Proposition 2.8. Suppose that $\varphi : E_1 \rightarrow E_2$ is a map of covers. Then $\varphi$ is a covering map.

Proof. We start by showing that $\varphi$ is surjective. Let $e \in E_2$. Let $b = p_2(e)$, and pick any $e' \in p_1^{-1}(b)$. Since $E_2$ is very connected, we can find a path $\alpha : \varphi(e') \rightsquigarrow e$ in $E_2$. We can push this path $\alpha$ down to a loop $p_2\alpha$ in $B$ and then lift this uniquely to a path $\tilde{\alpha}$ in $E_1$ starting at $e'$. Now $\varphi(\tilde{\alpha})$ is a lift of $p_2\alpha$ in $E_2$ starting at $\varphi(e')$, so by uniqueness of lifts, we must have $\varphi(\tilde{\alpha}) = \alpha$. In particular, $\varphi(\tilde{\alpha}(1)) = e$.

Now we show that $e$ has an evenly-covered neighborhood of $e$. We know that the point $p_2(e) \in B$ has an evenly covered neighborhood $U_2$ (with respect to $p_2$). Let $U_1$ be an evenly covered neighborhood, with respect to $p_1$, of $p_2(e)$. Write $U$ for the component of $U_1 \cap U_2$ containing $p_2(e)$. Then $p_2^{-1}(U) \cong \Pi V_i$. Let $V_0$ be the component containing $e$. Write $p_1^{-1}(U) \cong \Pi W_j$. Then, since $U$ is connected, each $V_i$ and $W_j$ must be connected. It follows that $\varphi$ takes each $W_j$ into a single $V_i$, so that $\varphi^{-1}(V_0) \subseteq p_1^{-1}(U)$ is a disjoint union of some of the $W_j$'s, and it follows that $\varphi$ restricts to a homeomorphism on each component because both $p_1$ and $p_2$ do so.

It follows that any universal cover $X \rightarrow B$ covers every other covering $E \rightarrow B$.

Remark 2.9. Recall that in the proof of Theorem 1.25, we ended up building a map of covers $\varphi : X \rightarrow X$ corresponding to any point in the fiber $F$, but we wanted to know it was in fact a homeomorphism. Prop 2.8 now gives us that it is a covering map, so that according to the homework, it suffices to show that the $\varphi$ we constructed was injective. This can be seen by verifying that it is injective on each fiber.

2.2. The monodromy action. Our next goal is to completely understand the possible covers of a given space $B$. There are two avenues of approach. On the one hand, Prop. 2.1 tells us that covering spaces give rise to subgroups of $\pi_1(B)$, so we can try to understand the collection of subgroups. Another approach, which we will look at next, focuses on the fiber $F = p^{-1}(b_0)$.

It will be convenient in what follows to write $G = \pi_1(B, b_0)$ and $F = p^{-1}(b_0) \subset E$. Given a loop $\gamma$ based at $b_0$ and a point $f \in F$, we will write $\tilde{\gamma}_f$ for the lift of $\gamma$ which starts at $f$.

Theorem 2.10. Let $p : E \rightarrow B$ be a covering and let $F = p^{-1}(b)$ be the fiber over the basepoint. Then the function

$$a : F \times \pi_1(B) \rightarrow F, \quad (f, [\gamma]) \mapsto \tilde{\gamma}_f(1)$$

specifies a transitive right action of $\pi_1(B)$ on the fiber $F$. This is called the monodromy action.

Proof. Recall that we have already showed this to be well-defined.

Let $c_{b_0}$ be the constant loop at $b_0$. Then the constant loop $c_f$ at $f$ in $E$ is a lift of $c_{b_0}$ starting at $f$, so by uniqueness it must be the only lift. Thus $f \cdot [c_{b_0}] = f$.

