CLASS NOTES
MATH 651 (SPRING 2018)
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Contents
1. The fundamental group - Examples 2
   1.1. The fundamental group of $S^1$ 2
   1.2. Fundamental group of spheres 4
   1.3. Dependence on the basepoint 5
   1.4. Fundamental group of $\mathbb{RP}^2$ 7
   1.5. Fundamental group of $S^1 \vee S^1$ 8
2. The theory of covering spaces 11
   2.1. Lifting Lemmas 11
   2.2. The monodromy action 14
   2.3. The classification of covers 16
   2.4. Existence of universal covers 18
3. The van Kampen Theorem 20
   3.1. The effect of attaching cells 27
   3.2. The classification of surfaces 29
4. Higher homotopy groups 35

Wed, Jan. 10

Here are a list of main topics for this semester:
(1) the fundamental group (topology $\leadsto$ algebra) (Hatcher - Ch. 1.1; Lee - Ch. 7, Ch. 8)
(2) the theory of covering spaces (Hatcher - Ch. 1.3; Lee - Ch. 11, Ch. 12)

Example 0.1.
(a) What spaces cover $\mathbb{R}$? Only $\mathbb{R}$ itself. Every covering map $E \rightarrow \mathbb{R}$ is a homeomorphism.
(b) What spaces cover $S^1$? There is the $n$-sheeted cover of $S^1$ by itself, for any nonzero integer $n$. (Wrap the circle around itself $n$ times.) There is also the exponential map $\mathbb{R} \rightarrow S^1$.
(c) What spaces cover $S^2$? Only $S^2$ itself. Every covering map $E \rightarrow S^2$ is a homeomorphism.
(d) What spaces cover $\mathbb{RP}^2$? There is the defining quotient map $S^2 \rightarrow \mathbb{RP}^2$ and the homeomorphisms.

(3) computation of the fundamental group via the Seifert-van Kampen theorem. (Hatcher - Ch. 1.2, Lee - Ch. 9, Ch. 10)
(4) classification of surfaces (compact, connected) and the Euler characteristic. (Lee - Ch. 6, Ch. 10)

Date: March 23, 2018.
The fundamental group, an algebraic object, will turn out to be crucial for understanding topics in geometric topology (coverings, surfaces).

1. The fundamental group - Examples

Our first major result in the course will be the computation of the fundamental group of the circle. In particular, we will show that it is nontrivial! The argument will involve a number of new ideas, and one thing I hope you will learn from this course is that computing fundamental groups is hard!

1.1. The fundamental group of $S^1$. Today, we begin the discussion of the fundamental group of $S^1$. We will need the following technical result that could have been included in the fall semester.

**Proposition 1.1. (Lebesgue number lemma) [Lee, 7.18]** Let $U$ be an open cover of a compact metric space $X$. Then there is a number $δ > 0$ such that any subset $A ⊆ X$ of diameter less than $δ$ is contained in an open set from the cover.

For any $n$, consider the loop in $S^1$ given by $γ_n(t) = e^{2πint}$. For today, we will denote the standard basepoint of $S^1$, the point $(1, 0)$, by the symbol $⋆$.

**Theorem 1.2.** The assignment $n ↦ γ_n$ is an isomorphism of groups

$$\Gamma : \mathbb{Z} \cong \pi_1(S^1, ⋆).$$

**Proof.** Let’s start by showing that it is a homomorphism. First note that $γ_0$ is the constant path at 1, which is the identity element of the fundamental group. Also, note that $γ_{−n}$ is the path-inverse of $γ_n$. It then remains to show that the path $γ_n \cdot γ_k$ is path-homotopic to $γ_{n+k}$ when $n$ and $k$ are non-negative.

