the case where $f = \mathrm{id}_B$, we refer to an $\tilde{f}$ as a map of bundles over B.

Example 17.5. Let $p : E \longrightarrow B$ be a bundle and let $f : X \longrightarrow B$ be a map. Then the natural map $\hat{f} : X \times_B E \longrightarrow E$ is a bundle map (over $f : X \longrightarrow B$). The $f_{V,V'}$ are again the transition functions of E.

Exercise 17.6. *Let $p : E \longrightarrow B$ and $p' : E' \longrightarrow B'$ be G-bundles with fiber F, and let $f : B \longrightarrow B'$ be a map. Then a bundle map $\hat{f} : E \longrightarrow E'$ is equivalent to a bundle isomorphism $E \longrightarrow f^*(E')$ over B.*

18. Wed, Feb. 25

Theorem 18.1. *Let $f_0, f_1 : X \longrightarrow B$ be homotopic maps, and let $p : E \longrightarrow B$ be a G-bundle. If X is paracompact, then $f_0^*(E) \cong f_1^*(E)$ as bundles over X.*

Recall from MA551 that any CW complex, for example, is paracompact Hausdorff.

Proof. We sketch the proof in the case of a compact, Hausdorff base. See [H2, Theorem 1.6] for the paracompact case.

Let $h : X \times I \longrightarrow B$ be a homotopy from f_0 to f_1. Then $f_0^*(E) = h^*(E)_{|X \times \{0\}}$ and similarly for $f_1^*(E)$. So without loss of generality we may replace B by $X \times I$, and we wish to show that the time 0 and time 1 restrictions of a bundle E on $X \times I$ are isomorphic.

Using compactness, one can show that there is a finite cover $\{U_1, \dots, U_n\}$ of X so that the restriction of E to each $U_i \times I$ is trivial. Let $\{\varphi_i\}_{i=1}^n$ be a partition of unity subordinate to the cover $\{U_i\}_{i=1}^n$. For each $0 \leq j \leq n$, define $\Phi_j = \sum_{i=1}^j \varphi_i$. Thus $\Phi_0 = 0$ and $\Phi_n = 1$ on X. For simplicity, we will assume $n = 2$, since that is enough to see the argument. Thus we have

$$\Phi_0 = 0 \leq \Phi_1 = \varphi_1 \leq \Phi_2 = 1$$

on X. For each $0 \leq j \leq n$, we define $X_j \subseteq X \times I$ to be the graph of Φ_j. Thus $X_0 = X \times \{0\}$ and $X_2 = X \times \{1\}$, and each X_j is homeomorphic to X via the projection. Finally, let E_j be the restriction of E to $X_j \cong X$. We claim that $E_0 \cong E_1 \cong E_2$.

To see that $E_0 \cong E_1$, recall that E is trivial on $U_1 \times I$. It follows that the trivialization of E on U_1 restricts to trivializations φ_{U_1} of E_0 and E_1 on U_1. Define $\alpha_{U_1} : (E_0)_{|U_1} \longrightarrow (E_1)_{|U_1}$ to be the composition

$$(E_0)_{|U_1} \xrightarrow{(\varphi_{U_1})_{|E_0}} F \times U_1 \xrightarrow{(\varphi_{U_1})^{-1}_{|E_1}} (E_1)_{|U_1}.$$

Now let $V_1 = X \setminus \mathrm{supp}(\varphi_1)$. Since φ_1 is supported inside U_1, it follows that $U_1 \cup V_1 = X$. Also, we have that $(E_0)_{|V_1} = (E_1)_{|V_1}$, since $X_0 \cap (V_1 \times I) = V_1 \times \{0\} = X_1 \cap (V_1 \times I)$. Now α_{U_1} on $(E_0)_{|U_1}$ glues together with id on $(E_0)_{|V_1}$ to give an isomorphism $E_0 \cong E_1$. □

Corollary 18.2. *Any bundle over a contractible space is trivial.*

We are headed towards the classification theorem, but first we need some preliminaries.

Proposition 18.3. *Let $p : E \longrightarrow B$ be a principal G-bundle. Then E admits a free right G-action such that $E/G \cong B$.*

Proof. Let $y \in E$ and let $y \in p^{-1}(V)$, where V is a trivializing neighborhood of B. We define the G-action on the neighborhood $p^{-1}(V)$ as the composition

$$p^{-1}(V) \times G \xrightarrow{\varphi_V \times \mathrm{id}} G \times V \times G \cong G \times G \times V \xrightarrow{m \times \mathrm{id}} G \times V \xrightarrow{\varphi_V^{-1}} p^{-1}(V).$$

It remains to check that this is independent of the choice of V. Thus suppose that $y \in p^{-1}(U \cap V)$. We now have two proposed definitions of the G-action on y, and we must check that they coincide.

$$\begin{tikzcd}
 & G \times (U\cap V) \times G \arrow[r, "\cong"] & G \times G \times (U \cap V) \arrow[r, "m\times\mathrm{id}"] & G \times (U \cap V) \arrow[dr, "\varphi_V^{-1}"] & \\
p^{-1}(U\cap V) \times G \arrow[ur, "\varphi_V \times \mathrm{id}"] \arrow[dr, "\varphi_U \times \mathrm{id}"'] & & & & p^{-1}(U \cap V). \\
 & G \times (U\cap V) \times G \arrow[r, "\cong"] \arrow[uu, "g_{U,V}\times\mathrm{id}\times\mathrm{id}"'] & G \times G \times (U \cap V) \arrow[r, "m\times\mathrm{id}"] \arrow[uu, "g_{U,V}\times\mathrm{id}\times\mathrm{id}"'] & G \times (U \cap V) \arrow[ur, "\varphi_U^{-1}"'] \arrow[uu, "g_{U,V}\times\mathrm{id}"'] &
\end{tikzcd}$$

The triangles commute by the definition of the transition functions $g_{U,V}$, and the square on the left clearly commutes. The square on the right commutes because $g_{U,V}$ means *left* multiplication by the element $g_{U,V}$.

We leave as an exercise to show that the action is free and that B is the space of orbits. □

19. Fri, Feb. 27

Denote by $\mathrm{Bun}_G(X, F)$ the set of isomorphism classes of G-bundles over X with fiber F. Theorem 18.1 implies that, given a bundle $p : E \longrightarrow B$, pullback of bundles defines a function

$$[X, B] \longrightarrow \mathrm{Bun}_G(X, F).$$
$$f \mapsto f^*(E)$$

In the case $F = G$, namely principal bundles, we write $\mathrm{Prin}_G(X) = \mathrm{Bun}_G(X, G)$.

Theorem 19.1 (Classification theorem)**.** *Let G be a topological group and let $p : E \longrightarrow B$ be a principal G-bundle in which E is contractible. Then the pullback construction*

$$[X, B] \longrightarrow \mathrm{Prin}_G(X)$$

is a bijection if X is paracompact.

For this reason, such a bundle $p : E \longrightarrow B$ is called a *universal* principal G-bundle.

The heavy lifting in the proof will be done by the following result.

Proposition 19.2. *Let $p : E \longrightarrow B$ be a principal G-bundle such that E is contractible. Let K be a regular CW complex and let $L \subseteq K$ be a subcomplex. Suppose given a bundle $p' : E' \longrightarrow K$ and a bundle map $\widehat{\alpha} : E'_{|L} \longrightarrow E$. Then there is an extension of $\widehat{\alpha}$ to a bundle map $\widehat{\beta} : E' \longrightarrow E$.*

$$\begin{tikzcd}
E'_{|L} \arrow[rrr, "\widehat{\alpha}"] \arrow[dr] \arrow[dd, "p'"'] & & & E \arrow[dd, "p"] \\
& E' \arrow[urr, dashed, "\widehat{\beta}"'] \arrow[dd, "p'"] & & \\
L \arrow[rrr, "\alpha"] \arrow[dr] & & & B \\
& K \arrow[urr, dashed, "\beta"'] & &
\end{tikzcd}$$

Proof. We assume without loss of generality that K is obtained from L by attaching a single n-cell. Since K is assumed to be regular, the characteristic map $\Phi : D^n \longrightarrow K$ for this cell is a homeomorphism onto its image, $\overline{e_n}$. By Corollary 18.2, the restriction $E'_{|\overline{e_n}}$ is trivial, and it also follows that the restriction to the boundary $\Phi(S^{n-1}) = \partial e^n$ of the cell is also trivial. By assumption, we already have a map of bundles $\widehat{\alpha} : E'_{|\partial e^n} \longrightarrow E$. Define a map

$$\lambda : S^{n-1} = \partial e^n \longrightarrow E$$
$$x \mapsto \widehat{\alpha}(x, e)$$

Since E is contractible, and in particular $\pi_{n-1}(E) = 0$, we can extend this to a map

$$\overline{\lambda} : e^n \longrightarrow E.$$

Finally, define a map

$$\Lambda : G \times D^n \longrightarrow E$$
$$(g, y) \mapsto \overline{\lambda}(y) \cdot g$$

We leave it as an exercise to show that this is a bundle map and that it is an extension of $\widehat{\alpha} : G \times S^{n-1} \longrightarrow E$. The main point is that a bundle map is given in the G-coordinate by left multiplication by some element $f_{U,V}$, and left multiplication by a fixed element is equivariant with respect to the right G-action. Now the bundle map $\widehat{\beta}$ is obtained by glueing Λ to the given map $\widehat{\alpha}$. $\square$

Proof. We follow the argument of [Ste2], §19. We assume that X is a regular CW complex. See [D] for a general proof in the paracompact case.

Surjectivity: Let $p : E' \longrightarrow X$ be a principal G-bundle. Using Proposition 19.2, taking $L = \emptyset$ and $K = X$ gives us a map of principle bundles $\widehat{\beta} : E' \longrightarrow E$. Recall from Exercise 17.6 that such a map of bundles corresponds to an isomorphism $E' \cong \beta^* E$.

Injectivity: Let $\lambda : f_0^*(E) \cong f_1^*(E)$ be an isomorphism. Consider the setup of Proposition 19.2 in which $E' \longrightarrow K$ is the bundle $f_0^*(E) \times I \longrightarrow X \times I$ and $L = X \times \{0, 1\}$. Then

$$E'_{|L} = f_0^*(E) \times \{0, 1\} = \Big(f_0^*(E) \times \{0\}\Big) \amalg \Big(f_0^*(E) \times \{1\}\Big).$$

We define $\widehat{\alpha} : f_0^*(E) \times \{0,1\}$ to be $\widehat{f_0}$ at time 0 and the composition $f_0^*(E) \xrightarrow{\widehat{\alpha}} f_1^*(E) \xrightarrow{\widehat{f_1}}$ at time 1. This is a bundle map, so by Proposition 19.2, we get a map of bundles

$$(\widehat{\beta}, \beta) : (f_0^*(E) \times I), X \times I) \longrightarrow (E, B).$$

The map β is the desired homotopy. $\square$

20. Monday, Mar. 2

Proposition 20.1. *Suppose that $p : E \longrightarrow B$ and $p' : E' \longrightarrow B'$ are both universal principal G-bundles. Then $B \simeq B'$.*

Proof. The principal bundle $p : E \longrightarrow B$ must be classified by some $f : B \longrightarrow B'$. That is, $E \cong f^*(E')$. Similarly, we have $E' \cong g^*(E)$ for some $g : B' \longrightarrow B$. Now we have

$$E \cong f^*(E') \cong f^*(g^*(E)) = (gf)^*(E).$$

By the classification theorem, it follows that $gf \simeq \mathrm{id}_B$. A similar argument shows that $fg \simeq \mathrm{id}_{B'}$. $\square$

Example 20.2. We have seen that EG is contractible, so it follows that $p : EG \longrightarrow BG$ is a universal principal G-bundle. For example, $\mathbb{RP}^\infty$ classifies principal $\mathbb{Z}/2\mathbb{Z}$-bundles, and $\mathbb{CP}^\infty$ classifies principal circle-bundles. For example, the Hopf bundle $S^3 \longrightarrow S^2 \cong \mathbb{CP}^1$ is classified by the inclusion $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$.

Example 20.3. According to the classification theorem, principal $\mathbb{Z}/2\mathbb{Z}$-bundles over S^1 are in bijective correspondence with $[S^1, B\mathbb{Z}/2\mathbb{Z}]$. Note that this is *unbased* homotopy classes of maps. Recall that for any path-connected space, $[S^1, X]$ can be identified with the set of conjugacy classes in $\pi_1(X)$. Since $\mathbb{Z}/2\mathbb{Z}$ is abelian, we get

$$[S^1, B\mathbb{Z}/2\mathbb{Z}] \cong \pi_1(B\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$

Thus there are precisely two principal $\mathbb{Z}/2\mathbb{Z}$-bundles over the circle: the trivial one and the double cover $S^1 \longrightarrow S^1$.

Example 20.4. We have already seen that $V_k(\mathbb{R}^n) \longrightarrow \mathrm{Gr}_k(\mathbb{R}^n)$ is a principal $O(k)$-bundle. We claim that when $n = \infty$, this is a universal bundle. To see this, note that we have principal bundles

$$O(n-1) \longrightarrow O(n) \longrightarrow O(n)/O(n-1) \cong S^{n-1}$$

and

$$O(n-k) \longrightarrow O(n) \longrightarrow O(n)/O(n-k) \cong V_k(\mathbb{R}^n).$$

These both follow from the general fact that given a Lie group G and a closed subgroup $H \le G$, the quotient map $G \longrightarrow G/H$ is a principal H-bundle. The proof is very similar to the argument we gave in Proposition 9.1.

Using the long exact sequence in homotopy and the fact that S^{n-1} is $(n-2)$-connected, we find that $\pi_i(O(n-1)) \to \pi_i(O(n))$ is an isomorphism for $i < n-2$ and a surjection for $i = n-2$. It follows that the composition $O(n-k) \hookrightarrow O(n-k+1) \hookrightarrow \ldots \hookrightarrow O(n)$

is an isomorphism for $i < n-k-1$ and a surjection for $i = n-k-1$. Plugging this into the long exact sequence

$$\longrightarrow \pi_i(O(n-k)) \longrightarrow \pi_i(O(n)) \longrightarrow \pi_i(V_k(\mathbb{R}^n)) \longrightarrow \pi_{i-1}(O(n-k)) \longrightarrow \pi_{i-1}(O(n)) \longrightarrow,$$

we see that $V_k(\mathbb{R}^n)$ is $(n-k-1)$-connected. It follows that $V_k(\mathbb{R}^\infty) = \operatorname{colim}_n V_k(\mathbb{R}^n)$ is weakly-contractible. Since it is CW, this implies it is contractible.

21. Wednesday, Mar. 4

Ok, so we know how to classify principal bundles. What about bundles with fiber F? Let $p : E \longrightarrow B$ be a principal G-bundle

Proposition 21.1. *A principal G-bundle is completely determined by its transition functions.*

Proof. Let $p : E \longrightarrow B$ be a principal G-bundle with transition functions $g_{U,V}$. Let $\mathcal{U} = \{U\}$ be a cover of B by trivializing neighborhoods . Define

$$E' = \coprod_{U \in \mathcal{U}} G \times U / \sim$$

where, for $g_1 \in G$ and $x \in U \cap V$, we identify the pair $(g_1, x) \in (G \times U)$ with the pair$(g_{U,V} g_1, x) \in (G \times V)$. The projections $G \times U \longrightarrow U$ glue together to define a map $E' \longrightarrow B$, and this becomes a principal G-bundle with transition maps given by the $g_{U,V}$. Moreover, the maps $\varphi_U^{-1} : G \times U \longrightarrow p^{-1}(U) \subseteq E$ glue together to define an isomorphism of bundles $E' \cong E$. □

Now given a principal G-bundle $p : E \longrightarrow B$ and a left G-space F, we can again form a new bundle

$$E' = \coprod_{U \in \mathcal{U}} F \times U / \sim$$

in the same manner. By construction, the fibers are now F instead of G. This is known as the **associated bundle** with fiber F to the principal bundle $p : E \longrightarrow B$. Another description of this principal bundle is

$$E' = E \times_G F,$$

where the notation $E \times_G F$ means the quotient of $E \times F$ under the relation

$$(y, v) \sim (y \cdot g^{-1}, g \cdot v),$$

using the right G-action on E and the left G-action on F.