Now let $\alpha$ and $\beta$ be loops at $b$. We wish to show that $(f \cdot \alpha) \cdot \beta = f \cdot (\alpha \cdot \beta)$. Let $f_2 = \tilde{\alpha}_f(1)$. Then $\tilde{\alpha}_f \cdot \tilde{\beta}_f$ is a (= the) lift of $\alpha \cdot \beta$ starting at $f$, so

$$f \cdot (\alpha \cdot \beta) = \tilde{\alpha}_f \cdot \tilde{\beta}_f(1).$$
On the other hand, \( f \cdot \alpha = \tilde{\alpha}_f(1) = f_2 \), so
\[
(f \cdot \alpha) \cdot \beta = f_2 \cdot \beta = \tilde{\beta} f_2(1)
\]

Finally, to see that this action is transitive, let \( f_1 \) and \( f_2 \) be points in the fiber \( F \). Let \( \gamma \) be a path in \( E \) from \( f_1 \) to \( f_2 \). Then \( \alpha = p \circ \gamma \) is a loop at \( b_0 \). Furthermore \( \tilde{\alpha}_f = \gamma \), so \( f_1 \cdot \alpha = \gamma(1) = f_2 \). □

Note that if we instead wrote path-composition in the “correct” order (i.e. in the same order as function composition), this would give a left action of \( \pi_1(B) \) on \( F \).

By the Orbit-Stabilizer theorem, since \( G \) acts transitively on \( F \), there is an isomorphism of right \( G \)-sets \( F \cong G_{e_0} \backslash G \), where \( G_{e_0} \leq G \) is the stabilizer of \( e_0 \).

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**Proposition 2.11.** The stabilizer of \( e \in F \) under the monodromy action is the subgroup \( p_*(\pi_1(E,e)) \leq \pi_1(B,b_0) \).

**Proof.** Let \( [\gamma] \in \pi_1(E,e) \). Then \( \gamma \) is a lift of \( p \circ \gamma \) starting at \( e \), so \( e \cdot p_*(\gamma) = \gamma(1) = e \). Thus \( p_*(\gamma) \) stabilizes \( e \).

On the other hand, let \( [\alpha] \in \pi_1(B,b_0) \) and suppose that \( e \cdot [\alpha] = e \). This means that \( \alpha \) lifts to a loop \( \tilde{\alpha} \) in \( E \). Thus \( \alpha = p \circ \tilde{\alpha} \) and \( [\alpha] \in p_*(\pi_1(E,e)) \). □

**Corollary 2.12.** Let \( p : E \rightarrow B \) be a covering. Then, writing \( H = p_*(\pi_1(E,e)) \) the map
\[
H \backslash \pi_1(B,b) \xrightarrow{\cong} F.
\]
\[
H \gamma \mapsto f \cdot \gamma
\]
is an identification of right \( \pi_1(B) \)-sets.

We have seen that any covering gives rise to a transitive \( G \)-set. We would also like to understand maps of coverings.

**Definition 2.13.** Let \( X \) and \( Y \) be (right) \( G \)-sets. A function \( f : X \rightarrow Y \) is said to be \( G \)-equivariant (or a map of \( G \)-sets) if \( f(xg) = f(x) \cdot g \) for all \( x \).

**Proposition 2.14.** Let \( \varphi : E_1 \rightarrow E_2 \) be a map of covers of \( B \). The induced map on fibers \( F_1 \rightarrow F_2 \) is \( \pi_1(B) \)-equivariant.

**Proof.** Let \( [\gamma] \in \pi_1(B) \) and \( f \in F_1 \). We have \( f \cdot [\gamma] = \tilde{\gamma}_f(1) \), where \( \tilde{\gamma}_f \) is the lift of \( \gamma \) starting at \( f \). Similarly, we have \( \varphi(f) \cdot [\gamma] = \tilde{\gamma}_{\varphi(f)}(1) \). But \( \varphi(\tilde{\gamma}) \) is a lift of \( \gamma \) starting at \( \varphi(\gamma(0)) = \varphi(f) \), so \( \tilde{\gamma}_{\varphi(f)} = \varphi(\tilde{\gamma}_f) \). Thus
\[
\varphi(f) \cdot [\gamma] = \tilde{\varphi}_f(1) = \varphi(\tilde{\gamma}_f)(1) = \varphi(\tilde{\gamma}_f(1)) = \varphi(f \cdot [\gamma]).
\]

□

**Proposition 2.15.** Let \( H, K \leq G \). Then every \( G \)-equivariant map \( \varphi : H \backslash G \rightarrow K \backslash G \) is of the form \( Hg \mapsto Kg \gamma \) for some \( \gamma \in G \) satisfying \( \gamma H \gamma^{-1} \leq K \).