For any $0 ≤ c ≤ 1$, we can define a path which first traverses $γ_n$ on the time interval $[0, c]$ and then traverses $γ_k$ on the time interval $[c, 1]$. Any two choices of $c$ gives homotopic paths. The choice $c = 1/2$ gives the usual path-composition $γ_n \cdot γ_k$, whereas the choice $c = n/(n + k)$ gives $γ_{n+k}$.

To show that $Γ$ is also a bijection, we will rely on the exponential map

$$p : \mathbb{R} \longrightarrow S^1$$

$$t \mapsto e^{2πit}.$$  

Note that $p^{-1}(⋆) = \mathbb{Z}$. One important property of this map that we will need is that we can cover $S^1$, say using the open sets $U_1 = S^1 \setminus \{(1,0)\}$ and $U_2 = S^1 \setminus \{(-1,0)\}$. On each of these open sets $U_i$, the preimage $p^{-1}(U_i)$ is a (countably infinite) disjoint union of subsets $V_{i,j}$ of $\mathbb{R}$, and $p$ restricts to a homeomorphism $p : V_{i,j} \cong U_i$.

If $f : X \longrightarrow S^1$ is a map from some space $X$, then by a **lift** $\tilde{f} : X \longrightarrow \mathbb{R}$ we mean simply a map such that $p \circ \tilde{f} = f$.
Lemma 1.3. Let $\gamma: I \to S^1$ be a loop at $\ast$ and let $n \in \mathbb{Z}$. Then there is a unique lift $\hat{\gamma}: I \to \mathbb{R}$ such that $\hat{\gamma}(0) = n$.

Proof. By the Lebesgue number lemma applied to $I$, there is a subdivision of $I$ into subintervals $[s_i, s_{i+1}]$ such that each subinterval is contained in a single $\gamma^{-1}(U_i)$.

Consider the first such subinterval $[0, s_1] \subseteq \gamma^{-1}(U_2)$. Now our lifting problem simplifies to that on the right. The interval $[0, s_1]$ is connected, so the image of $\hat{\gamma}$ must lie in a single component $V_{1,j}$. And we have no choice of the component since we have already decided that $\hat{\gamma}(0)$ must be $n$. Call the component $V_{2,0}$.

Now our lifting problem reduces to lifting against the homeomorphism $p_{2,0}: V_{2,0} \cong U_2$, and we define our lift on $[0, s_1]$ to be the composite $p_{2,0}^{-1} \circ \gamma$. Now play the same game with the next interval $[s_1, s_2]$. We already have a lift at the point $s_1$, so this forces the choice of component at this stage. By induction, at each stage we have a unique choice of lift on the subinterval $[s_k, s_{k+1}]$. Piecing these all together gives the desired lift $\hat{\gamma}: I \to \mathbb{R}$. ■

Thus given a loop $\gamma$ at $\ast$, there is a unique lift $\hat{\gamma}: I \to \mathbb{R}$ that starts at $0$. The endpoint of the lift $\hat{\gamma}$ must also be in $p^{-1}(0) = \mathbb{Z}$. We claim that the function $\gamma \mapsto w(\gamma) = \hat{\gamma}(1)$ is inverse to $\Gamma$. First we must show it is well-defined.

Lemma 1.4. Let $h: \gamma \simeq \delta$ be a path-homotopy between loops at $\ast$ in $S^1$. Then there is a unique lift $\hat{h}: I \times I \to \mathbb{R}$ such that $\hat{h}(0, 0) = 0$.

Proof. We already know about the unique lift $\hat{\gamma}$ on $I \times 0$. On $0 \times I$, the only possible lift is the constant lift. Now use the Lebesgue number lemma again to subdivide the compact square $I \times I$ so that every subsquare is mapped by $\gamma$ into one of the $U_i$. Using the same argument as above, we get a unique lift on each subsquare, starting from the bottom left square and moving along each row systematically. ■

Note that the lift $\hat{h}$ is a path-homotopy between the lifts $\hat{\gamma}$ and $\hat{\delta}$. This is because $\hat{h}(0, t)$ and $\hat{h}(1, t)$ are lifts of constant paths. By the uniqueness of lifts, according to Lemma 1.3, the lift of a constant path must be a constant path. It follows that $\hat{\gamma}(1) = \hat{\delta}(1)$. This shows that the function $w: \pi_1(S^1) \to \mathbb{Z}$ is well-defined.