The above discussion implies that the associated bundle construction gives a surjection

$$\mathrm{Prin}_G(X) \twoheadrightarrow \mathrm{Bun}_G(X, F).$$

If we further assume that the G-action on F is **effective** (also known as faithful), meaning that the only group element satisfying $g \cdot v = v$ for all $v \in F$ is $g = e$, then we get a bijection. The reason we need this extra hypothesis is that if the action on fibers is not effective, then the transition functions are not uniquely determined by the local trivializations.

Corollary 21.2. *Let $p : E \longrightarrow B$ be a universal principal G-bundle and let X be paracompact. If G acts effectively on F, then pullback of bundles defines a bijection*

$$[X, B] \xrightarrow{\cong} \mathrm{Bun}_G(X, F)$$
$$f \mapsto f^*(E \times_G F)$$

Example 21.3. Consider the $\mathbb{Z}/2\mathbb{Z}$-action on I in which the nonidentity element acts as $t \mapsto 1 - t$. Then the $\mathbb{Z}/2\mathbb{Z}$ bundle with fiber I associated to the double cover $2 : S^1 \longrightarrow S^1$ is the Möbius bundle.

Example 21.4. In the case $G = O(k)$ and $F = \mathbb{R}^k$, the k-plane bundle associated to the principal bundle $V_k(\mathbb{R}^\infty) \longrightarrow \mathrm{Gr}_k(\mathbb{R}^\infty)$ is the tautological bundle (as an $O(k)$-bundle). It follows that the tautological bundle is a universal vector bundle.

Cohomology

We have seen that principal bundles are classified by maps $X \longrightarrow BG$. Suppose now that G is discrete, so that $BG \simeq K(G, 1)$.

For any CW complex X and space Y, there is a $\pi_1(Y)$-action on the set $[X, Y]_*$ of based homotopy classes of based maps (see section 4A of [H]). Furthermore, if Y is path-connected, then there is a bijection [H, Prop 4A.2]

$$[X, Y] \cong ([X, Y]_*)/\pi_1(Y).$$

Considering the case $Y = K(G, 1)$, we get a bijection

$$[X, K(G, 1)] \cong ([X, K(G, 1)]_*)/G.$$

22. Friday, Mar. 6

ANOTHER SNOW DAY...
#yagottabekiddingme

23. Monday, Mar. 9

Proposition 23.1. *Let X be a connected CW complex. Then the assignment*

$$\begin{aligned} [X, K(G, 1)]_* &\longrightarrow \mathrm{Hom}(\pi_1(X), G) \\ f \quad &\mapsto \quad f_* \end{aligned}$$

is a bijection.

Proof. We give a sketch in the case where X has a single 0-cell. See [H, Prop 1B.9] for a complete proof.

We start with surjectivity. Thus let $\varphi : \pi_1(X) \longrightarrow G$ be a homomorphism. We wish to show that $\varphi = f_*$ for some map $f : X \longrightarrow K(G, 1)$. Let $X^1 \subseteq X$ be the 1-skeleton, which is necessarily a wedge of circles. Inclusion of this circle $S^1 \hookrightarrow X^1 \hookrightarrow X$ defines an element $\alpha \in \pi_1(X)$, and we define f on this circle to be any representative of the class $\varphi(\alpha) \in G \cong \pi_1(K(G, 1))$.

In order to extend f over the 2-skeleton X^2, we need to know that for each 2-cell e^2_β, the restriction of f_1 to the boundary of this 2-cell is null homotopic in $K(G, 1)$. Let

$$\gamma : S^1 \hookrightarrow X^1 \hookrightarrow X$$

be the restriction of the characteristic map Φ_β to the boundary. Then γ is null-homotopic in X since it extends to a map from a disk. Thus $\gamma = 0 \in \pi_1(X)$, and $\varphi(\gamma) = 0 \in G$. It follows that f_1 must take this loop in X^1 to a contractible loop in $K(G,1)$, so that f_1 extends over the 2-cell e^2_β.

Now assume that f has already been extended to the n-skeleton, where $n \geq 2$, and let e^{n+1}_δ be an $(n+1)$-cell in X. To extend f over this cell, we again need the restriction of f to the boundary of the cell to be null in $K(G,1)$. But restriction defines an element of $\pi_n(K(G,1)) = 0$, so the extension exists.

For injectivity, suppose that $f_* = f'_*$. It is easy to see that the restrictions f_1 and f'_1 of the two maps to the 1-skeleton must be homotopic, since again the 1-skeleton is just a wedge of circles. Now we have a map

$$h : (X^1 \times I) \cup (X \times \{0,1\}) \longrightarrow K(G,1).$$

Recall that cells of a product $X \times Y$ correspond to products of cells $e^n_\alpha \times e^k_\beta$ of X and Y. We have yet to define h on cells of $X \times I$ of the form $e^n_\alpha \times e^1$, where $n \geq 2$. These cells have dimension $n + 1 \geq 3$. So the argument from above applies again here: the restriction of h to the boundary of such a cell would be an n-cell in $K(G,1)$, where $n \geq 2$, and so is null-homotopic. $\square$

Combining the proposition with what came before, we have

$$[X, K(G,1)] \cong ([X, K(G,1)]_*)/G \cong \mathrm{Hom}(\pi_1(X), G)/G$$

if X is path-connected. The latter quotient here is given by the conjugation action of G on the hom set. Note that when G is abelian, this conjugation is trivial, and we have

$$\mathrm{Hom}_{Gp}(\pi_1(X), G) \cong \mathrm{Hom}_{Ab}(\pi_1(X)_{ab}, G) \cong \mathrm{Hom}_{Ab}(H_1(X), G) \cong H^1(X; G).$$

The first isomorphism is the universal property of abelianization, the second is Hurewicz, and the last is the Universal Coefficient Theorem. To sum up, we have shown

Proposition 23.2. *Suppose that G is abelian and that X is path-connected. Then there are bijections*

$$\mathrm{Prin}_G(X) \cong \mathrm{Hom}(\pi_1(X), G) \cong H^1(X; G).$$

In fact, the statement is also true in the nonabelian case, but requires defining a version of nonabelian cohomology.

Example 23.3. In the case $G = \mathbb{Z}/2\mathbb{Z}$, recall that principal bundles correspond to real line bundles, by changing the fiber. Thus real line bundles are classified by $H^1(-;\mathbb{Z}/2\mathbb{Z})$. The cohomology class corresponding to a line bundle ζ is denoted $w_1(\zeta) \in H^1(-;\mathbb{Z}/2\mathbb{Z})$ and is called the **first Stiefel-Whitney class** of the bundle. This is the beginning of the subject of charateristic classes.

24. Wednesday, Mar. 11

We now consider the analogue for $K(G,n)$ with $n \geq 2$. Since $K(G,n)$ is $(n-1)$-connected, it follows that $H_{n-1}(K(G,n);\mathbb{Z}) \cong 0$. The universal coefficient theorem then gives

$$H^n(K(G,n); G) \cong \mathrm{Hom}(H_n(K(G,n);\mathbb{Z}), G) \cong \mathrm{Hom}(\pi_n(K(G,n)), G) \cong \mathrm{Hom}(G, G).$$

Thus the identity homomorphism $\mathrm{id} : G \longrightarrow G$ corresponds to an element $\iota_n \in H^n(K(G,n);G)$.

Theorem 24.1. *Let X be a connected CW complex. Then the assignment $f \mapsto f^*(\iota_n)$ defines a bijection*

$$[X, K(G,n)] \cong H^n(X;G).$$

Proof. For the case $n = 0$, the space $K(G,n)$ is simply G with the discrete topology, and homotopy classes of maps to G are constant functions, which coincides with $H^0(X;G)$. We already handled the $n = 1$ case above. We thus now assume $n \geq 2$ and that G is abelian. Recall also as discussed above that since $K(G,n)$ is simply connected ($n \geq 2$), then the natural map

$$[X, K(G,n)]_* \longrightarrow [X, K(G,n)]$$

is a bijection, so we work from now on only with based maps.

There are two approaches to this result. The first is very much like the proof of Proposition 23.1 above. That is, think of $\alpha \in H^n(X;G)$ as represented by a cocycle $z_\alpha : C_n(X) \longrightarrow G$. We want to build a map $f : X \longrightarrow K(G,n)$ such that the composition

$$C_n(X) \xrightarrow{f_*} C_n(K(G,n)) \longrightarrow G$$

is cohomologous to z_α. For the purpose of this proof, we may take $K(G,n)$ to be a CW complex with no cells (other than the basepoint) in dimensions below n. In dimension n, the cocyle tells you what to do, and you argue that this can be extended over higher-dimensional cells. The extension to higher cells is part of the story of "obstruction theory".

There is a completely different proof from the point of view of generalized cohomology theories. Recall that a generalized cohomology theory for based CW complexes is a contravariant functor

$$\widetilde{E}^*(-) : CWTop_* \longrightarrow GrAb$$

from based CW complexes and based maps to graded abelian groups satisfying the following axioms:

(**homotopy**) if $f \simeq g$, then $\widetilde{E}(f) = \widetilde{E}(g)$

(**additivity**) for any wedge sum $\bigvee_i X_i$, the inclusions $X_i \hookrightarrow \bigvee_i X_i$ induce an isomorphism

$$\widetilde{E}^*\Big(\bigvee_i X_i\Big) \cong \prod_i \widetilde{E}^*(X_i).$$

(**exactness**) for any based subcomplex $A \subseteq X$, the sequence

$$\widetilde{E}(X/A) \longrightarrow \widetilde{E}(X) \longrightarrow \widetilde{E}(A)$$

is exact.

(**suspension**) for each n, there is a natural isomorphism

$$\sigma^n : \widetilde{E}^n(X) \cong \widetilde{E}^{n+1}(\Sigma X).$$

If the reduced cohomology theory further satisfies the

(**dimension**) $\widetilde{E}^0(S^0) \cong A$ and $\widetilde{E}^n(S^0) = 0$ for $n \neq 0$,

then there is an isomorphism $\widetilde{E}^n(X) \cong \widetilde{H}^n(X;A)$.

We claim that the collection $[-, K(G,n)]_*$, as n varies, defines a reduced cohomology theory. Since each $K(G,n)$ can be built as a topological abelian group, each set $[X, K(G,n)]_*$ inherits the structure of an abelian group. The functoriality is just given by composition: namely, given a "cohomology class" $\alpha : Y \longrightarrow K(G,n)$ and a map $f : X \longrightarrow Y$, we get a class $f^*(\alpha) = \alpha \circ f : X \longrightarrow K(G,n)$. Since we are working with homotopy classes of maps, the **homotopy** axiom holds. The **additivity** axiom follows easily from the universal property of the wedge.

25. Friday, Mar. 13

For the **exactness** axiom, consider the mapping cone $X \cup_A C(A)$. The subcomplex $C(A)$ is contractible, so the quotient

$$X \cup_A C(A) \longrightarrow \Big(X \cup_A C(A)\Big)/C(A) \cong X/A$$

is a homotopy equivalence. But now, for any space Y, the sequence

$$[X \cup_A C(A), Y]_* \longrightarrow [X, Y]_* \longrightarrow [A, Y]_*$$

is exact since a map out of X which is null when restricted to A is precisely the same as a map out of the mapping cone. For the **suspension** axiom, recall that for any based spaces X and Y, we have an adjunction

$$[\Sigma X, Y]_* \cong [X, \Omega Y]_*,$$

where ΩY is the space of based loops in Y. Taking $X = S^n$ shows that $\pi_{n+1}(Y) \cong \pi_n(\Omega Y)$, so that $\Omega K(G, n+1)$ is a $K(G,n)$. Finally, the desired suspension isomorphism is

$$[X, K(G,n)]_* \cong [X, \Omega K(G, n+1)]_* \cong [\Sigma X, K(G, n+1)]_*.$$

We have shown that the collection of functors $[-, K(G,n)]$ determine a reduced cohomology theory. The **dimension** axiom is quickly verified:

$$[S^0, K(G,n)]_* \cong \pi_0(K(G,n)) \cong \begin{cases} G & n = 0 \\ 0 & \text{else.} \end{cases}$$

We conclude that $[X, K(G,n)] \cong \widetilde{H}^n(X; G)$. □

26. Monday, Mar. 23

Remark 26.1. There is an analogue of Theorem 24.1 for homology as well. It states that

$$\widetilde{H}_n(X; A) \cong \operatorname*{colim}_k \pi_{n+k}(X \wedge K(A, k)).$$

Example 26.2. The space $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$ and classifies principal S^1-bundles. By change-of-fiber, this also classifies complex line bundles. Thus complex line bundles are classified by $H^2(-; \mathbb{Z})$. The cohomology class corresponding to a complex line bundle ξ is denoted $c_1(\xi)$ and is called the **first Chern class** of ξ.

The theorem implies that principal $K(G,n)$-bundles over X are classified by $H^{n+1}(X; G)$. But the types of bundles that are studied most often are vector bundles, or, equivalently, $O(n)$-bundles (or $U(n)$-bundles in the complex case). We have $O(1) = \mathbb{Z}/2\mathbb{Z}$, and we have already seen that (real) line bundles are classified by maps into $\mathbb{RP}^\infty$ and also by the cohomology group $H^1(X; \mathbb{Z}/2\mathbb{Z})$.

For $O(k)$-bundles, we have seen in Example 20.4 that $\mathrm{Gr}_k(\mathbb{R}^\infty)$ is a model for $BO(k)$. In order to produce cohomology classes from bundles, we need a calculation of $H^*(BO(k))$.

Consider the inclusion of the diagonal orthogonal matrices $O(1)^k \hookrightarrow O(k)$. Applying our classifying space functor, we get a map of spaces $BO(1)^k \longrightarrow BO(k)$. Since $O(1) \cong \mathbb{Z}/2\mathbb{Z}$, this is a map

$$\lambda : (\mathbb{RP}^\infty)^k \longrightarrow BO(k).$$

Passing to cohomology with $\mathbb{F}_2$-coefficients for convenience, we get a map

$$H^*(BO(k);\mathbb{F}_2) \longrightarrow H^*((\mathbb{RP}^\infty)^k;\mathbb{F}_2) \cong \otimes_k H^*(\mathbb{RP}^\infty;\mathbb{F}_2) = \mathbb{F}_2[x_1,\ldots,x_k],$$

where the x_i are polynomial generators in degree 1. Furthermore, the group of diagonal matrices $O(1)^k$ is closed under conjugation by permutation matrices (reordering the basis elements). This action extends to an action on cohomology, which simply permutes the generators x_i. There is a corresponding action on $H^*(BO(k);\mathbb{F}_2)$.

Lemma 26.3. *Let $g \in G$ and let $c_g : G \longrightarrow G$ be the conjugation homomorphism. Then $Bc_g \simeq \mathrm{id}_{BG}$.*

Sketch. The functions

$$h_i(g_1,\ldots,g_n) = [g_1,\ldots,g_i,g^{-1},c_g(g_{i+1}),\ldots,c_g(g_n)]$$

define a simplicial homotopy on the bar construction. □

It follows that the induced action of the symmetric group Σ_k on $H^*(BO(k);\mathbb{F}_2)$ is trivial, so that λ^* must take values in the Σ_k-fixed subring $\mathbb{F}_2[x_1,\ldots,x_k]^{\Sigma_k}$.

Theorem 26.4. *The invariant subring $\mathbb{F}_2[x_1,\ldots,x_k]^{\Sigma_k}$ is a polynomial algebra on the elementary symmetric functions*

$$\sigma_i(x_1,\ldots,x_k) = \sum_{1\le j_1<\cdots<j_i\le n} x_{j_1}\cdots x_{j_i}.$$

This is a standard (and important) result covered in many algebra textbooks. Let us abbreviate $w_i = \sigma_i(x_1,\ldots,x_k)$. We have now produced an algebra map

$$H^*(BO(k);\mathbb{F}_2) \longrightarrow \mathbb{F}_2[w_1,\ldots,w_k].$$

In fact, this is an isomorphism.

27. Wed, Mar. 25

We will now study this cohomology ring from another perspective. Recall that we have a principal $O(k)$-bundle

$$O(k) \longrightarrow V_k(\mathbb{R}^\infty) \longrightarrow \mathrm{Gr}_k(\mathbb{R}^\infty) \simeq BO(k),$$

with $V_k(\mathbb{R}^\infty) \simeq *$. *Spectral sequences* will give us a method for relating the cohomology groups of a fiber space, total space, and base space in a bundle.

