Math 752 - Hopf algebras
Worksheet 6
Spring 2017

1. (From Worksheet 5) Recall that $\mathcal{A}(1) \subseteq \mathcal{A}$ is the subalgebra generated by $[1]$ and $[2]$. The algebra generators $[1]$ and $[2]$ give rise to elements $h_0 \in \text{Ext}_{\mathcal{A}(1)}^1 \mathcal{A}(1)$ and $h_1 \in \text{Ext}_{\mathcal{A}(1)}^{1,2} \mathcal{A}(1)$ which can be thought of as the extensions.

\[ h_0 : \bullet \quad \rightarrow \quad \rightarrow \quad \rightarrow \quad \bullet \quad \text{and} \quad h_1 : \rightarrow \quad \rightarrow \quad \rightarrow \quad \bullet. \]

(a) Show that $h_0 h_1 = 0$ in $\text{Ext}_{\mathcal{A}(1)}^{2,3} \mathcal{A}(1)$ using the extension approach.

(b) Show that $h_0 h_1 = 0$ in $\text{Ext}_{\mathcal{A}(1)}^{2,3} \mathcal{A}(1)$ from the cobar complex. Recall that $\mathcal{A}(1) \vee \sim = \mathbb{F}_2[z_1, z_2]/(z_1^4, z_2^2)$, with $z_1$ primitive and $\Delta(z_2) = z_1^2 \otimes z_1$.

(c) Show (using either method) that $h_1^3 = 0$.

2. Recall that in Wednesday’s class, we started to investigate the Cartan-Eilenberg spectral sequence for the central extension

\[ E(Q_1) \longrightarrow \mathcal{A}(1) \longrightarrow \mathcal{A}(1)/Q_1, \]

with $H^*(E(Q_1)) \cong \mathbb{F}_2[v_1]$ and $H^*(\mathcal{A}(1)/Q_1) \cong \mathbb{F}_2[h_0, h_1]$.

(a) Last time, we established the differential $d_2(v_1) = h_0 h_1$. Deduce that $E_3 \cong \mathbb{F}_2[v_1^2, h_0, h_1]/h_0 h_1$.

(b) Either by considering the cobar complex or by building a minimal resolution, show that $v_1^2$ cannot survive to $H^2(\mathcal{A}(1))$. Use this to deduce the differential $d_3(v_1^2) = h_1^2$.

(c) Deduce from (b) that $E_4$ has a vector space basis

\[ E_4 \cong \mathbb{F}_2 \{ h_0^n \cdot v_1^{4j}, h_1^k \cdot v_1^{4j}, h_0^n v_1^{4j} \cdot h_0 v_1^2 \}_{n, j \geq 0, 2 \geq k \geq 0} \]

and show that $E_4 = E_\infty$ by showing that $v_4^4$ cannot support any differentials.