It remains to show that $w$ is the inverse of $\Gamma$.

First note that $\delta_n(s) = ns$ is a path in $\mathbb{R}$ starting at $0$, and $p \circ \delta_n(s) = e^{2\pi i ns} = \gamma_n(s)$, so $\delta_n$ is a lift of $\gamma_n$ starting at $0$. By uniqueness of lifts (Lemma 1.3), $\delta_n$ must be $\tilde{\gamma}_n$. Therefore

$$w \circ \Gamma(n) = w(\gamma_n) = \tilde{\gamma}_n(1) = \delta(1) = n.$$

It remains to check that $\Gamma(w(\gamma)) = [\gamma]$ for any loop $\gamma$. Consider lifts $\Gamma(\tilde{\gamma}(\gamma))$ and $\gamma$. These are both paths in $\mathbb{R}$ starting at $0$ and ending at $\tilde{\gamma}(1) = w(\gamma)$ (this uses that $w \circ \Gamma(n) = n$). But any two such paths are homotopic (use a straight-line homotopy)! Composing that homotopy with the exponential map $p$ will produce a path-homotopy $\Gamma(w(\gamma)) \simeq_p \gamma$ as desired. ■
Recall that the open hemisphere $U$, following: given a path in $S^1 \times S^1 \times \cdots \times S^1$ (n times). Then $\pi_1(T^n) \cong \mathbb{Z}^n$.

**Theorem 1.6.** (Borsuk-Ulam Theorem) For every continuous map $f : S^2 \to \mathbb{R}^2$, there is an antipodal pair of points $\{x, -x\} \subset S^2$ such that the $f(x) = f(-x)$.

Proof. Suppose not. Then we can define a map $g : S^2 \to S^1$ by

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}.$$ 

Then $g$ satisfies $g(-x) = -g(x)$. Let $\gamma : S^1 \to S^1$ be the restriction to the equator. Note that since $\gamma$ extends over the northern (or southern) hemisphere, the loop $\gamma$ is null. We also write $\delta$ for the composition $I \to S^1 \to S^1$.

The equation $g(-z) = -g(z)$ means that $\gamma(z) = -\gamma(z)$ or $\delta(t + \frac{1}{2}) = -\delta(t)$. Denote by $\tilde{\delta}$ a lift to a path in $\mathbb{R}$. Then $\tilde{\delta}$ must satisfy the equation $\tilde{\delta}(t + \frac{1}{2}) = \tilde{\delta}(t) + \frac{1}{2} + k$ for some integer $k$. In particular, we find that

$$\tilde{\delta}(1) = \tilde{\delta}(\frac{1}{2}) + \frac{1}{2} + k = \tilde{\delta}(0) + 1 + 2k.$$ 

Thus the degree of $\gamma$ is the odd integer $1 + 2k$. This contradicts that $\gamma$ is null.

**Application:** At any point in time, there are two polar opposite points on Earth having the same temperature and same barometric pressure. (Or pick any two continuously varying parameters)

**Corollary 1.7.** The sphere $S^2$ is not homeomorphic to any subspace of $\mathbb{R}^2$.

Proof. According to the theorem, there is no continuous injection $S^2 \to \mathbb{R}^2$.

1.2. **Fundamental group of spheres.** We saw that $S^1$ has a nontrivial fundamental group, but in contrast we will see that the higher spheres all have trivial fundamental groups. A (path-connected) space with trivial fundamental group is said to be **simply connected**.

**Theorem 1.8.** The n-sphere $S^n$ is simply connected if $n \geq 2$.

This follows from the following theorem.

