Wed, Jan. 17

Using problem 4 from Homework I, we get the following result.

**Corollary 1.5.** Let \( T^n \) denote the \( n \)-torus \( T^n = S^1 \times S^1 \times \cdots \times S^1 \) (\( n \) times). Then \( \pi_1(T^n) \cong \mathbb{Z}^n \).

**Theorem 1.6.** (Borsuk-Ulam Theorem) For every continuous map \( f : S^2 \rightarrow \mathbb{R}^2 \), there is an antipodal pair of points \( \{x, -x\} \subset S^2 \) such that the \( f(x) = f(-x) \).

**Proof.** Suppose not. Then we can define a map \( g : S^2 \rightarrow S^1 \) by

\[
g(x) = \frac{f(x) - f(-x)}{||f(x) - f(-x)||}.
\]

Then \( g \) satisfies \( g(-x) = -g(x) \). Let \( \gamma : S^1 \rightarrow S^1 \) be the restriction to the equator. Note that since \( \gamma \) extends over the northern (or southern) hemisphere, the loop \( \gamma \) is null. We also write \( \delta \) for the composition \( I \rightarrow S^1 \xrightarrow{\gamma} S^1 \).

The equation \( g(-z) = -g(z) \) means that \( \gamma(-z) = -\gamma(z) \) or \( \delta(t + \frac{1}{2}) = -\delta(t) \). Denote by \( \tilde{\delta} \) a lift to a path in \( \mathbb{R} \). Then \( \tilde{\delta} \) must satisfy the equation \( \tilde{\delta}(t + \frac{1}{2}) = \tilde{\delta}(t) + \frac{1}{2} + k \) for some integer \( k \). In particular, we find that

\[
\tilde{\delta}(1) = \tilde{\delta} \left( \frac{1}{2} \right) + \frac{1}{2} + k = \tilde{\delta}(0) + 1 + 2k.
\]

Thus the degree of \( \gamma \) is the odd integer \( 1 + 2k \). This contradicts that \( \gamma \) is null. 

**Application:** At any point in time, there are two polar opposite points on Earth having the same temperature and same barometric pressure. (Or pick any two continuously varying parameters)

**Corollary 1.7.** The sphere \( S^2 \) is not homeomorphic to any subspace of \( \mathbb{R}^2 \).

**Proof.** According to the theorem, there is no continuous injection \( S^2 \rightarrow \mathbb{R}^2 \).

1.2. **Fundamental group of spheres.** We saw that \( S^1 \) has a nontrivial fundamental group, but in contrast we will see that the higher spheres all have trivial fundamental groups. A (path-connected) space with trivial fundamental group is said to be **simply connected.**

**Theorem 1.8.** The \( n \)-sphere \( S^n \) is simply connected if \( n \geq 2 \).

This follows from the following theorem.

**Theorem 1.9.** Any continuous map \( S^1 \rightarrow S^n \) is path-homotopic to one that is not surjective.

Let’s first use this to deduce the statement about \( n \)-spheres. Let \( \gamma \) be a loop in \( S^n \). We know it is path-homotopic to a loop \( \delta \) that is not surjective. But recall that \( S^n - \{P\} \cong \mathbb{R}^n \). Thus we can contract \( \delta \) using a straight-line homotopy in the complement of any missed point. It remains to prove the latter theorem.

**Proof.** There are a number of ways to prove this result. For instance, it is an easy consequence of “Sard’s Theorem” from differential topology. Here is a proof using once again the Lebesgue number lemma.

Let \( \{U, V\} \) be the covering of \( S^n \), where \( U \) is the upper (open) hemisphere, and \( V \) is the complement of the North pole. Let \( \gamma : S^1 \rightarrow S^n \) be a loop. By Lebesgue, we can subdivide the interval \( I \) into finitely many subintervals \( [s_i, s_{i+1}] \) such that on each subinterval, \( \gamma \) stays within either \( U \) or \( V \). We will deform \( \gamma \) so that it misses the North pole. On the subintervals that are mapped into \( V \), nothing needs to be done.

Suppose \( [s_i, s_{i+1}] \) is not mapped into \( V \), so that \( \gamma \) passes through the North pole on this segment. Recall that the open hemisphere \( U \) is homeomorphic to \( \mathbb{R}^n \). The problem thus reduces to the following: given a path in \( \mathbb{R}^n \), show it is path-homotopic to one not passing through the origin.
This is simple. First, any path is homotopic to the straight-line path. If that does not pass through the origin, great. If it does, just wiggle it a little, and it won’t any more. ■

Corollary 1.10. The infinite sphere $S^\infty$ is simply connected.

Proof. Consider a loop $\alpha$ in $S^\infty$. The image of $\alpha$ is then a compact subset of the CW complex $S^\infty$. It follows (see Hatcher, A.1) that the image of $\alpha$ is contained in a finite union of cells. In other words, the image of $\alpha$ is contained in some $S^n$. By the above, $\alpha$ is null-homotopic in $S^n$ and therefore in $S^\infty$ as well. ■

Fri, Jan. 19

You showed on your homework that $S^\infty$ is contractible, and this in fact implies simply connected, as the next result shows.

Theorem 1.11. Let $f : X \rightarrow Y$ be a homotopy equivalence. Then, for any choice of basepoint $x \in X$, the induced map

$$f_* : \pi_1(X, x) \xrightarrow{\sim} \pi_1(Y, f(x))$$

is an isomorphism.