A simple case is the case of a covering space $F \longrightarrow E \longrightarrow B$. Here, F is discrete and therefore only has cohomology in degree 0. Recall that if $p : E \longrightarrow B$ is an n-sheeted covering (so F has n points), there is a **transfer homomorphism** $p_! : \mathrm{H}^i(E) \longrightarrow \mathrm{H}^i(B)$ such that the composition

$$\mathrm{H}^i(B) \xrightarrow{p^*} \mathrm{H}^i(E) \xrightarrow{p_!} \mathrm{H}^i(B)$$

is multiplication by n. The transfer is most easily defined using singular cohomology. We define a map $p^\# : C_i(B) \longrightarrow C_i(E)$ as follows: given a singular n-chain $f : \Delta^i \longrightarrow B$, we have n different lifts $\tilde{f}_j : \Delta^i \longrightarrow E$, for $j = 1, \dots, n$, one lift for each of the n lifts of the initial vertex of Δ^i. Then $p^\#(f) = \sum_j \tilde{f}_j$. Dualizing gives a map $p_\# : C^i(E) \longrightarrow C^i(B)$ which descends to give the transfer map in cohomology. Since each $\tilde{f}_j$ is a lift of f, it follows that $p \circ \tilde{f}_j = f$, so that the composition

$$C_i(B) \xrightarrow{p^\#} C_i(E) \xrightarrow{p_*} C_i(B)$$

is $f \mapsto n \cdot f$. It then follows that $p_! \circ p^* = n \cdot \mathrm{id}$ in $\mathrm{H}^i(B)$.

Example 27.1. Consider the universal cover $EG \longrightarrow BG$, where G is a finite group. Let $n = |G|$. Then this is an n-sheeted cover. By the above, we have that the composition

$$\mathrm{H}^i(BG) \xrightarrow{p^*} \mathrm{H}^i(EG) \xrightarrow{p_!} \mathrm{H}^i(BG)$$

is multiplication by n. But $EG \simeq *$, so $\mathrm{H}^i(EG) = 0$ if $i > 0$. It follows that all higher cohomology groups of BG are n-torsion. Examples of this are

$$\mathrm{H}^i(BC_2; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd}, \end{cases}$$

or more generally

$$\mathrm{H}^i(BC_n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd}. \end{cases}$$

Corollary 27.2. *Let G be a finite group and let k be a field of characteristic 0 or p, where p does not divide $|G|$. Then $\mathrm{H}^i(BG; k) = 0$ for $i > 0$.*

Proof. Consider the covering $EG \longrightarrow BG$. We know that the composition

$$\mathrm{H}^i(BG; k) \xrightarrow{p^*} \mathrm{H}^i(EG; k) \xrightarrow{p_!} \mathrm{H}^*(BG; k)$$

is multiplication by $|G|$. By hypothesis, $|G| \neq 0$ in k, so the composition is an isomorphism. On the other hand, $\mathrm{H}^i(EG; k) = 0$, so the result follows. □

Spectral Sequences

A short introduction to spectral sequences is given in [C]. Other nice sources are [BT], [H3], and [McC].

One typical situation in which a spectral sequence arises is the computation of the homology of a chain complex which is equipped with a filtration. Let

$$\dots C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} C_{n-2} \xrightarrow{d} \dots \xrightarrow{d} C_0$$

be a chain complex, and let $F^0 \subseteq F^1 \subseteq F^2 \subseteq \cdots \subseteq C_*$ be an increasing sequences of subcomplexes. It follows that each quotient F^i/F^{i-1} inherits a differential from F^i and is therefore a complex. This quotient, called the **associated graded**, is often denoted $\mathrm{gr}^i_F(C_*)$. The hope or goal is to somehow

Goal: Recover the groups $\mathrm{H}_i(C_*)$ from the simpler groups $\mathrm{H}_j(\mathrm{gr}^k C_*)$.

Let's start by thinking about a 1-step filtration $F^0 \subseteq F^1 = C_*$. In this case we only have

$$\mathrm{gr}^0(C_*) = F^0, \qquad \text{and} \qquad \mathrm{gr}^1(C_*) = C_*/F^0.$$

The short exact sequence

$$0 \longrightarrow F^0C_* \longrightarrow C_* \longrightarrow C_*/F^0C_* \longrightarrow 0$$

gives rise to a long exact sequence

$$\dots \longrightarrow \mathrm{H}_i(F^0) \longrightarrow \mathrm{H}_i(C_*) \longrightarrow \mathrm{H}_i(C_*/F^0) \xrightarrow{\partial} \mathrm{H}_{i-1}(F^0) \longrightarrow \dots$$

Thus, if we can understand the homology of the associated graded pieces and the connecting homomorphism between them, we can perhaps recover the homology of the original complex C_*.

28. Fri, Mar. 27

We finished last time by talking about the homology of a complex equipped with a 1-step filtration. Let us describe this in a slightly different way that will generalize to the spectral sequence context.

If we define

$$E^0_{p,q} := F^pC_{p+q}/F^{p-1}C_{p+q},$$

then the differential on the associated graded complex takes the form

$$d^0 : E^0_{p,q} \longrightarrow E^0_{p,q-1}.$$

The index p is called the **filtration degree**, and q is the **complementary degree**. The usual degree is $p+q$, referred to here as the **total degree**.

We can take homology of the complex $(E^0_{*,*}, d^0)$, which we write as $E^1_{*,*}$. That is, $E^1_{p,q}$ will be the homology at $E_0^{p,q}$. Typically, $E^1_{*,*}$ will not capture the homology groups $\mathrm{H}_*(C_*)$, and there will be a further differential

$$d^1 : E^1_{p,q} \longrightarrow E^1_{p-1,q}.$$

We can again take homology to define groups $E^2_{p,q}$. In general, we continue in this way, producing groups $E^n_{p,q}$ (said to be in the n**th page** of the spectral sequence) and differentials

$$d^n : E^n_{p,q} \longrightarrow E^n_{p-n,q+n-1}.$$

For any particular pair (p,q), these groups will stabilize, and we write $E^\infty_{p,q}$ for this stable value.

Consider the case of a one-step filtration $F^0 \subseteq F^1 = C_*$. Then E^0 is concentrated in two columns, with

$$E^0_{0,q} = F^0C_q, \qquad E^0_{1,q} = C_{1+q}/F^0C_{1+q}.$$

$$E^0:\quad \begin{array}{|c|c|c|c|} \hline F^0C_3 & \frac{F^1C_4}{F^0C_4} & 0 & 0 \\ \downarrow & \downarrow & & \\ F^0C_2 & \frac{F^1C_3}{F^0C_3} & 0 & 0 \\ \downarrow & \downarrow & & \\ F^0C_1 & \frac{F^1C_2}{F^0C_2} & 0 & 0 \\ \downarrow & \downarrow & & \\ F^0C_0 & \frac{F^1C_1}{F^0C_1} & 0 & 0 \\ \hline \end{array} \qquad E^1:\quad \begin{array}{|cc|c|c|} \hline E^1_{0,3} \leftarrow & E^1_{1,3} & 0 & 0 \\ \hline E^1_{0,2} \leftarrow & E^1_{1,2} & 0 & 0 \\ \hline E^1_{0,1} \leftarrow & E^1_{1,1} & 0 & 0 \\ \hline E^1_{0,0} \leftarrow & E^1_{1,0} & 0 & 0 \\ \hline \end{array}$$

The d^1-differential is just the boundary homomorphism in the long exact sequence associated to the short exact sequence

$$0 \longrightarrow F^0C_* \longrightarrow C_* \longrightarrow C_*/F^0C_* \longrightarrow 0.$$

In the case of a one-step filtration, the spectral sequence contains the same information as this long exact sequence. If we were to continue the spectral sequence, at the E^2-stage, the d^2-differentials would move two steps to the left and one step up, so there would simply be no room to have a nontrivial differential. The same is true in all later stages, so we say that the spectral sequence **collapses** or degenerates at E^2, and we write $E^2 = E^\infty$. The goal of the computation was $\mathrm{H}_*(C_*)$, and (*assuming we are working over a field*) we recover this via

$$\mathrm{H}_n(C_*) \cong E^2_{0,n} \oplus E^2_{1,n-1}.$$

Again, this is the same decomposition we would get from the long exact sequence

$$\longrightarrow \mathrm{H}_n(F^0C_*) \longrightarrow \mathrm{H}_n(C_*) \longrightarrow \mathrm{H}_n(C_*/F^0C_*) \longrightarrow$$

To see how the general story goes, let's consider next a two-step filtration $F^0 \subseteq F^1 \subseteq F^2$. Again, the goal is to recover $\mathrm{H}_*(C_*)$ from the homology of the associated graded pieces. We define $E^1_{p,q}$ as before, and d^1 is as defined in the diagram

$$\begin{tikzcd}
 & & \mathrm{H}_n(F^1/F^0) \arrow[d, "d^1"'] \\
 & & \mathrm{H}_{n-1}(F^0) \arrow[d, "\iota_*"] \\
\mathrm{H}_n(C_*) \arrow[r] & \mathrm{H}_n(C_*/F^1C_*) \arrow[r, "\partial"] \arrow[dr, dashed, "d^1"'] & \mathrm{H}_{n-1}(F^1) \arrow[d] \\
 & & \mathrm{H}_{n-1}(F^1/F^0) \arrow[d, "d^1"'] \\
 & & \mathrm{H}_{n-2}(F^0)
\end{tikzcd}$$

In order to define $d^2 : E^2_{2,n} \longrightarrow E^2_{0,n+1}$, consider $x \in E^1_{2,n}$ such that $d^1(x) = 0 \in \mathrm{H}_{n+1}(F^1/F^0)$. By exactness of the column on the right, this means that $\partial(x) = \iota_*(z)$ for some $z \in \mathrm{H}_{n+1}(F^0)$. This z is only well-defined up to the image of $d^1 : \mathrm{H}_n(F^1/F^0) \longrightarrow \mathrm{H}_{n-1}(F^0)$.

Thus we have defined

$$\begin{array}{ccc} (\ker d^1 : E^1_{2,n} & \longrightarrow E^1_{1,n}) \xrightarrow{d^2} & E^1_{0,n+1}/d^1(E^1_{1,n+1}) \\ \| & & \| \\ E^2_{2,n} & & E^2_{0,n+1} \end{array}$$

Here, the picture is

$$E^0: \begin{array}{|c|c|c|c|} \hline F^0C_3 & \frac{F^1C_4}{F^0C_4} & \frac{F^2C_4}{F^1C_4} & 0 \\ \downarrow & \downarrow & \downarrow & \\ F^0C_2 & \frac{F^1C_3}{F^0C_3} & \frac{F^2C_3}{F^1C_3} & 0 \\ \downarrow & \downarrow & \downarrow & \\ F^0C_1 & \frac{F^1C_2}{F^0C_2} & \frac{F^2C_2}{F^1C_2} & 0 \\ \downarrow & \downarrow & \downarrow & \\ F^0C_0 & \frac{F^1C_1}{F^0C_1} & \frac{F^2C_1}{F^1C_1} & 0 \\ \hline \end{array} \qquad E^1: \begin{array}{|c|c|c|c|} \hline E^1_{0,3} \leftarrow & E^1_{1,3} \leftarrow & E^1_{2,3} & 0 \\ \hline E^1_{0,2} \leftarrow & E^1_{1,2} \leftarrow & E^1_{2,2} & 0 \\ \hline E^1_{0,1} \leftarrow & E^1_{1,1} \leftarrow & E^1_{2,1} & 0 \\ \hline E^1_{0,0} \leftarrow & E^1_{1,0} \leftarrow & E^1_{2,0} & 0 \\ \hline \end{array} \qquad E^2: \begin{array}{|c|c|c|c|} \hline E^2_{0,3} & E^2_{1,3} & E^2_{2,3} & 0 \\ \hline E^2_{0,2} & E^2_{1,2} & E^2_{2,2} & 0 \\ \hline E^2_{0,1} & E^2_{1,1} & E^2_{2,1} & 0 \\ \hline E^2_{0,0} & E^2_{1,0} & E^2_{2,0} & 0 \\ \hline \end{array}$$

This idea generalizes to define the higher differentials.

29. Mon, Mar. 30

Again, if the groups $E^r_{p,q}$ stabilize (this will always happen for a finite filtration), then we write $E^\infty_{p,q}$ for these stable values. This is the final output of the spectral sequence. To relate this to our goal of computing $\mathrm{H}_*(C_*)$, note that we can define a filtration on $\mathrm{H}_*(C_*)$ by defining $F^p\mathrm{H}_*(C_*)$ to be the image of the map $\mathrm{H}_*(F^pC_*) \longrightarrow \mathrm{H}_*(C_*)$. Then typically we will have

$$E^\infty_{p,q} \cong F^p\mathrm{H}_{p+q}(C_*)/F^{p-1}\mathrm{H}_{p+q}(C_*).$$

Thus we recover some associated-graded version of the homology groups we were after. If we are working throughout with coefficients in a field, so that all homology groups are vector spaces, then we get the homology vector spaces by taking the direct sum of the graded pieces. But in general, this is a nontrivial problem.

Example 29.1. Consider the chain complex defined by

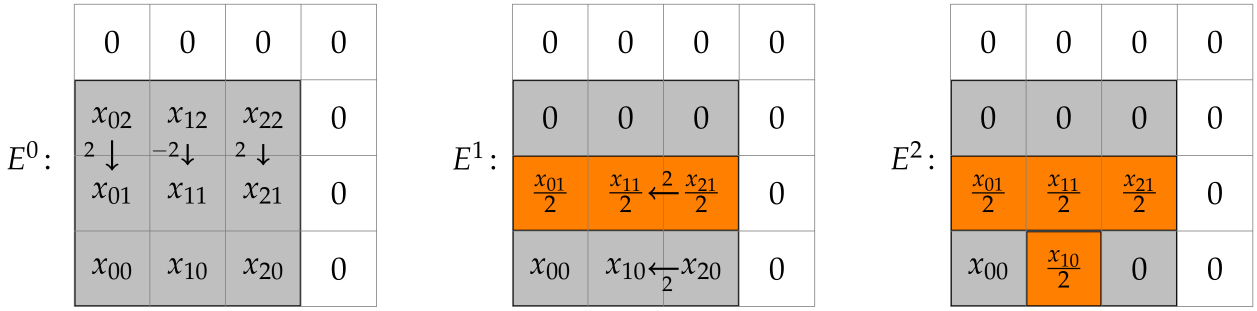
$$C_0 = \mathbb{Z}\{x_{00}\}, \quad C_1 = \mathbb{Z}\{x_{10}, x_{01}\}, \quad C_2 = \mathbb{Z}\{x_{20}, x_{11}, x_{02}\},$$

$$C_3 = \mathbb{Z}\{x_{21}, x_{12}\}, \quad C_4 = \mathbb{Z}\{x_{22}\}$$

and differentials as specified in

$$C_4 \xrightarrow{\begin{pmatrix} 2 \\ 2 \end{pmatrix}} C_3 \xrightarrow{\begin{pmatrix} 0 & 0 \\ 2 & -2 \\ 0 & 0 \end{pmatrix}} C_2 \xrightarrow{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}} C_1 \xrightarrow{0} C_0.$$

This is the cellular chain complex for $\mathbb{RP}^2 \times \mathbb{RP}^2$. We introduce a filtration by the first subscript. Thus F^0 only contains the span of $\{x_{00}, x_{01}, x_{02}\}$. Next, F^1 further includes the generators x_{10}, x_{11}, and x_{12}. Finally, $F^2 = C_*$. This is a two-step filtration.



Here every square that is filled in orange gives a copy of $\mathbb{Z}/2\mathbb{Z}$. On the E^2-page, there is no room for d^2-differentials or any higher differentials. It follows that $E^2 = E^\infty$. We read off

$$\mathrm{H}_0(C_*) \cong \mathbb{Z}, \qquad \mathrm{H}_2(C_*) \cong \mathbb{Z}/2\mathbb{Z}, \qquad \mathrm{H}_3(C_*) \cong \mathbb{Z}/2\mathbb{Z}.$$

For $\mathrm{H}_1(C_*)$, there are two pieces that contribute, $E^2_{1,0}$ and $E^2_{0,1}$, and we get a short exact sequence

$$0 \longrightarrow E^2_{0,1} \longrightarrow \mathrm{H}_1(C_*) \longrightarrow E^2_{1,0} \longrightarrow 0.$$

In this case, it turns out that the short exact sequence splits (though we don't see this automatically from the spectral sequence), so that $\mathrm{H}_1(C_*) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. These are the nonzero homology groups of $\mathbb{RP}^2 \times \mathbb{RP}^2$, as can be verified from the Kunneth formula. The fact that there was no d^2 differential in this example corresponds to the fact that there was no differential on any class in C_* that lowered filtration by 2.