**Theorem 1.9.** Any continuous map $S^1 \to S^n$ is path-homotopic to one that is not surjective.

Let’s first use this to deduce the statement about $n$-spheres. Let $\gamma$ be a loop in $S^n$. We know it is path-homotopic to a loop $\delta$ that is not surjective. But recall that $S^n - \{P\} \cong \mathbb{R}^n$. Thus we can contract $\delta$ using a straight-line homotopy in the complement of any missed point. It remains to prove the latter theorem.

Proof. There are a number of ways to prove this result. For instance, it is an easy consequence of “Sard’s Theorem” from differential topology. Here is a proof using once again the Lebesgue number lemma.

Let $\{U, V\}$ be the covering of $S^n$, where $U$ is the upper (open) hemisphere, and $V$ is the complement of the North pole. Let $\gamma : S^1 \to S^n$ be a loop. By Lebesgue, we can subdivide the interval $I$ into finitely many subintervals $[s_i, s_{i+1}]$ such that on each subinterval, $\gamma$ stays within either $U$ or $V$. We will deform $\gamma$ so that it misses the North pole. On the subintervals that are mapped into $V$, nothing needs to be done.

Suppose $[s_i, s_{i+1}]$ is not mapped into $V$, so that $\gamma$ passes through the North pole on this segment. Recall that the open hemisphere $U$ is homeomorphic to $\mathbb{R}^n$. The problem thus reduces to the following: given a path in $\mathbb{R}^n$, show it is path-homotopic to one not passing through the origin.
This is simple. First, any path is homotopic to the straight-line path. If that does not pass through the origin, great. If it does, just wiggle it a little, and it won’t any more.

**Corollary 1.10.** The infinite sphere $S^\infty$ is simply connected.

**Proof.** Consider a loop $\alpha$ in $S^\infty$. The image of $\alpha$ is then a compact subset of the CW complex $S^\infty$. It follows (see Hatcher, A.1) that the image of $\alpha$ is contained in a finite union of cells. In other words, the image of $\alpha$ is contained in some $S^n$. By the above, $\alpha$ is null-homotopic in $S^n$ and therefore in $S^\infty$ as well.

**Fri, Jan. 19**

You showed on your homework that $S^\infty$ is contractible, and this in fact implies simply connected, as the next result shows.

**Theorem 1.11.** Let $f : X \to Y$ be a homotopy equivalence. Then, for any choice of basepoint $x \in X$, the induced map

$$f_* : \pi_1(X, x) \xrightarrow{\cong} \pi_1(Y, f(x))$$

is an isomorphism.

At first glance, this might seem obvious, since we have a quasi-inverse $g : Y \to X$ to $f$, and so we would expect $g_*$ to be the inverse of $f_*$. But note that there is no reason that $g(f(x))$ would be $x$ again, so $g_*$ does not even map to the correct group to be the inverse of $f_*$. We need to employ some sort of change-of-basepoint to deal with this. So we take a little detour to address this issue.

1.3. Dependence on the basepoint.

Although we often talk about “the fundamental group” of a space $X$, this group depends on the choice of basepoint for the loops. One thing at least should be clear: if we want to understand $\pi_1(X, x_0)$, only the path component of $x_0$ in $X$ is relevant. Any other path component can be ignored. More precisely, if $PC_x$ denotes the path-component of a point $x$, then for any choice of basepoint $x_0$, we get an isomorphism of groups

$$\pi_1(PC_{x_0}, x_0) \cong \pi_1(X, x_0).$$

For this reason, we will often assume from now on that our spaces are path-connected.