**Proof.** Since \( H \backslash G \) is a transitive \( G \)-set, an equivariant map out of it is determined by the value at any point. Suppose we stipulate
\[
He \mapsto K \gamma.
\]
Then equivariance would force
\[
Hg \mapsto K \gamma g.
\]
Is this well-defined? Since \( Hg = Hh g \) for any \( h \in H \), we would need \( K \gamma g = K \gamma h g \). Multiplying by \( g^{-1} \gamma^{-1} \) gives \( K = K \gamma h \gamma^{-1} \). Since \( h \in H \) is arbitrary, this says that \( \gamma H \gamma^{-1} \leq K \). □
Corollary 2.16. A $G$-equivariant map $\varphi : H \backslash G \rightarrow K \backslash G$ exists if and only if $H$ is conjugate in $G$ to a subgroup of $K$. The two orbits are isomorphic (as right $G$-sets) if and only if $H$ is conjugate to $K$.

Notation. Given covers $(E_1, p_1)$ and $(E_2, p_2)$ of $B$, we denote by $\text{Map}_B(E_1, E_2)$ the set of covering homomorphisms $\varphi : E_1 \rightarrow E_2$. Given two right $G$-sets $X$ and $Y$, we denote by $\text{Hom}_G(X, Y)$ the set of $G$-equivariant maps $X \rightarrow Y$.

The following theorem classifies covering homomorphisms.

Theorem 2.17. Let $E_1$ and $E_2$ be coverings of $B$. Then Proposition 2.14 induces a bijection

$$\text{Map}_B(E_1, E_2) \cong \text{Hom}_G(F_1, F_2).$$

Proof. The key is that a covering homomorphism is a lift in the diagram to the right. Uniqueness of lifts gives injectivity in the theorem. For surjectivity, we use the lifting criterion Prop 2.5. Thus suppose given a $G$-equivariant map $\lambda : F_1 \rightarrow F_2$ and fix a point $e_1 \in F_1$. Let $e_2 = \lambda(e_1) \in F_2$. The lifting criterion will provide a lift if we can verify that

$$(p_1)_* (\pi_1(E_1, e_1)) \leq (p_2)_* (\pi_1(E_2, e_2)).$$

But remember that according to Prop 2.11, these are precisely the stabilizers of $e_1$ and $e_2$, respectively. Writing $H_1$ and $H_2$ for these groups, the map $\lambda : F_1 \rightarrow F_2$ corresponds to a map

$$\tilde{\lambda} : H_1 \backslash G \rightarrow H_2 \backslash G.$$

According to Prop 2.15, this means that $\gamma H_1 \gamma^{-1} \leq H_2$, where $\tilde{\lambda}(H_1 e) = H_2 \gamma$. The fact that $\lambda(e_1) = e_2$ means that $\gamma = e$. So $H_1 \leq H_2$ as desired. ■

Corollary 2.18. If $E$ is a cover of $B$, then we have group isomorphisms

$$\text{Aut}_B(E) \cong \text{Aut}_G(H \backslash G, H \backslash G) \cong N_G(H)/H,$$

where $N_G(H)$ is the normalizer of $H$ in $G$, consisting of those elements of $G$ which conjugate $H$ to itself.

Proof. Theorem 2.17 gives the first bijection. By Corollary 2.15, we have a surjective group homomorphism $N_G(H) \rightarrow \text{Aut}_G(H \backslash G, H \backslash G)$, and it remains only to identify the kernel. But $\gamma \in N_G(H)$ lies in the kernel if $Hg \Rightarrow H\gamma g$ is the identity map of $H \backslash G$, which happens just if $\gamma \in H$. So we conclude that the kernel is $H$. ■

The quotient group $N_G(H)/H$ is known as the Weyl group of $H$ in $G$ and is sometimes denoted $W_G(H)$.