The issue we had with $\mathrm{H}_1(C_*)$ in this example comes up often in spectral sequences, and is what is called an "extension problem". If we work with field coefficients, these problems go away, since every short exact sequence of vector spaces automatically splits.

30. Wed, Apr. 1

There is another approach to spectral sequences, using what is known as an **exact couple**. This is a pair of abelian groups D and E, together with homomorphisms making the diagram

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

exact at each vertex. Then the composite $E \xrightarrow{k} D \xrightarrow{j} E$ defines a differential on E, since the composition

$$E \xrightarrow{k} D \xrightarrow{j} E \xrightarrow{k} D \xrightarrow{j}$$

is zero. Define $E' = \mathrm{H}_*(E, jk)$ and define $D' = i(D) = \ker(j)$. Then define

$$i' : D' \longrightarrow D', \qquad j' : D' \longrightarrow E', \qquad k' : E' \longrightarrow D'$$

as follows. The homomorphism i' is simply the restriction of i to $D' = i(D)$.

Next, j' is defined by the formula $j'(i(x)) = j(x)$. To see that this is well defined, suppose that $i(w) = 0$. We wish to see that $j(w) = 0$ in E'. By exactness, we know that

$w = k(z)$ for some $z \in E$. So now $j(w) = jk(z)$, which is a boundary under the differential jk on E.

Finally, for a cycle $z \in E$ representing a homology class, we define $k'([z]) = k(z)$. Since z is a cycle, this means that $jk(z) = 0$. By exactness, this means that $k(z) = i(w)$ for some $w \in D$. In other words, $k(z)$ is an element of D'. To see that k' is well-defined, assume that z is a boundary, so that $z = jk(y)$ for some $y \in E$. Then $k'(z) = k(z) = k(jk(y)) = 0$. We conclude that k' is well-defined.

Lemma 30.1. *The 5-tuple (D', E', i', j', k') again defines an exact couple.*

This is called the **derived couple** associated to the exact couple (D, E).

Typically, the groups D and E will be bigraded. Suppose that the bidegrees of the maps i, j, and k are given by

$$|i| = (1, -1), \qquad |j| = (0, 0), \qquad |k| = (-1, 0).$$

Proposition 30.2. *Let (D, E, i, j, k) be as described above. Then successively taking the derived couple yields a spectral sequence, with*

$$E^r_{p,q} = E^{(r-1)}_{p,q}, \text{the } (r-1)\text{-fold derived group.}$$

The differential d^r is the composition $j^{(r-1)}k^{(r-1)}$.

One important early application of this idea is the Serre spectral sequence. For a CW complex B, we write B^p for the p-skeleton.

Proposition 30.3 (Serre spectral sequence)**.** *Let $E \xrightarrow{p} B$ be a (Serre) fibration. Then there is a spectral sequence*

$$E^1_{s,t} = \mathrm{H}_{s+t}(p^{-1}(B^s), p^{-1}(B^{s-1}); A) \Rightarrow \mathrm{H}_{s+t}(E; A).$$

The notation means that the spectral sequence converges to the homology groups of E. Again, this means that the E^∞ terms are the associated-graded for some filtration on $\mathrm{H}_*(E; A)$.

As stated thus far, this is not so useful. What we typically want for a spectral sequence is a well-understood E^2-term.

Theorem 30.4 (Serre spectral sequence)**.** *Let $F \to E \xrightarrow{p} B$ be a (Serre) fibration (see p. 18). If B is simply-connected, then in the Serre spectral sequence there is an isomorphism*

$$E^2_{s,t} \cong \mathrm{H}_s(B; \mathrm{H}_t(F; A)).$$

Before we discuss the proof, we look at some examples.

Example 30.5. Suppose we have a fiber sequence $S^n \longrightarrow S^k \longrightarrow S^j$, with $j \geq 2$. What can we conclude about the numbers n, k, and j? We think of a fibration as some sort of twisted product, so we at least expect $k = n + j$. From the Serre spectral sequence, we know the E^2-term is

$$E^2: \quad \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline x_n & 0 & 0 & x_n \otimes y_j \\ \hline 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & y_j \\ \hline \end{array}$$

The class $x_n \otimes y_j$ cannot support any differentials (we call such a class a **permanent cycle**), and it also cannot be the target of any differentials. It follows that this must contribute to the top degree homology of the total space S^k, so $k = n + j$. We know that we cannot have homology in intermediate degrees, so the classes x_n and y_j cannot survive the spectral sequence. The only way this can happen is if $d^j(y_j) = \pm x_n$, which would force $n = j - 1$. So we conclude that the only possible such fiber sequences take the form

$$S^{j-1} \longrightarrow S^{2j-1} \longrightarrow S^j.$$

The Hopf fibration η gives an example when $j = 2$, and there are analogous Hopf fibrations ν and σ when $j = 4$ and $j = 8$.

31. Fri, Apr. 3

We now sketch the proofs of the results announced last time regarding the Serre spectral sequence, mostly following [H3].

Proof of Proposition 30.3. To simplify notation, we write $Y^s := p^{-1}(B^s)$. We form an exact couple, taking

$$D_{s,t} = \mathrm{H}_{s+t}(Y^s; A) \qquad \text{and} \qquad E_{s,t} = \mathrm{H}_{s+t}(Y^s, Y^{s-1}; A).$$

The map

$$E_{s,t} = \mathrm{H}_{s+t}(Y^s, Y^{s-1}; A) \xrightarrow{k} \mathrm{H}_{s+t-1}(Y^{s-1}; A) = D_{s-1,t}$$

is the connecting homomorphism. The map

$$D_{s,t} = \mathrm{H}_{s+t}(Y^s; A) \xrightarrow{i} \mathrm{H}_{s+t}(Y^{s+1}; A) = D_{s+1,t-1}$$

is induced by the inclusion $Y^s \longrightarrow Y^{s+1}$. Similarly, the map

$$D_{s,t} = \mathrm{H}_{s+t}(Y^s; A) \xrightarrow{j} \mathrm{H}_{s+t}(Y^s, Y^{s-1}; A) = D_{s+1,t-1}$$

is the map in the long exact sequence for the pair (Y^s, Y^{s-1}).

It remains to show that $E^\infty_{s,t}$ is isomorphic to $F^s\mathrm{H}_{s+t}(E; A)/F^{s-1}\mathrm{H}_{s+t}(E; A)$ for some filtration on $\mathrm{H}_{s+t}(E; A)$. We define the filtration by

$$F^s\mathrm{H}_{s+t}(E; A) := \mathrm{im}\left(\mathrm{H}_{s+t}(Y^s; A) \longrightarrow \mathrm{H}_{s+t}(E; A)\right).$$

Recall that by definition, $D^r_{s,t}$ is the image of

$$i^{r-1} : \mathrm{H}_{s+t}(Y^{s-r+1}; A) \longrightarrow \mathrm{H}_{s+t}(Y^s; A) = D^1_{s,t}.$$

Thus, for r large enough and s and t fixed, these groups are 0. Consider the exact sequence

$$D^r_{s+r-2,t-r+2} \xrightarrow{i_r} D^r_{s+r-1,t-r+1} \xrightarrow{j_r} E^r_{s+t} \xrightarrow{k_r} D^r_{s-1,t}.$$

For s and t fixed and r large enough, we know that the group on the right is 0. Thus E^r_{s+t} is a quotient of $D^r_{s+r-1,t-r+1}$. But this latter group is the image of $\mathrm{H}_{s+t}(Y^s; A)$ in $\mathrm{H}_{s+t}(Y^{s+r-1}; A)$. We claim that for r large enough, we have

$$\mathrm{H}_{s+t}(Y^{s+r-1}; A) \cong \mathrm{H}_{s+t}(Y^{\infty}; A) = \mathrm{H}_{s+t}(E; A).$$

We have shown that $E^r_{s,t}$, for r large enough, is the associated graded in the desired filtration on $\mathrm{H}_{s+t}(E; A)$. To verify the claim, note that $B^{s+r-1} \hookrightarrow B$ induces an isomorphism in homotopy below level $s+r-1$ and a surjection at level $s+r-1$. By a problem from HW4, it follows that the same is true of $Y^{s+r-1} \hookrightarrow Y^{\infty} = E$. By a relative version of the Hurewicz theorem, it follows that the induced map on homology is also an isomorphism below level $s+r-1$. Taking $r > t+1$ gives the claim. $\square$

Sketch of Theorem 30.4. We will identity the E^2-term with the *cellular* homology of B, with coefficients in the homology of F. Recall that the cellular chains may be defined as

$$C_s(B; A) := \mathrm{H}_s(B^s, B^{s-1}; \mathbb{Z}) \otimes A,$$

where the relative homology group is defined using singular homology. The differential is induced by

$$\mathrm{H}_s(B^s, B^{s-1}; \mathbb{Z}) \xrightarrow{\partial} \mathrm{H}_{s-1}(B^{s-1}; \mathbb{Z}) \longrightarrow \mathrm{H}_{s-1}(B^{s-1}, B^{s-2}; \mathbb{Z}).$$

To prove the theorem, it suffices to produce isomorphisms

$$\begin{tikzcd}
\mathrm{H}_{s+t}(Y^s, Y^{s-1}; A) \arrow[r, "d^1"] \arrow[d, "\Lambda_{s,t}"'] & \mathrm{H}_{s+t-1}(Y^{s-1}, Y^{s-2}; A) \arrow[d, "\Lambda_{s-1,t}"] \\
\mathrm{H}_s(B^s, B^{s-1}; \mathbb{Z}) \otimes \mathrm{H}_t(F; A) \arrow[r, "\partial_{\mathrm{cell}}"'] & \mathrm{H}_{s-1}(B^{s-1}, B^{s-2}; \mathbb{Z}) \otimes \mathrm{H}_t(F; A)
\end{tikzcd}$$

making the square commute. Recall that the group $\mathrm{H}_s(B^s, B^{s-1}; \mathbb{Z})$ of cellular s-chains on B is a free abelian group on the set of s-cells of B. Thus the bottom row of the square could also be written

$$\bigoplus_{\alpha \in \mathrm{Cells}_s(B)} \mathrm{H}_t(F; A) \xrightarrow[\partial_{\mathrm{cell}}]{} \bigoplus_{\beta \in \mathrm{Cells}_{s-1}(B)} \mathrm{H}_t(F; A)\ .$$

For each s-cell e^s_α in B, we have a characteristic map $\Phi_\alpha : D^s \longrightarrow B^s$, and by a homework problem, we may pull back the fibration p along Φ_α to obtain a fibration

$$\begin{tikzcd}
\widetilde{D^s_\alpha} = \Phi_\alpha^{-1}(p) \arrow[r, "\widetilde{\Phi_\alpha}"] \arrow[d, "\widetilde{p}"'] & Y^s \arrow[d, "p"] \\
D^s \arrow[r, "\Phi_\alpha"'] & B^s.
\end{tikzcd}$$

By an excision argument, the map of pairs

$$\coprod_{\alpha}(\widetilde{D^s_\alpha}, \widetilde{S^{s-1}_\alpha}) \longrightarrow (Y^s, Y^{s-1})$$

induces an isomorphism

$$\bigoplus_{\alpha} \mathrm{H}_{s+t}(\widetilde{D^s_\alpha}, \widetilde{S^{s-1}_\alpha}; A) \xrightarrow{\cong} \mathrm{H}_{s+t}(Y^s, Y^{s-1}; A).$$

Next, if $F_\alpha \subseteq \widetilde{D^s_\alpha}$ denotes $\widetilde{p}^{-1}(\mathbf{e}_1)$ and $\widetilde{D^{s-1}_{\alpha,+}}$ denotes the preimage $\widetilde{p}^{-1}(D^s_+)$ of a hemisphere, there is a chain of isomorphisms

$$\mathrm{H}_{s+t}(\widetilde{D^s_\alpha}, \widetilde{S^{s-1}_\alpha}; A) \cong \mathrm{H}_{s+t-1}(\widetilde{D^{s-1}_{\alpha,+}}, \widetilde{S^{s-2}_\alpha}; A) \cong \ldots \cong \mathrm{H}_t(F_\alpha; A).$$

Now recall from the homework that a choice of path $\gamma\Phi_\alpha(\mathbf{e}_1) \rightsquigarrow b_0$ yields a homotopy equivalence $F_\alpha \simeq F$. Different choices of paths may yield different choices of equivalences, but *since B is assumed to be simply-connected*, any two paths are homotopic, and those any two equivalences are homotopic. It follows that we get a well-defined isomorphism $\mathrm{H}_t(F_\alpha; A) \cong \mathrm{H}_t(F; A)$. Putting all of this together gives

$$\begin{aligned}\Lambda_{s,t} : \mathrm{H}_{s+t}(Y^s, Y^{s-1}; A) \cong \bigoplus_{\alpha} \mathrm{H}_{s+t}(\widetilde{D^s_\alpha}, \widetilde{S^{s-1}_\alpha}; A) \cong \bigoplus_{\alpha} \mathrm{H}_t(F_\alpha; A) &\cong \bigoplus_{\alpha} \mathrm{H}_t(F; A) \\ &\cong \mathrm{H}_s(B^s, B^{s-1}; \mathbb{Z}) \otimes \mathrm{H}_t(F; A).\end{aligned}$$

To see that d^1 agrees with the cellular differential, recall that d^1 is defined to be the composition

$$\mathrm{H}_{s+t}(Y^s, Y^{s-1}; A) \xrightarrow{\partial} \mathrm{H}_{s+t-1}(Y^{s-1}; A) \longrightarrow \mathrm{H}_{s+t-1}(Y^{s-1}, Y^{s-2}; A),$$

which is the same formula we wrote down for the cellular differential above. $\square$

Remark 31.1. The condition that B is simply-connected is stronger than what is needed. Glancing back at the sketch given above, this assumption was only used to obtain a well-defined isomorphism $\mathrm{H}_t(F_\alpha; A) \cong \mathrm{H}_t(F; A)$. For any path-connected B, a loop (at the basepoint) in B gives rise to a self-homotopy equivalence $F \simeq F$ and thus an automorphism $\mathrm{H}_t(F; A) \cong \mathrm{H}_t(F; A)$. In other words, we have an action of $\pi_1(B)$ on the homology groups of the fiber. What we need to make the above argument work is that this action of $\pi_1(B)$ on the homology groups $\mathrm{H}_t(F; A)$ is trivial.

32. Monday, Apr. 6

We have given the homological version of the Serre spectral sequence, but there is an analogous spectral sequence in cohomology. In a cohomological spectral sequence, we write $E_r^{p,q}$, and the differentials take the form

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}.$$

The cohomological Serre spectral sequence has the form

$$E_2^{s,t} \cong \mathrm{H}^s(B; \mathrm{H}^t(F; A)) \Rightarrow \mathrm{H}^{s+t}(E; A).$$

What's more, if we take coefficients in a commutative ring R, so that we have cup products, then this turns out to be a spectral sequence of algebras. That is, each page $E_r^{*,*}$ has a (bigraded) algebra structure, and each differential d_r satisfies the **Leibniz rule**

$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^{|x|} x \cdot d_r(y).$$

Remark 32.1. Keeping track of signs in homological algebra can be a nasty business, but there is a principle that tends to work well:

> **Koszul sign rule:** Whenever you permute an element a past an element x, you should introduce the sign $(-1)^{|a||x|}$.

For example, in the above Leibniz rule, in the second term we have moved the d_r, which is a degree 1 map, past the element x. We therefore decorate that term with the sign $(-1)^{1\cdot|x|} = (-1)^{|x|}$.