Under this assumption that $X$ is path-connected, how does the fundamental group depend on the choice of base point? Suppose that $x_0$ and $x_1$ are points in $X$. How can we compare loops based at $x_0$ to loops based at $x_1$? Since $X$ is path-connected, we may choose some path $\alpha$ in $X$ from $x_0$ to $x_1$. Then we may use the change-of-basepoint technique that we discussed at the end of the fall semester. If $\gamma$ is a loop based at $x_0$, we get a loop $\overline{\alpha} \cdot \gamma \cdot \alpha$ based at $x_1$. Let us write $\Phi_\alpha(\gamma)$ for this loop. The same argument we gave in the case $X = S^1$ generalizes to give

**Proposition 1.12.**

1. The operation $\Phi_\alpha$ gives a well-defined operation on homotopy-classes of loops.
2. The operation $\Phi_\alpha$ only depends on the homotopy-class of $\alpha$.
3. The operation $\Phi_\alpha$ induces an isomorphism of groups

$$\Phi_\alpha : \pi_1(X, x_0) \cong \pi_1(X, x_1)$$

with inverse induced by $\Phi_\overline{\alpha}$.

So, as long as $X$ is path-connected, the isomorphism-type of the fundamental group of $X$ does not depend on the basepoint. For example, once we know that $\pi_1(\mathbb{R}^2, 0) = \langle e \rangle$, it follows that the same would be true with any other choice of basepoint. More generally, we know that any convex subset of $\mathbb{R}^n$ is simply connected.
Proposition 1.13. Let \( h \) be a homotopy between maps \( f, g : X \xrightarrow{} Y \). For a chosen basepoint \( x_0 \in X \), define a path \( \alpha \) in \( Y \) by \( \alpha(s) = h(x_0, s) \). Then the diagram to the right commutes.

\[ \pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(Y, g(x_0)) \]

\[ \Phi_\alpha \sim \Phi_\alpha \]

Proof. For any loop \( \gamma \) in \( X \) based at \( x_0 \), we want a path-homotopy \( H : \Phi_\alpha(f \circ \gamma) \simeq_p g \circ \gamma \). For convenience, let us write \( y_0 = g(x_0) \). For each \( t \), let \( \alpha_t \) denote the path \( \alpha_t(s) = \alpha(1 - (1 - s)t) \).

Then the function

\[ H_t = \overline{\alpha_t} \cdot (h_t \circ \gamma) \cdot \alpha_t \]

defines a path-homotopy \( \overline{\alpha} \cdot g(\gamma) \cdot e_{y_0} \simeq_p \overline{\alpha} \cdot f(\gamma) \cdot \alpha = \Phi_\alpha(f(\gamma)). \]

Proof of Theorem 1.11. Let \( g : Y \longrightarrow X \) be a quasi-inverse to \( f \). Then \( g \circ f \simeq \text{id}_X \), so Prop 1.13 gives us a diagram

\[ \pi_1(X, x_0) \xrightarrow{\text{id}_*} \pi_1(X, x_0) \xrightarrow{(gf)_*} \pi_1(X, gf(x_0)) \]

Now \((gf)_*\) must be an isomorphism since the other two maps in the diagram are isomorphisms. Since \((gf)_* = g_* \circ f_*\), the map \( f_* \) must be injective and similarly \( g_* \) must be surjective.

But now we can swap the roles of \( f \) and \( g \), getting a diagram

\[ \pi_1(Y, f(x_0)) \xrightarrow{\text{id}_*} \pi_1(Y, f(x_0)) \xrightarrow{(fg)_*} \pi_1(Y, fgf(x_0)) \]

It then follows that \( g_* : \pi_1(Y, f(x_0)) \longrightarrow \pi_1(X, gf(x_0)) \) is injective. Since we already showed it is surjective, we deduce that it is an isomorphism. Now going back to our first diagram, we get

\[ g_* \circ f_* = \Phi_\alpha, \quad \text{or} \quad f_* = g_*^{-1} \circ \Phi_\alpha, \]

so that \( f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0)) \) is an isomorphism.