Fri, Feb. 9

2.3. The classification of covers. We have almost shown that working with covers of $B$ is the same as working with transitive right $G$-sets (technically, we are heading to an “equivalence of categories”). All that is left is to show that for every $G$-orbit $F$, there is a cover $p : E \rightarrow B$ whose fiber is $F$ as a $G$-set.

We assume that $B$ has a universal cover $q : X \rightarrow B$. Recall that we showed in Theorem 1.25 that the group of deck transformations of $X$ is isomorphic to $G$.

Proposition 2.19. The (left) action of $G$ on $X$ via deck transformations is free and properly discontinuous.
Proof. Let \( x \in X \) and suppose \( gx = x \) for some \( g \in G \). Recall that here \( g \) is a covering homomorphism \( X \to X \) and thus a lift of \( q : X \to B \). By the uniqueness of lifts, since \( g \) looks like the identity at the point \( x \), it must be the identity. This shows the action is free.

Again, let \( x \in X \). We want to find a neighborhood \( V \) of \( x \) such that only finitely many translates \( gV \) meet \( V \). Consider \( b = q(x) \). Let \( U \) be an evenly-covered neighborhood of \( b \). Then \( q^{-1}(U) \cong \coprod V_i \) and \( x \in V_j \) for some \( j \). Recall that \( G \) freely permutes the pancakes \( V_i \). In particular, the only translate of \( V_j \) that meets \( V_j \) is the identity translate \( eV_j \).

According to Homework IV.2, this means that the quotient map \( X \to G \backslash X \) is a cover. Actually, the cover \( X \to B \) factors through a homeomorphism \( G \backslash X \cong B \). If we consider the action of a subgroup \( H \subseteq G \), it is still free and properly discontinuous. So we get a covering

\[
q_H : X \to G \backslash X = X_H
\]

for every \( H \). Moreover, the universal property of quotients gives an induced map

\[
p_H : H \backslash X \to B.
\]

**Proposition 2.20.** The map \( p_H : H \backslash X \to B \) is a covering map, and the fiber \( F \) is isomorphic to \( H \backslash G \) as a \( G \)-set.

**Proof.** Let \( b \in B \). Then we have a neighborhood \( U \) which is evenly-covered by \( q \). Recall again that the \( G \)-action, and therefore also the \( H \)-action, simply permutes the pancakes in \( p^{-1}(U) \). We thus get an action of \( H \) on the indexing set \( I \) for the pancakes in \( p^{-1}(U) \). If we write \( W_i = q_H(V_i) \), we thus have the diagram

\[
\begin{CD}
q^{-1}(U) @>q_H>> p_H^{-1}(U) @>p_H>> U \\
\cong @. \cong @. \cong \\
\coprod_{i \in I} V_i @>>> \coprod_{j \in H \backslash I} W_j @>>> U
\end{CD}
\]

To see that the restriction of \( p_H \) to a single \( W_j \) gives a homeomorphism, we use the fact that \( q_H : V_j \to W_j \) is a homeomorphism, since \( q_H : X \to X_H \) is a covering, and that \( q : V_j \to U \) is a homeomorphism. It follows that \( p_H = q \circ q_H^{-1} \) is a homeomorphism.

For the identification of the fiber \( F \subseteq X_H \), notice that the \( H \)-action on \( X \) acts on each fiber separately, and the quotient of this action on the fiber of \( X \) gives precisely \( H \backslash G \).

**Example 2.21.** Suppose that \( G = \Sigma_3 \), the symmetric group on 3 letters, and let \( H = \{e, (12)\} \subseteq G \). If we take an evenly-covered neighborhood \( U \) in \( B \), then the situation described in the proof above is given in the picture to the right.

As an aside, note that \( X_H \) here is an example of a covering in which the deck transformations do not act transitively on the fibers.

To sum up, we have shown that if \( B \) has a universal cover, then the assignment \((E, p) \mapsto F\) gives an “equivalence of categories” between coverings of \( B \) (Cov\(_B\)) and \( G \)-orbits (Orb\(_G\)).