Example 32.2. To see how the machine works, we apply this to the fiber sequence

$$S^1 \longrightarrow S^\infty \longrightarrow \mathbb{CP}^\infty.$$

In this example, we will deduce the cohomology (ring) of $\mathbb{CP}^\infty$ from scratch. We know that $\mathbb{CP}^\infty$ is simply-connected, so we know the E_2 of the cohomology spectral sequence is

$$E_2^{s,t} \cong \mathrm{H}^s(\mathbb{CP}^\infty; \mathrm{H}^t(S^1;\mathbb{Z})) \cong \begin{cases} \mathrm{H}^s(\mathbb{CP}^\infty;\mathbb{Z}) & t = 0,1 \\ 0 & \text{otherwise.} \end{cases}$$

We know the spectral sequence converges to $\mathrm{H}^*(S^\infty;\mathbb{Z})$, which is concentrated in degree 0. Note that we must have

$$\mathrm{H}^1(\mathbb{CP}^n;\mathbb{Z}) \cong E_2^{1,0} = 0$$

since $E_2^{1,0} = E_\infty^{1,0}$ for degree reasons. Next, we know that

$$\mathrm{H}^1(S^\infty;\mathbb{Z}) \cong \mathrm{H}^2(S^\infty;\mathbb{Z}) \cong 0,$$

so the differential

$$d_2 : \mathbb{Z} \cong E_2^{0,1} \longrightarrow E_2^{2,0}$$

must be an isomorphism. Let y be a generator of $E_2^{0,1} \cong \mathrm{H}^1(S^1;\mathbb{Z})$ and let x be the corresponding generator for $\mathrm{H}^2(\mathbb{CP}^n;\mathbb{Z}) \cong E_2^{2,0}$. The class $x \otimes y$ generates $E_2^{2,1}$, and again the differential $d_2 : E_2^{2,1} \longrightarrow E_2^{4,0}$ must be an isomorphism. On the other hand, the Leibniz rule gives

$$d_2(xy) = d_2(x)y + xd_2(y) = 0 + x^2 = x^2.$$

Since d_2 is an isomorphism, we conclude that x^2 is a generator for $E_2^{4,0} \cong \mathrm{H}^4(\mathbb{CP}^\infty;\mathbb{Z})$. We can continue in this way and find that x^n is a generator for $\mathrm{H}^{2n}(\mathbb{CP}^\infty;\mathbb{Z})$. The E_2-page is given by

$$E_2 : \quad \begin{tikzcd} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ y \arrow[drr] & 0 & xy \arrow[drr] & 0 & x^2y \arrow[drr] & 0 & x^3y & \cdots \\ 1 & 0 & x & 0 & x^2 & 0 & x^3 & \cdots \end{tikzcd}$$

We conclude that

$$\mathrm{H}^*(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}[x], \qquad \deg(x) = 2.$$

A similar analysis shows that

$$\mathrm{H}^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[x]/x^{n+1}, \qquad \deg(x) = 2.$$

Again, $\mathbb{CP}^\infty$ is a classifying space for complex line bundles, and knowledge of its cohomology gives us understanding of characteristic classes (here, the first Chern class). We would like to do the same for classifying spaces $BU(n)$ for complex n-plane bundles and $BO(n)$ for real n-plane bundles. We start with the complex case.

The strategy is to consider the fibration (bundle) $G \longrightarrow EG \longrightarrow BG$. Since $EG \simeq *$, the Serre spectral sequence relates the cohomology of BG to the cohomology of G. So our first step is to calculate $\mathrm{H}^*(U(n);\mathbb{Z})$.

Proposition 32.3. *$\mathrm{H}^*(U(n);\mathbb{Z}) \cong E(x_1, x_3, x_5, \ldots, x_{2n-1})$ for $n \geq 2$, where $\deg(x_j) = j$.*

In the statement of the proposition, $E(x_1, x_3, x_5, \ldots, x_{2n-1})$ means the *exterior algebra* generated by the classes $x_1, x_3, x_5, \ldots, x_{2n-1}$. By definition, this is the quotient of the polynomial algebra on these classes by the ideal $(x_1^2, x_3^2, x_5^2, \ldots, x_{2n-1}^2)$.

33. Wed, Apr. 8

Proof. We prove this by induction. In the base case $n = 1$, recall that $U(1) \cong S^1$, so that the cohomology ring is exterior on the class x_1. Now assume by induction that we know

$$\mathrm{H}^*(U(n-1);\mathbb{Z}) \cong E(x_1, x_3, \ldots, x_{2n-3}).$$

Much like in the (real) orthogonal case, we have a transitive action of $U(n)$ on S^{2n-1} with stabilizer $U(n-1) \subseteq U(n)$. It follows that we have a fibration sequence

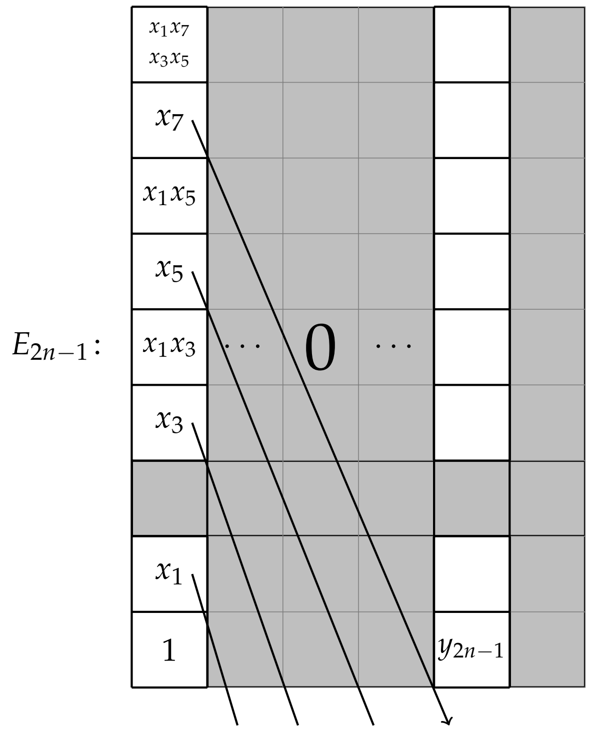
$$U(n-1) \longrightarrow U(n) \longrightarrow S^{2n-1}.$$

Since S^{2n-1} is simply-connected, we have a Serre spectral sequence

$$E_2^{*,*} = \mathrm{H}^*(S^{2n-1};\mathrm{H}^*(U(n-1);\mathbb{Z})) \Rightarrow \mathrm{H}^*(U(n);\mathbb{Z}).$$

This spectral sequence is concentrated in two columns: $s = 0$ and $s = 2n-1$. It follows that the only possible differential is a d_{2n-1}. But this differential lowers t by $2n-2$, so d_{2n-1} must vanish on the generators $x_1, x_3, \ldots, x_{2n-3}$ of the cohomology of $U(n-1)$.

Since it vanishes on multiplicative generators, it must vanish on all classes.



We conclude that $E_2 = E_{2n-1} = E_\infty$, so that

$$\mathrm{H}^*(U(n);\mathbb{Z}) \cong E(x_1, x_3, x_5, \ldots, x_{2n-3})[y_{2n-1}]/y_{2n-1}^2 \cong E(x_1, x_3, x_5, \ldots, x_{2n-1}).$$

□

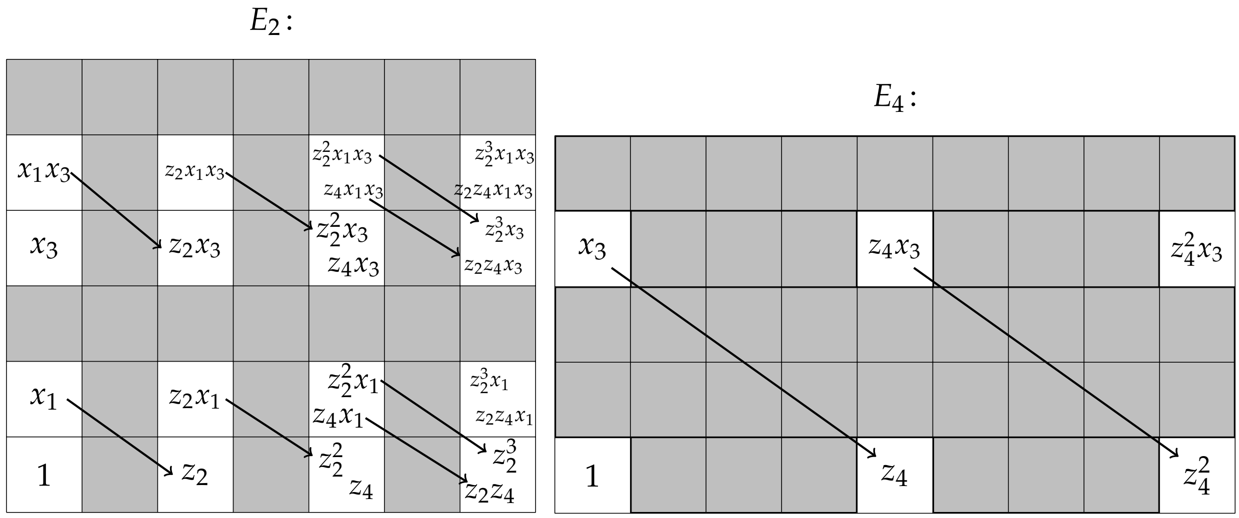
Proposition 33.1. *Let $n \geq 1$. Then $\mathrm{H}^*(BU(n);\mathbb{Z}) \cong \mathbb{Z}[z_2, z_4, \ldots, z_{2n}]$, where $\deg(z_i) = i$.*

Proof. We have a fiber sequence $U(n) \longrightarrow EU(n) \longrightarrow BU(n)$. In order to apply the Serre spectral sequence, we need to know $\pi_1(BU(n))$. By the long exact sequence in homotopy, $\pi_1(BU(n)) \cong \pi_0(U(n))$. By Prop 32.3, we know that $U(n)$ is path-connected, so that $BU(n)$ is simply-connected. Let us consider the case $n = 2$. Then

$$\mathrm{H}^*(U(2);\mathbb{Z}) \cong E(x_1, x_3).$$

Since $EU(n) \simeq *$, the class $x_1 \in E_2^{0,1}$ cannot survive the spectral sequence, and the only possibility is that $d_2(x_1) \neq 0$. Define $z_2 := d_2(x_1)$. Then the class x_1z_2 cannot be hit by any differential, so we conclude that $d_2(x_1z_2) = z_2^2$ must be nonzero.

Note that $\mathrm{H}^3(BU(2);\mathbb{Z}) = 0$ since no classes in $E_2^{0,3}$ can be hit by a differential (even on a later page). Now x_3 cannot survive the spectral sequence, and the only possibility is that $d_4(x_3) = z_4$ for some $z_4 \in \mathrm{H}^4(BU(2);\mathbb{Z})$. Again, $d_4(z_4x_3) = z_4^2$ and this is the only nontrivial differential that could involve the class z_4x_3, so that we conclude $z_4^2 \neq 0$.



We have shown that $\mathrm{H}^*(BU(2);\mathbb{Z}) \cong \mathbb{Z}[z_1, z_2]$.

34. Fri, Apr. 10

Recall that last time we were in the midst of the computation of $\mathrm{H}^*(BU(n);\mathbb{Z})$. We gave the argument for $n = 2$ (the $n = 1$ case, where $BU(1) = \mathbb{CP}^\infty$ having been handled previously).

The strategy for general n is essentially the same. We have a Serre spectral sequence with

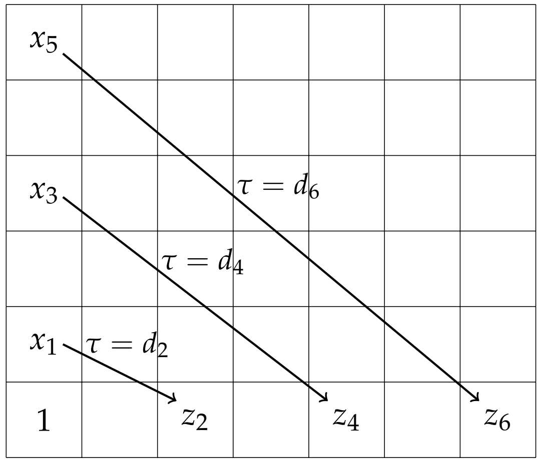
$$E_2^{s,t} \cong \mathrm{H}^s(BU(n); \mathrm{H}^t(U(n);\mathbb{Z})) \Rightarrow \mathrm{H}^{s+t}(EU(n);\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & s,t = 0 \\ 0 & \text{else,} \end{cases}$$

where $\mathrm{H}^*(U(n);\mathbb{Z})$ is the exterior algebra $E(x_1, x_3, \ldots, x_{2n-1})$.

The argument begins as in the $n = 2$ case: there is a polynomial generator $z_2 \in E_2^{2,0}$ and a differential $d_2(x_1) = z_2$.

> **Idea:** Each exterior generator x_i, is a d_r-cycle for $2 \le r \le i$, and $z_{i+1} := d_{i+1}(x_i)$ is a (nonzero) polynomial generator for the cohomology of $\mathrm{H}^*(BU(n);\mathbb{Z})$.

By the way, any element $x \in E_2^{0,t}$ satisfying $d_r(x) = 0$ for $2 \le r \le t$ and $d_{t+1}(x) \neq 0 \in E_{t+1}^{t+1,0}$ is called a **transgressive** element, and the differential $d_{t+1} : E_{t+1}^{0,t} \longrightarrow E_{t+1}^{t+1,0}$ is referred to as the **transgression** and is sometimes denoted τ.



Unfortunately, we do not know that all of the x_i's are transgressive as desired. The first ambiguous case is x_9, where there is a potential differential

$$d_2(x_9) \stackrel{?}{=} z_2 x_3 x_5.$$

We know that we do have the differential $d_2(x_1x_3x_5) = z_2x_3x_5$, so if x_9 does support this d_2, then the class $x_9' := x_9 + x_1x_3x_5$ is a d_2-cycle. Moreover,

$$\begin{aligned}(x_9')^2 = (x_9 + x_1x_3x_5)(x_9 + x_1x_3x_5) &= x_9^2 + x_9x_1x_3x_5 + x_1x_3x_5x_9 + x_1x_3x_5x_1x_3x_5 \\ &= x_9^2 - x_1x_3x_5x_9 + x_1x_3x_5x_9 - x_1^2x_3^2x_5^2 = 0,\end{aligned}$$

where we have used the facts that (1) each x_i squares to zero and (2) that $x_ix_j = -x_jx_i$ if $i \neq j$ and i and j are both odd. The effect is that we can replace the exterior algebra generator x_9 by the transgressive exterior algebra generator x_9'. It is a theorem from the much-celebrated thesis of Borel [Bo] that this can always be done if the cohomology of the fiber is exterior on odd-dimensional classes and the total space is contractible. In this way, the $\tau(x_i)$, after possibly replacing some x_i's if necessary, give polynomial generators for $\mathrm{H}^*(BU(n);\mathbb{Z})$ as desired. $\square$

Again, one reason to care about this computation is that it gives rise to characteristic classes for complex vector bundles.

Definition 34.1. Let $p : E \longrightarrow B$ be an n-dimensional complex vector bundle. As we have discussed, this is classified by a homotopy class of maps $f_p : B \longrightarrow BU(n)$. Define the ith Chern class of the bundle p to be

$$c_i(p) := f_p^*(z_{2i}) \in \mathrm{H}^{2i}(B;\mathbb{Z}).$$

These are **characteristic classes** for complex vector bundles, and the Chern classes are the principle tools used to study complex bundles.

35. Monday, Apr. 13

There is an analogous story for real vector bundles, coming from the cohomlogy of $BO(n)$. This cohomology ring has 2-torsion (this should not be surprising, given that

$SO(3) \cong \mathbb{RP}^3$), and it is standard to consider cohomology with $\mathbb{F}_2$-coefficients. Again, the strategy will be to use the Serre spectral sequence arising from the fibration

$$O(n) \longrightarrow EO(n) \longrightarrow BO(n).$$

However, $\pi_1(BO(n)) \cong \pi_0(O(n)) \cong \mathbb{Z}/2\mathbb{Z}$, which complicates the spectral sequence. On the other hand, we have the connected subgroup $SO(n) \subseteq O(n)$, and it follows that $BSO(n)$ is simply-connected. We therefore start by computing the cohomology of $SO(n)$, which will allow us to compute the cohomology of $BSO(n)$.

We will not completely describe the cohomology ring $\mathrm{H}^*(SO(n); \mathbb{F}_2)$.