At first glance, this might seem obvious, since we have a quasi-inverse $g : Y \rightarrow X$ to $f$, and so we would expect $g_*$ to be the inverse of $f_*$. But note that there is no reason that $g(f(x))$ would be $x$ again, so $g_*$ does not even map to the correct group to be the inverse of $f_*$. We need to employ some sort of change-of-basepoint to deal with this. So we take a little detour to address this issue.

1.3. Dependence on the basepoint.

Although we often talk about “the fundamental group” of a space $X$, this group depends on the choice of basepoint for the loops. One thing at least should be clear: if we want to understand $\pi_1(X, x_0)$, only the path component of $x_0$ in $X$ is relevant. Any other path component can be ignored. More precisely, if $PC_x$ denotes the path-component of a point $x$, then for any choice of basepoint $x_0$, we get an isomorphism of groups

$$\pi_1(PC_{x_0}, x_0) \cong \pi_1(X, x_0).$$

For this reason, we will often assume from now on that our spaces are path-connected.

Under this assumption that $X$ is path-connected, how does the fundamental group depend on the choice of basepoint? Suppose that $x_0$ and $x_1$ are points in $X$. How can we compare loops based at $x_0$ to loops based at $x_1$? Since $X$ is path-connected, we may choose some path $\alpha$ in $X$ from $x_0$ to $x_1$. Then we may use the change-of-basepoint technique that we discussed at the end of the fall semester. If $\gamma$ is a loop based at $x_0$, we get a loop $\overline{\pi} \cdot \gamma \cdot \alpha$ based at $x_1$. Let us write $\Phi_\alpha(\gamma)$ for this loop. The same argument we gave in the case $X = S^1$ generalizes to give

Proposition 1.12.

1. The operation $\Phi_\alpha$ gives a well-defined operation on homotopy-classes of loops.
2. The operation $\Phi_\alpha$ only depends on the homotopy-class of $\alpha$.
3. The operation $\Phi_\alpha$ induces an isomorphism of groups

$$\Phi_\alpha : \pi_1(X, x_0) \cong \pi_1(X, x_1)$$

with inverse induced by $\Phi_{\overline{\pi}}$.

So, as long as $X$ is path-connected, the isomorphism-type of the fundamental group of $X$ does not depend on the basepoint. For example, once we know that $\pi_1(\mathbb{R}^2, 0) = \langle e \rangle$, it follows that the same would be true with any other choice of basepoint. More generally, we know that any convex subset of $\mathbb{R}^n$ is simply connected.
Proposition 1.13. Let $h$ be a homotopy between maps $f, g : X \rightarrow Y$. For a chosen basepoint $x_0 \in X$, define a path $\alpha$ in $Y$ by $\alpha(s) = h(x_0, s)$. Then the diagram to the right commutes.

Proof. For any loop $\gamma$ in $X$ based at $x_0$, we want a path-homotopy $H : \Phi_\alpha(f \circ \gamma) \simeq g \circ \gamma$. For convenience, let us write $y_0 = g(x_0)$. For each $t$, let $\alpha_t$ denote the path $\alpha_t(s) = \alpha(1 - (1 - s)t)$.

Then the function $H_t = \overline{\alpha_t} \cdot (h_t \circ \gamma) \cdot \alpha_t$
defines a path-homotopy $\overline{\alpha_0} \cdot g(y_0) \cdot c_{y_0} \simeq p \cdot f(\gamma) \cdot \alpha = \Phi_\alpha(f(\gamma))$.

Proof of Theorem 1.11. Let $g : Y \rightarrow X$ be a quasi-inverse to $f$. Then $g \circ f \simeq \text{id}_X$, so Prop 1.13 gives us a diagram

$$\pi_1(X, x_0) \xrightarrow{\text{id}_*} \pi_1(X, x_0) \xrightarrow{(gf)_*} \pi_1(X, gf(x_0))$$

Now $(gf)_*$ must be an isomorphism since the other two maps in the diagram are isomorphisms.

Since $(gf)_* = g_* \circ f_*$, the map $f_*$ must be injective and similarly $g_*$ must be surjective.

But now we can swap the roles of $f$ and $g$, getting a diagram

$$\pi_1(Y, f(x_0)) \xrightarrow{\text{id}_*} \pi_1(Y, f(x_0)) \xrightarrow{(fg)_*} \pi_1(Y, gff(x_0))$$

It then follows that $g_* : \pi_1(Y, f(x_0)) \rightarrow \pi_1(X, gf(x_0))$ is injective. Since we already showed it is surjective, we deduce that it is an isomorphism. Now going back to our first diagram, we get

$$g_* \circ f_* = \Phi_\alpha, \quad \text{or} \quad f_* = g_*^{-1} \circ \Phi_\alpha,$$

so that $f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, f(x_0))$ is an isomorphism.

So far, we know a number of simply connected spaces ($\mathbb{R}^n$, $S^n$ for $n \geq 2$), and we know that $\pi_1(T^n) \cong \mathbb{Z}^n$ for any $n \geq 1$. Can there be torsion in the fundamental group? For example, is it possible that for some nontrivial loop $\gamma$ in $X$, winding around the loop twice gives a trivial loop? The next example will have this property.