Definition 35.1. A set $\{x_i\}$ of elements of a cohomology ring $\mathrm{H}^*(X; \mathbb{F}_2)$ is said to be a **simple system of generators** if the element 1 and the monomials $x_{i_1} x_{i_2} \cdots x_{i_k}$, with $i_1 < i_2 < \cdots < i_k$, form an $\mathbb{F}_2$-basis for $\mathrm{H}^*(X; \mathbb{F}_2)$.

Example 35.2.

(1) In an exterior algebra $E(x_1, x_2, \ldots, x_n)$, the elements $\{x_1, x_2, \ldots, x_n\}$ form a simple system of generators.
(2) In a polynomial algebra $\mathbb{F}_2[x]$, the elements $\{x, x^2, x^4, x^8, \ldots\}$ form a simple system of generators.

Proposition 35.3. $\mathrm{H}^*(SO(n); \mathbb{F}_2)$ *has a simple system of generators* $\{x_1, \ldots, x_{n-1}\}$*, where* $|x_i| = i$*.*

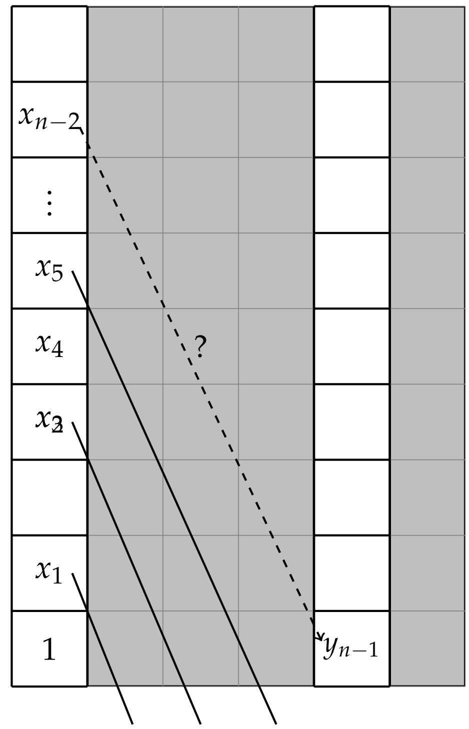
Proof. We prove this by induction. The base case is $SO(2) \cong S^1$. We know the cohomology is exterior on a class in degree 1, so we are done.

For the induction step, recall that we have a fiber sequence $SO(n-1) \longrightarrow SO(n) \longrightarrow S^{n-1}$. Note that now $n \geq 3$, so that S^{n-1} is simply-connected. We therefore have a Serre spectral sequence with understood E_2-term. On our system of generators in the fiber $SO(n-1)$, the only possible nonzero differential is

$$d_{n-1}(x_{n-2}) \stackrel{?}{=} y_{n-1},$$

where $y_{n-1} \in \mathrm{H}^{n-1}(S^{n-1}; \mathbb{F}_2)$ is the generator.

E_{n-1}:



To understand this possible differential, consider the fiber sequence

$$S^{n-2} \longrightarrow V_2(\mathbb{R}^n) \longrightarrow S^{n-1},$$

which can be thought of as the fiber sequence

$$SO(n-1)/SO(n-2) \longrightarrow SO(n)/SO(n-2) \longrightarrow SO(n)/SO(n-1)$$

if $n > 2$. The latter description makes it clear that there is a map of fiber sequences

$$\begin{tikzcd} SO(n-1) \arrow[r] \arrow[d] & SO(n) \arrow[r] \arrow[d] & S^{n-1} \arrow[d, equal] \\ S^{n-2} \arrow[r] & V_2(\mathbb{R}^n) \arrow[r] & S^{n-1} \end{tikzcd}$$

It follows that there is a *map of spectral sequences* from the cohomological Serre spectral sequence for the $V_2(\mathbb{R}^n)$ fibration to the spectral sequence for the $SO(n)$-fibration. Moreover, in the $V_2(\mathbb{R}^n)$ spectral sequence, there is only one possible differential, the same

$$d_{n-1} : E_2^{0,n-2} \longrightarrow E_2^{n-1,0}.$$

These E_2 groups are both $\mathbb{F}_2$ (we are working throughout with $\mathbb{F}_2$-coefficients), corresponding to the top homology groups of the two spheres. And these groups map isomorphically to the corresponding groups in the $SO(n)$-spectral sequence. So to understand the differential in the $SO(n)$ spectral sequence, it suffices to understand it in the $V_2(\mathbb{R}^n)$ spectral sequence.

36. Wednesday, Apr. 15

Combining Example 20.4 and Proposition 36.2 below with the Hurewicz theorem gives that $\mathrm{H}_{n-2}(V_2(\mathbb{R}^n))$ is either $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$, depending on the parity of n. In either case, Universal Coefficients gives that

$$\mathrm{H}^{n-2}(V_2(\mathbb{R}^n);\mathbb{F}_2) \cong \mathrm{Hom}(\mathrm{H}_{n-2}(V_2(\mathbb{R}^n),\mathbb{F}_2) \oplus \mathrm{Ext}(\mathrm{H}_{n-3}(V_2(\mathbb{R}^n),\mathbb{F}_2) \cong \mathbb{F}_2.$$

It follows that the d_{n-1} differential in question must be zero, since the group $E_2^{0,n-2} \cong \mathbb{F}_2$ must survive to E_∞. □

Remark 36.1. The proof strategy for computing the cohomology of $SO(n)$ is similar to that for $U(n)$, so it is worth commenting on why the result is weaker: we do not get a description of the cohomology ring here. The case of $SO(3) \cong \mathbb{RP}^3$ is instructive. From the spectral sequence, we get that E_∞ is exterior on a class in degree 1 and a class in degree 3. But we know that $\mathrm{H}^*(\mathbb{RP}^3;\mathbb{F}_2) \cong \mathbb{F}_2[x]/x^4$. The point is that the E_∞-term of the spectral sequence only recovers the *associated graded* groups of $\mathrm{H}^*(SO(n);\mathbb{F}_2)$ for some filtration, and so an algebra relation like $x_1^2 = 0$ in the E_∞-page only means that x_1^2 has lower filtration in $\mathrm{H}^*(SO(n);\mathbb{F}_2)$. The reason this situation did not arise for the cohomology of $U(n)$ was that we were getting a free commutative algebra, so there were no relations to "resolve".

Proposition 36.2. *The homotopy group $\pi_{n-2}(V_2(\mathbb{R}^n))$ is $\mathbb{Z}/2\mathbb{Z}$ if n is odd and $\mathbb{Z}$ if n is even.*

Proof. We follow the argument in [W, § IV.10]. To prove the proposition, we again use the comparison of fiber sequences for $SO(n)$ and $V_2(\mathbb{R}^n)$ as in Prop 35.3 above. On the level of homotopy groups, we get the diagram of long exact sequences

$$\begin{tikzcd}
\pi_{n-1}(S^{n-1}) \arrow[r, "\partial^1"] \arrow[d, equal] & \pi_{n-2}(SO(n-1)) \arrow[r] \arrow[d, "p_*"] & \pi_{n-2}(SO(n)) \arrow[r] \arrow[d] & \pi_{n-2}(S^{n-1}) = 0 \arrow[d, equal] \\
\pi_{n-1}(S^{n-1}) \arrow[r, "\partial^2"] & \pi_{n-2}(S^{n-2}) \arrow[r] & \pi_{n-2}(V_2(\mathbb{R}^n)) \arrow[r] & \pi_{n-2}(S^{n-1}) = 0
\end{tikzcd}$$

Thus the homotopy group in question is cyclic, and it remains only to determine the map ∂^2.

Define $f : S^{n-2} \longrightarrow SO(n-1)$ as follows: $f(\mathbf{x})$ is the orthogonal transformation which first reflects (multiplies by -1) in the last coordinate and second reflects across the hyperplane orthogonal to $\mathbf{x}$. Note that since $f(\mathbf{x})$ is a composition of two reflections, it follows that it is in $SO(n-1)$. Below, we will also write f_{n-2} for this map.

Lemma 36.3. $p_*([f]) = 1 + (-1)^{n+1}$.

Proof. We write $g = p \circ f$, so that $g(\mathbf{x}) = f(\mathbf{x})(\mathbf{e}_1)$. Then $p_*([f]) = [g]$.
Case I: n even. Note that $f(-\mathbf{x}) = f(\mathbf{x})$, so $g(-\mathbf{x}) = \mathbf{x}$. If we denote by $a : S^{n-2} \longrightarrow S^{n-2}$ the antipodal map, then the equation says that $g \circ a = g$. Since $[a] = (-1)^{n-1}$ and n is even, we get that

$$[g] = [g][a] = [g](-1)^{n-1} = -[g],$$

so that $[g]$ must be 0.

37. Fri, Apr. 17

Case II; n odd. Divide S^{n-2} into two hemispheres D_r^{n-2} and D_ℓ^{n-2}, using the first coordinate, and consider S^{n-3} as the equator where $x_1 = 0$. Note that g is constant at $\mathbf{e}_1$ on this equator. Thus we can define g_r and g_ℓ to be functions agreeing with g on one hemisphere and constant at $\mathbf{e}_1$ on the other, and $[g] = [g_r] + [g_\ell]$. But $g_r = g_\ell \circ a$, so $[g_r] = [g_\ell](-1)^{n-1} = [g_\ell]$. Thus $[g] = 2[g_\ell]$. Furthermore, it is simple to verify that the only solution to $g(\mathbf{x}) = -\mathbf{x}$ is $\mathbf{x} = \mathbf{e}_1$, so that there are no solutions to $g_\ell(\mathbf{x}) = -\mathbf{x}$. It follows that we can define a (normalized) straight-line homotopy from g_ℓ to id, so that $[g_\ell] = 1$. □

Lemma 37.1. $\partial^1(\mathrm{id}_{S^{n-1}}) = [f]$.

Proof. Let $f_{n-1} : S^{n-1} \longrightarrow SO(n)$ be defined just as f_{n-2} above. Then

$$D_\ell^{n-1} \hookrightarrow S^{n-1} \xrightarrow{f_{n-1}} SO(n) \xrightarrow{p} S^{n-1}$$

is $(g_{n-1})_\ell$, which by the argument in Lemma 36.3, is a degree 1 map. On the other hand, the restriction of f_{n-1} to the $x_1 = 0$ equator is f_{n-2}. □

The lemmas prove the proposition, as the image of $\partial^2 = p_* \circ \partial^1$ is either the subgroup of index two or the trivial subgroup, which gives the desired conclusion about $\pi_{n-2}(V_2(\mathbb{R}^n))$. □

Proposition 37.2. *Let $n \geq 1$. Then $\mathrm{H}^*(BSO(n); \mathbb{F}_2) \cong \mathbb{F}_2[w_2, w_3, w_4, \dots]$, where $\deg(w_i) = i$.*

Proof. The idea is the same as for Prop 33.1. We use the fibration sequence

$$SO(n) \longrightarrow ESO(n) \longrightarrow BSO(n).$$

Lemma 37.3. $\mathrm{H}^*(SO(n); \mathbb{F}_2)$ *has a simple system of transgressive generators.*

Proof. We have already seen that it has a simple system of generators $\{x_1, \dots, x_{n-1}\}$. It remains to show that these generators can be chosen to be transgressive. Each x_i can be chosen to be in the image of $\mathrm{H}^i(V_{n-i}(\mathbb{R}^n); \mathbb{F}_2) \longrightarrow \mathrm{H}^i(SO(n); \mathbb{F}_2)$. (The cohomology of the Stiefel manifold is one-dimensional in degree i according to homework 5.) Furthermore, we have a map of fiber sequences

$$\begin{tikzcd}
SO(n) \arrow[r] \arrow[d] & ESO(n) \arrow[r] \arrow[d] & BSO(n) \arrow[d, equal] \\
V_{n-i}(\mathbb{R}^n) \arrow[r] & ESO(n)/SO(i) \arrow[r] & BSO(n)
\end{tikzcd}$$

In the spectral sequence for the second fibration, the nonzero class in $\mathrm{H}^i(V_{n-i}(\mathbb{R}^n); \mathbb{F}_2)$ is automatically transgressive, as there are no classes in lower (positive) degree in the fiber. It follows that $x_i \in \mathrm{H}^i(SO(n); \mathbb{F}_2)$, which is the image of this transgressive class in $\mathrm{H}^i(V_{n-i}(\mathbb{R}^n); \mathbb{F}_2)$, must also be transgressive. □

Now the argument is much like the $BU(n)$ case. Each generator x_i is transgressive. Let $w_{i+1} = d_{i+1}(x_i)$. Then $d_{i+1}(x_i w_{i+1}) = w_{i+1}^2$ by the Leibniz rule. But no differential can hit $x_i w_{i+1}$, so it cannot be a permanent cycle (else it would survive to E_∞). This means that w_{i+1}^2 cannot be zero. The same argument works for higher powers, and we get the desired conclusion. $\square$

Proposition 37.4. $\mathrm{H}^*(BO(n);\mathbb{F}_2) \cong \mathbb{F}_2[w_1, w_2, \ldots, w_n]$, *where* $\deg(w_i) = i$.

We only give a sketch. The idea is to use the Serre spectral sequence for the fibration

$$BSO(n) \longrightarrow BO(n) \longrightarrow B\mathbb{Z}/2\mathbb{Z}.$$

One issue is that $B\mathbb{Z}/2\mathbb{Z}$ is not simply-connected, and one must show that $\pi_1(B\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ acts trivially on $\mathrm{H}^*(BSO(n);\mathbb{F}_2)$. This then gives a spectral sequence with E_2-term

$$\mathbb{F}_2[x_1] \otimes \mathbb{F}_2[w_2, \ldots, w_n]$$

and the claim is that this collapses at E_2. It suffices to show that each w_i is transgressive. For w_2, you use the map of spectral sequences arising from the map of fiber sequences

$$\begin{tikzcd}
BSO(2) \arrow[r] \arrow[d] & BO(2) \arrow[r] \arrow[d] & B\mathbb{Z}/2\mathbb{Z} \\
BSO(n) \arrow[r] & BO(n) \arrow[r] & B\mathbb{Z}/2\mathbb{Z}.
\end{tikzcd}$$

But $O(2) \cong SO(2) \times \mathbb{Z}/2\mathbb{Z}$ as topological groups, so $BO(2) \cong BSO(2) \times B\mathbb{Z}/2\mathbb{Z}$, so we get the desired answer for $BO(2)$. This shows that the generator in $\mathrm{H}^2(BSO(2);\mathbb{F}_2)$ is transgressive, which implies that the same must be true for $w_2 \in \mathrm{H}^2(BSO(n);\mathbb{F}_2)$. Analogous comparisons of spectral sequences are needed for the higher w_i's.

Definition 37.5. Let $p : E \longrightarrow B$ be an n-dimensional real vector bundle. As we have discussed, this is classified by a homotopy class of maps $f_p : B \longrightarrow BO(n)$. Define the ith Stiefel-Whitney class of the bundle p to be

$$w_i(p) := f_p^*(w_i) \in \mathrm{H}^i(B;\mathbb{F}_2).$$

Remark 37.6. An n-plane bundle is said to be **orientable** if all of its transition functions can be taken in $SO(n)$. In that case, the classification map $B \longrightarrow BO(n)$ factors through $BSO(n)$, so that $w_1(p) = 0$. In fact, this is an if-and-only-if statement. The first Stiefel-Whitney class detects orientability.

38. Monday, April 20

J. P Serre made great advances in the homotopy groups of spheres in his thesis [Se]. We will now discuss some applications of spectral sequences to computations of these homotopy groups.

Definition 38.1. Let X be a connected CW complex. A **Whitehead tower** for X is a diagram

$$\begin{tikzcd}
\vdots \arrow[d] & & \\
P_{>3}(X) \arrow[d, "q_2"] \arrow[ddrr, "f_3"] & & \\
P_{>2}(X) \arrow[d, "q_1"] \arrow[drr, "f_2"] & & \\
P_{>1}(X) \arrow[d, "q_0"] \arrow[rr, "f_1"] & & X \\
P_{>0}(X) = X \arrow[urr, "f_0"'] & &
\end{tikzcd}$$

such that

(1) Each $P_j X$ is j-connected
(2) Each f_j induces an isomorphism in homotopy in dimensions $> j$

For instance, $P_{>1}(X)$ is a universal cover of X. More generally, we call $P_{>n}(X)$ the n-connected cover of X.

Remark 38.2. Note that it follows that each q_j also induces isomorphisms in homotopy in dimensions $> j+1$.

Proposition 38.3. *After replacing each q_j up to homotopy by a fibration, we see that*

$$K(\pi_j(X), j-1) \simeq \operatorname{fiber}(P_{>j}(X) \longrightarrow P_{>j-1}(X)).$$

Proof. This follows from the long exact sequence in homotopy

$$\ldots \longrightarrow \pi_{k+1}(P_{>j-1}) \longrightarrow \pi_k(F) \longrightarrow \pi_k(P_{>j}(X)) \longrightarrow \pi_k(P_{>j-1}) \longrightarrow \ldots.$$

□

Proposition 38.4. *Whitehead towers exist!*

Proof. Start by letting $q_0 : P_{>1}(X) \longrightarrow P_{>0}(X) = X$ be the universal cover. We can build a $K(\pi_2(X), 2)$ from $P_{>1}(X)$ by first attaching 4-cells to kill π_3, then attaching 5-cells to kill π_4, etc. We then have an inclusion $g : P_{>1}(X) \longrightarrow K(\pi_2(X), 2)$. Define

$$P_{>2}(X) = \operatorname{fiber}(P_{>1}(X) \longrightarrow K(\pi_2(X), 2)).$$

It follows from the long exact sequence in homotopy that $P_{>2}(X)$ has the desired properties. We can then, in general, define

$$P_{>n+1}(X) = \operatorname{fiber}(P_{>n}(X) \longrightarrow K(\pi_n(X), n).$$

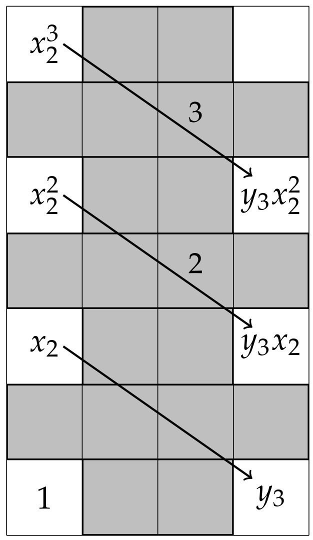
$\square$

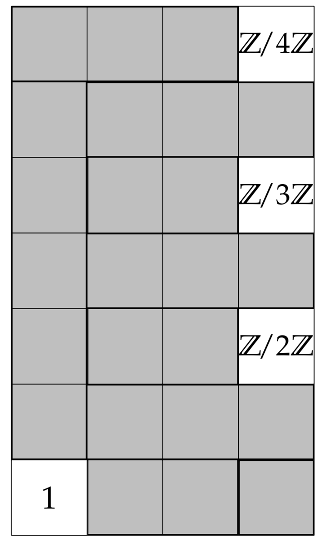
Proposition 38.5. $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. Consider the Whitehead tower for S^3. The fiber sequence we get from the start of the tower is

$$K(\mathbb{Z},2) \longrightarrow P_{>3}(S^3) \longrightarrow S^3.$$

The resulting Serre spectral sequence looks like

$E_2 = E_3$: 

$E_4 = E_\infty$: 

Note that we must have $d_3(x_2) = \pm y_3$ since the target $P_{>3}(S^3)$ is 3-connected, which implies that $\mathrm{H}^3(P_{>3}(S^3);\mathbb{Z}) = 0$. We have that

$$\mathrm{H}^5(P_{>3};\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}, \qquad \mathrm{H}^7(P_{>3}(S^3);\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}, \qquad \mathrm{H}^9(P_{>3}(S^3);\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}.$$

From the Universal Coefficients theorem, it follows that

$$\mathrm{H}_4(P_{>3};\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}, \qquad \mathrm{H}_6(P_{>3}(S^3);\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}, \qquad \mathrm{H}_8(P_{>3}(S^3);\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}.$$

In particular, $\pi_4(P_{>3}(S^3)) \cong \mathbb{Z}/2\mathbb{Z}$ by the Hurewicz theorem. But, according to the definition of the Whitehead tower, $\pi_4(P_{>3}(S^3)) \cong \pi_4(S^3)$. $\square$

39. Wed, Apr. 22

Remark 39.1. We showed in Proposiiton 12.2 that $\pi_n(S^3) \cong \pi_n(S^2)$ for $n \geq 3$. It follows now that $\pi_4(S^2) \cong \mathbb{Z}/2\mathbb{Z}$ as claimed in Example 1.1.

Remark 39.2. Suspension gives us a homomorphism

$$\mathbb{Z} \cong \pi_3(S^2) \longrightarrow \pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z},$$

and the Freudenthal Suspension Theorem says that if we keep suspending, the result will not change any more. This "stable" value $\pi_{n+1}(S^n)$, for $n \geq 3$ is the first *stable homotopy group of spheres*, $\pi_1^s(\mathbb{S}) \cong \mathbb{Z}/2\mathbb{Z}$.

On the first day of class, we asked whether a space can be identified merely by specifying its homotopy groups. That is, we asked if we always have an identification

$$X \overset{?}{\simeq} \prod_{n \geq 0} K(\pi_n(X), n).$$

Consider $X = S^2$. Here we could ask if

$$S^2 \overset{?}{\simeq} K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 3) \times P_{>3}(S^2).$$

But we know that the homology of S^2 is concentrated in degrees zero and two, whereas this is not true for the product on the right. For instance, $K(\mathbb{Z}, 3)$ will contribute a $\mathbb{Z}$-summand in degree 3, and $P_{>3}(S^2)$ will contribute an $\mathbb{F}_2$ in degree 4.

Similarly, we see that

$$S^3 \overset{?}{\simeq} K(\mathbb{Z}, 3) \times K(\mathbb{F}_2, 4) \times P_{>4}(S^3)$$

cannot hold, since the homology on the right is much larger. In general, a space which is equivalent to a product of Eilenberg-Mac Lane spaces is called a **generalized Eilenberg-Mac Lane** space, or simply **GEM** for short.

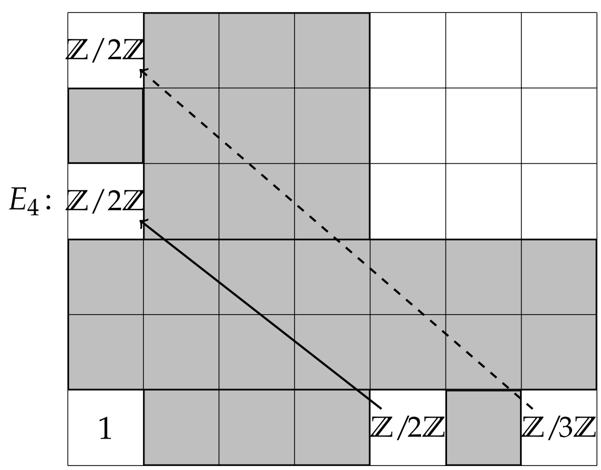
We next extend our computation to $\pi_5(S^3)$. This will rely on Proposition 40.1 below.

Proposition 39.3. *$\pi_5(S^3) \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. We again use the Whitehead tower. We have a fiber sequence

$$K(\pi_4(S^3), 3) \longrightarrow P_{>4}(S^3) \longrightarrow P_{>3}(S^3).$$

By Prop 38.5 and Prop 40.1, the $E_2 = E_4$-term of the homology spectral sequence looks like



Since we know the target $P_{>4}(S^3)$ is 4-connected, we must have the nonzero d_4 as displayed. Next, there is a potential d^6-differential as displayed, but of course this must be zero since there is no nontrivial homomorphism $\mathbb{Z}/3\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$. We conclude that

$$\pi_5(S^3) \cong \pi_5(P_{>4}(S^3))\mathrm{H}_5(P_{>4}(S^3)) \cong \mathbb{Z}/2\mathbb{Z}.$$

$\square$

Corollary 39.4. *$\pi_5(S^2) \cong \mathbb{Z}/2\mathbb{Z}$.*

Proposition 39.5. *In degrees ≤ 5, we have*

$$\mathrm{H}^*(K(\mathbb{F}_2, 2); \mathbb{F}_2) \cong \mathbb{F}_2[y_2, y_3, y_5] \quad \text{and} \quad \mathrm{H}^*(K(\mathbb{F}_2, 2); \mathbb{Z}) \cong \mathbb{Z}[y_3, y_5]/(2y_3, 4y_5).$$

Again, we emphasize that we are only claiming the isomorphism in degrees up to 5. The statement is definitely false in higher degrees. For instance, the ring $\mathrm{H}^*(K(\mathbb{F}_2, 2); \mathbb{F}_2)$ is known to be polynomial in infinitely many generators.

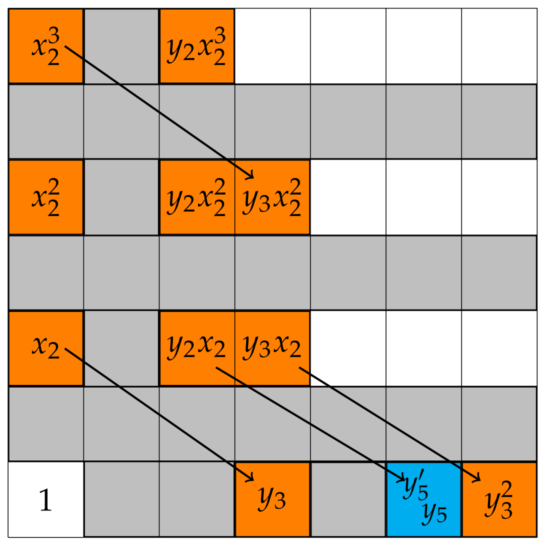
Proof. We start with the $\mathbb{F}_2$-calculation. In either case, we use the fiber sequence

$$\Omega K(\mathbb{F}_2, 2) \longrightarrow P_* K(\mathbb{F}_2, 2) \longrightarrow K(\mathbb{F}_2, 2).$$

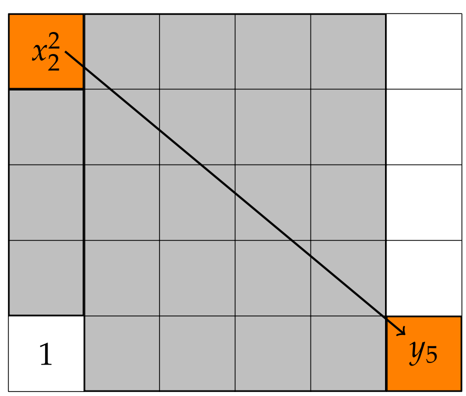
Recall that $\Omega K(\mathbb{F}_2, 2) \simeq K(\mathbb{F}_2, 1) \simeq \mathbb{RP}^\infty$. We know that $\mathrm{H}^*(\mathbb{RP}^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[x_1]$. We also know that $K(\mathbb{F}_2, 2)$ is simply-connected and has $\pi_2 \cong \mathbb{F}_2$. It follows that $\mathrm{H}^1(K(\mathbb{F}_2, 2); \mathbb{F}_2) = 0$. We define $y_2 := d_2(x_1)$. Then x_1^2 is transgressive, and we define $y_3 := d_3(x_1^2)$. Similarly, x_1^4 is transgressive, and we define $y_5 := d_5(x_1^4)$. The argument here is very similar to previous examples we have considered.

Now consider the $\mathbb{Z}$-calculation. One new phenomenon in this spectral sequence is that the rows of the E_2-term are not all isomorphic. Recall that in general, the E_2 term is given by $E_2^{s,t} = \mathrm{H}^s(B, \mathrm{H}^t(F; \mathbb{Z}))$. Here, we know that $\mathrm{H}^*(\mathbb{RP}^\infty; \mathbb{Z}) \cong \mathbb{Z}[x_2]/(2x_2)$, so that $\mathrm{H}^t(\mathbb{RP}^\infty; \mathbb{Z})$ is $\mathbb{Z}$ when $t = 0$ but is 2-torsion when t is positive and even. It follows that our previous calculation appears as the nonzero (even) rows of the integral E_2-page.

E_3:



E_5:



40. Friday, Apr. 24

Note that we have two classes, y_5 and y'_5 in $\mathrm{H}^5(K(\mathbb{F}_2, 2); \mathbb{Z})$. We know that y'_5 is 2-torsion, and that y_5-is 2-torsion once we have collapsed y'_5. Thus there are two possibilities: either H^5 is $\mathbb{Z}/2\mathbb{Z}\{y'_5\} \oplus \mathbb{Z}/2\mathbb{Z}\{y_5\}$ or $\mathrm{H}^5 \cong \mathbb{Z}/4\mathbb{Z}\{y_5\}$, where $y'_5 = 2y_5$.

To answer this question, recall that the short exact sequence of coefficients

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{F}_2 \longrightarrow 0$$

gives rise to a long exact sequence

$$\ldots \longrightarrow \mathrm{H}^n(X;\mathbb{Z}) \xrightarrow{2} \mathrm{H}^n(X;\mathbb{Z}) \xrightarrow{r} \mathrm{H}^n(X;\mathbb{F}_2) \xrightarrow{\beta} \mathrm{H}^{n+1}(X;\mathbb{Z}) \xrightarrow{2} \ldots.$$

The connecting homomorphism β is known as the Bockstein homomorphism. Writing $X = K(\mathbb{F}_2;2)$, we have

$$0 = \mathrm{H}^4(X;\mathbb{Z}) \xrightarrow{r} \mathbb{F}_2\{y_2^2\} = \mathrm{H}^4(X;\mathbb{F}_2) \xrightarrow{\beta} \mathrm{H}^5(X;\mathbb{Z}) \xrightarrow{2} \mathrm{H}^5(X;\mathbb{Z}) \xrightarrow{r} \mathrm{H}^5(X;\mathbb{F}_2) \cong \mathbb{F}_2\{x_2x_3, x_5\}$$

It follows that we only have one 2-torsion class in $\mathrm{H}^5(X;\mathbb{Z})$ (since this is the image of ∂), which must be the class y_5' we already know about. Since y_5 is 2-torsion once we have collapsed y_5', we conclude that $\mathrm{H}^5(X;\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}\{y_5\}$ and $y_5' = 2y_5$.

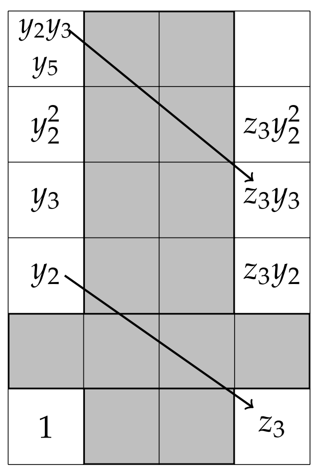
□

Proposition 40.1. *$\mathrm{H}_4(K(\mathbb{F}_2,3);\mathbb{Z}) = 0$ and $\mathrm{H}_5(K(\mathbb{F}_2,3);\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. We calculate cohomology using the path-loop fibration

$$\Omega K(\mathbb{F}_2,3) \longrightarrow P_*K(\mathbb{F}_2,3) \longrightarrow K(\mathbb{F}_2,3),$$

with $\Omega K(\mathbb{F}_2,3) \simeq K(\mathbb{F}_2,2)$. Since $K(\mathbb{F}_2,3)$ is 2-connected and has $\pi_3 \cong \mathbb{F}_2$, Universal Coefficients gives that $\mathrm{H}^3(K(\mathbb{F}_2,3);\mathbb{F}_2) \cong \mathbb{F}_2\{z_3\}$. Again, we start with some discussion of the case of $\mathbb{F}_2$-coefficients. It turns out we will only need the $E_2 = E_3$-page, which is (partially) given by

E_3:


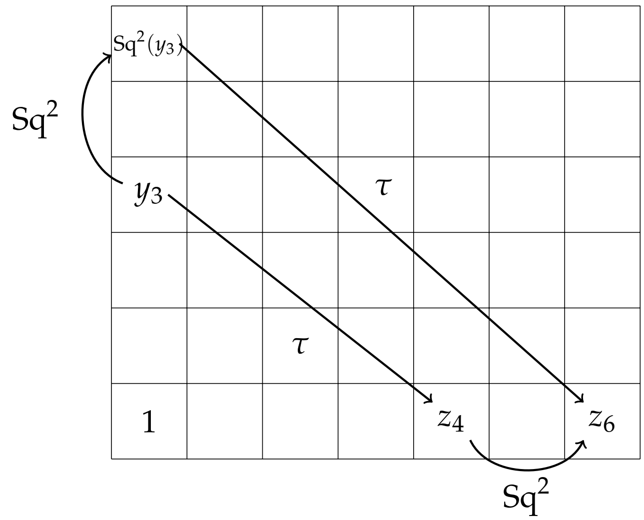
One explanation for why y_3 and y_5 do not support a d_3 comes from **cohomology operations**. For our purposes, what we need to know is that there are certain natural transformations

$$\mathrm{Sq}^i : \mathrm{H}^n(X;\mathbb{F}_2) \longrightarrow \mathrm{H}^{n+i}(X;\mathbb{F}_2)$$

know as **Steenrod operations**. For instance, $\mathrm{Sq}^n : \mathrm{H}^n(X;\mathbb{F}_2) \longrightarrow \mathrm{H}^{2n}(X;\mathbb{F}_2)$ is the squaring operation $x \mapsto x^2$.

Theorem 40.2 (Kudo's transgression theorem)**.** *If a class $y \in \mathrm{H}^n(F;\mathbb{F}_2)$ is transgressive, with $\tau(y) = z$, then $\mathrm{Sq}^i(y)$ is also transgressive, and*

$$\tau(\mathrm{Sq}^i(y)) = \mathrm{Sq}^i(\tau(y)) = \mathrm{Sq}^i(z).$$

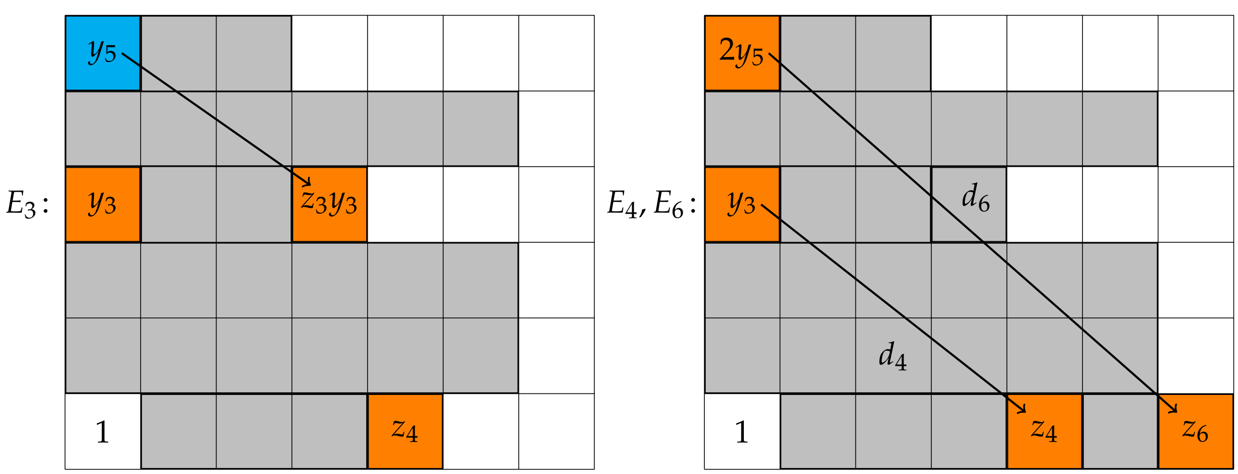


The classes y_2, y_3, and y_5 in $\mathrm{H}^*(K(\mathbb{F}_2,2);\mathbb{F}_2)$ arose as the trangression of the classes x_1, $x_1^2 = \mathrm{Sq}^1(x_1)$, and $x_1^4 = \mathrm{Sq}^2(\mathrm{Sq}^1(x_1))$, respectively. It follows that $y_3 = \mathrm{Sq}^1(y_2)$ and $y_5 = \mathrm{Sq}^2(y_3)$. In particular, these are transgressive in the current spectral sequence. We also remark that the operation Sq^1 is related to the Bockstein by the commutative triangle

$$\begin{tikzcd}
\mathrm{H}^n(X;\mathbb{F}_2) \arrow[r, "\beta"] \arrow[dr, "\mathrm{Sq}^1"'] & \mathrm{H}^{n+1}(X;\mathbb{Z}) \arrow[d, "r"] \\
& \mathrm{H}^{n+1}(X;\mathbb{F}_2).
\end{tikzcd}$$

41. Monday, Apr. 27

We now turn to the integral picture. The portion of the $E_2 = E_3$-page we need is



To see that $d_3(y_5) \neq 0$, we compare to the $\mathbb{F}_2$-coefficient calculation above.

$$\begin{tikzcd} & & H^6(X;\mathbb{Z}) \arrow[d, "r"] \\ H^5(X;\mathbb{Z}) \arrow[r, "r"] & H^5(X;\mathbb{F}_2) \arrow[ur, "\beta"] \arrow[r, "\mathrm{Sq}^1"'] & H^6(X;\mathbb{F}_2) \end{tikzcd}$$

According to our calculations, when $X = K(\mathbb{F}_2, 2)$ these groups are

$$\begin{tikzcd} & & H^6(X;\mathbb{Z}) \arrow[d, "r"] \\ \mathbb{Z}/4\mathbb{Z}\{y_5\} \arrow[r, "r"] & \mathbb{F}_2\{y_5\} \oplus \mathbb{F}_2\{y_2y_3\} \arrow[ur, "\beta"] \arrow[r, "\mathrm{Sq}^1"'] & \mathbb{F}_2\{y_2^3\} \oplus \mathbb{F}_2\{y_3^2\} \end{tikzcd}$$

We want to understand $r(y_5)$. We know that $\beta(r(y_5)) = 0$. To understand Sq^1, we will use (1) the relations $\mathrm{Sq}^1\mathrm{Sq}^2 = \mathrm{Sq}^3$ and $\mathrm{Sq}^1\mathrm{Sq}^1 = 0$ (these are examples of Adem relations) and (2) the Cartan formula $\mathrm{Sq}^n(xy) = \sum_{i+j=n} \mathrm{Sq}^i(x)\mathrm{Sq}^j(y)$, where Sq^0 is the identity operation. Recalling that $y_3 = \mathrm{Sq}^1(y_2)$, we have

$$\mathrm{Sq}^1(y_5) = \mathrm{Sq}^1\mathrm{Sq}^2(y_3) = \mathrm{Sq}^3(y_3) = y_3^2 \qquad \text{and}$$
$$\mathrm{Sq}^1(y_2y_3) = \mathrm{Sq}^1(y_2)y_3 + y_2\mathrm{Sq}^1(y_3) = y_3^2 + y_2\mathrm{Sq}^1\mathrm{Sq}^1(y_2) = y_3^2.$$

Since $y_5 + y_2y_3$ is the only nonzero class in the kernel of Sq^1, we conclude that

$$r(y_5) = y_5 + y_2y_3.$$

Since, in the $\mathbb{F}_2$-coefficient spectral sequence, we have $d_3(y_5) = 0$ and $d_3(y_2y_3) = z_3y_3$, it follows that $d_3(y_5) = z_3y_3$ in the $\mathbb{Z}$-coefficient spectral sequence as well.

We now know that, taking $B = K(\mathbb{F}_2, 3)$, we have

$$H^4(B;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\{z_4\}, \qquad H^5(B;\mathbb{Z}) = 0, \qquad \text{and} \quad H^6(B;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\{z_6\}.$$

Universal coefficients then gives that

$$H_3(B;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}, \qquad H_4(B;\mathbb{Z}) \cong 0, \qquad \text{and} \quad H_5(B;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$

□

Remark 41.1. The Freudenthal theorem says that the suspension

$$\mathbb{Z}/2\mathbb{Z} \cong \pi_5(S^3) \longrightarrow \pi_6(S^4)$$

is surjective, and that the target is in the stable range. In fact, this is an isomorphism, and $\pi_2^s(\mathbb{S}) \cong \mathbb{Z}/2\mathbb{Z}$. The generator is

$$S^5 \xrightarrow{\Sigma^2\eta} S^4 \xrightarrow{\Sigma\eta} S^3.$$

42. Wed, Apr. 29

Theorem 42.1 (Serre)**.** *The higher homotopy groups of spheres $\pi_k(S^n)$, for $k > n$, are all finite, except that*

$$\pi_{4j-1}(S^{2j}) \cong \mathbb{Z} \oplus \textit{finite}.$$

We will assume the following result, also from Serre's thesis.

Theorem 42.2 (Serre)**.** *The homotopy groups of a simply-connected finite-type CW complex are finitely generated.*

Remark 42.3. The simply-connected hypothesis is needed, as $\pi_2(S^1 \vee S^2)$ is not finitely-generated, as can be seen by considering the universal cover.

We will also need the following two results.

Proposition 42.4. $\mathrm{H}^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \begin{cases} \mathbb{Q}[x_n] & n \text{ even} \\ E_{\mathbb{Q}}(x_n) & n \text{ odd.} \end{cases}$

Proposition 42.5. *Let $n > 0$. Then $\widetilde{\mathrm{H}}^i(K(A, n); \mathbb{Q}) = 0$ if A is finite abelian.*

Proof of Theorem 42.1. Note that a finitely-generated abelian group A is finite if and only if $A \otimes \mathbb{Q} = 0$. So it suffices to show that the rational homotopy groups $\pi_k(S^n) \otimes \mathbb{Q}$ vanish.

We first show that S^n, for n odd, has finite higher homotopy groups. Consider the Whitehead tower.

$$\dots \longrightarrow S^n\langle q \rangle \longrightarrow \dots \longrightarrow S^n\langle n \rangle \longrightarrow S^n\langle n-1 \rangle = S^n.$$

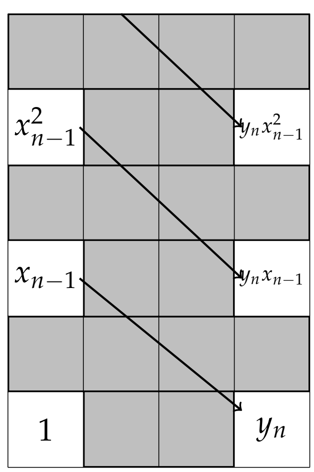
Claim: $\widetilde{\mathrm{H}}_i(S^n\langle q \rangle; \mathbb{Q}) = 0$ if $q \geq n$,

We first show this for $q = n$. We have a fiber sequence

$$K(\mathbb{Z}, n-1) \longrightarrow S^n\langle n \rangle \longrightarrow S^n.$$

Since $S^n\langle n \rangle$ is n-connected, we must have $d_n(x_{n-1}) = y_n$, and we then see by the Leibniz rule that nothing survives the spectral sequence.

E_n:



For the induction step, suppose that $S^n\langle q \rangle$ has no rational cohomology. It follows that it also has no rational homology. By the Hurewicz theorem, we conclude that

$$\pi_{q+1}(S^n) \otimes \mathbb{Q} \cong \pi_{q+1}(S^n\langle q \rangle) \otimes \mathbb{Q} \cong \mathrm{H}_{q+1}(S^n\langle q \rangle; \mathbb{Q}) = 0.$$

We now have a fiber sequence

$$K(\pi_{q+1}(S^n), q) \longrightarrow S^n\langle q+1\rangle \longrightarrow S^n\langle q\rangle$$

in which both the base and fiber have trivial rational cohomology. It follows that the same must be true of the total space. We now get, as above, that

$$\pi_{q+2}(S^n) \otimes \mathbb{Q} \cong \pi_{q+2}(S^n\langle q+1\rangle) \otimes \mathbb{Q} \cong \mathrm{H}_{q+2}(S^n\langle q+1\rangle; \mathbb{Q}) = 0.$$

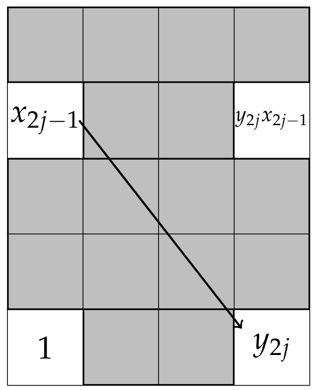
43. Fri, May 1

The even case is similar. Let $n = 2j$. Then our first fiber sequence is

$$K(\mathbb{Z}, 2j-1) \longrightarrow S^{2j}\langle 2j\rangle \longrightarrow S^{2j}.$$

Now our spectral sequence looks like

E_{2j}:



so that $S^{2j}\langle 2j\rangle$ has (reduced) rational cohomology only in degree $4j-1$. It follows (assuming $j > 1$) that

$$\pi_{2j+1}(S^{2j}) \otimes \mathbb{Q} \cong \pi_{2j+1}(S^{2j}\langle 2j\rangle) \otimes \mathbb{Q} \cong \mathrm{H}_{2j+1}(S^{2j}\langle 2j\rangle; \mathbb{Q}) = 0,$$

so that $\pi_{2j+1}(S^{2j})$ must be finite. Using the fiber sequence

$$K(\pi_{q+1}(S^{2j}), q) \longrightarrow S^{2j}\langle q+1\rangle \longrightarrow S^{2j}\langle q\rangle$$

for $q < 4j-2$, we conclude that the fiber has trivial rational cohomology by Prop 42.5, so that $\mathrm{H}^*(S^{2j}\langle q+1\rangle; \mathbb{Q}) \cong \mathrm{H}^*(S^{2j}\langle q\rangle; \mathbb{Q}) \cong E_{\mathbb{Q}}(y_{4j-1})$. In the last case considered, we get $\mathrm{H}^*(S^{2j}\langle 2j-2\rangle; \mathbb{Q}) \cong E_{\mathbb{Q}}(y_{4j-1})$. It then follows that

$$\pi_{4j-1}(S^{2j}) \otimes \mathbb{Q} \cong \pi_{4j-1}(S^{2j}\langle 4j-2\rangle) \otimes \mathbb{Q} \cong \mathrm{H}_{4j-1}(S^{2j}\langle 4j-2\rangle; \mathbb{Q}) \cong \mathbb{Q}.$$

Thus in the fiber sequence

$$K(\pi_{4j-1}(S^{2j}), 4j-2) \longrightarrow S^{2j}\langle 4j-1\rangle \longrightarrow S^{2j}\langle 4j-2\rangle$$

the fiber has polynomial cohomology, and we conclude as in the n odd case that the connected cover $S^{2j}\langle 4j-1\rangle$, as well as all higher connected covers, have trivial rational cohomology. It follows that all of the higher homotopy groups are finite. □

Remark 43.1. The Hopf map $\eta : S^3 \longrightarrow S^2$ generates $\pi_3(S^2) \cong \mathbb{Z}$. There are similarly defined maps $\nu \in \pi_7(S^4)$ and $\sigma \in \pi_{15}(S^8)$ that do the job in those dimensions. But these are special cases (the "Hopf invariant one maps").

When $2n \neq 2,4,8$, a generator of the free summand in $\pi_{4n-1}(S^{2n})$ can be found as follows. Let $f : S^{4n-1} \longrightarrow S^{2n} \vee S^{2n}$ be the attaching map for the top cell in $S^{2n} \times S^{2n}$, and let $c : S^{2n} \vee S^{2n} \longrightarrow S^{2n}$ be the "fold" map, which restricts to the identity on each wedge summand. Then the composition

$$S^{4n-1} \xrightarrow{f} S^{2n} \vee S^{2n} \xrightarrow{c} S^{2n}$$

is nontrivial and generates the free summand of $\pi_{4n-1}(S^{2n})$ if $2n \neq 2,4,8$. In the special dimensions $2n = 2,4,8$, this map is twice a generator.

Shortly after his thesis, Serre was also able to prove the following theorem.

Theorem 43.2. *Suppose that X is a finite, simply-connected complex with at least one nontrivial homotopy group. Then X has infinitely many nontrivial homotopy groups.*

In fact, he showed that infinitely many of the homotopy groups contain a subgroup isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$. The proof uses cohomology with $\mathbb{F}_2$-coefficients and a comparison of the Poincaré series for the $\mathbb{F}_2$-cohomology of Eilenberg-Mac Lane spaces with that for connected covers of X.

Further directions

We have seen that spectral sequences can help to calculate some homotopy groups of spheres. Where they have had much more impact is in the computation of the *stable* homotopy groups of spheres. One of the main tools for this is the Adams spectral sequence, which takes the form

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_{t-s}^{\mathrm{stab}}(S^0) = \pi_{t-s+k}(S^k), \qquad k >> 0.$$

Here $\mathcal{A}$ is the Steenrod algebra of all (stable) cohomology operations (aka Steenrod operations).

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