So far, we know a number of simply connected spaces (\( \mathbb{R}^n, S^n \) for \( n \geq 2 \)), and we know that \( \pi_1(T^n) \cong \mathbb{Z}^n \) for any \( n \geq 1 \). Can there be torsion in the fundamental group? For example, is it possible that for some nontrivial loop \( \gamma \) in \( X \), winding around the loop twice gives a trivial loop? The next example will have this property.
1.4. **Fundamental group of** \( \mathbb{R}P^2 \). Recall that the real projective plane \( \mathbb{R}P^2 \) is defined as the quotient of \( S^2 \) by the equivalence relation \( x \sim -x \). The equivalence classes are precisely the sets of pairs of antipodal points. Another way to think about this is that each pair of antipodal points corresponds to a straight line through the origin. We will determine \( \pi_1(\mathbb{R}P^2) \). Today, we’re going to calculate \( \pi_1(\mathbb{R}P^2) \), but first I want to discuss a result about contractibility of paths.

**Proposition 1.14.**

1. Let \( \alpha \in \pi_1(X, x_0) \). Then \( \alpha \simeq_x c_{x_0} \) if and only if \( \alpha : S^1 \to X \) extends to a map \( D^2 \to X \).

2. Let \( \alpha \) and \( \beta \) be paths in \( X \) from \( x \) to \( y \). Then \( \alpha \simeq_p \beta \) if and only if the loop \( \alpha \ast \bar{\beta} \) is null.

**Proof.**

1. \( (\Rightarrow) \) This follows from Homework II.1.

   \( (\Leftarrow) \) Again using Homework II.1, we may assume given a homotopy \( h : \alpha \simeq c_x \). Since \( h \) is not assumed to be a path-homotopy, the formula \( \gamma(s) = h(0, s) \) defines a possible nontrivial path. The picture

\[
\begin{array}{c|c|c|c|c}
\gamma & c_{x_0} & h & \gamma & h_3 \\
\hline
h_1 & \gamma & h & \gamma & h_3 \\
\gamma & c_{x_0} & h & \gamma & h_3 \\
\end{array}
\]

where \( h_1(s, t) = \gamma(st) \) and \( h_3(s, t) = \gamma(st) \), defines a path-homotopy \( H : \alpha \simeq_p \gamma \cdot c_x \cdot \gamma \).

2. The point is that \( \alpha \simeq_p \beta \implies \alpha \bar{\beta} \simeq_p \beta \bar{\beta} \simeq_p c_x \)

and similarly

\[
\alpha \bar{\beta} \simeq_p c_x \implies \alpha \simeq_p \alpha \bar{\beta} \beta \simeq_p c_x \beta \simeq_p \beta
\]

Recall that for \( S^1 \), the exponential map \( p : \mathbb{R} \to S^1 \) was key. The analogue of that map for \( \mathbb{R}P^2 \) will be the quotient map

\[
q : S^2 \to \mathbb{R}P^2.
\]

Note that in this case, the “fiber” (the preimage of the basepoint) consists of two points. Another ingredient that was used for \( S^1 \) was that it has a nice cover. The same is true for \( \mathbb{R}P^2 \); there is a cover of \( \mathbb{R}P^2 \) by open sets \( U_1, U_2, U_3 \) such that each preimage \( q^{-1}(U_i) \) is a disjoint union \( V_{i,1} \sqcup V_{i,2} \) such that on each component \( V_{i,j} \), the map \( q \) gives a homeomorphism \( q : V_{i,j} \cong U_i \). For instance, \( U_1 \) consists of points \( q(x,y,z) \) with \( x \neq 0 \). Then \( q^{-1}(U_1) \) is the disjoint union of the left and right open hemispheres in \( S^2 \). On each hemisphere \( \mathcal{H} \), \( q \) restricts to a homeomorphism \( q : \mathcal{H} \cong U_1 \).

For any point \( x \in q^{-1}(1) = \{-1, 1\} \), we define a loop \( \Gamma(x) \) at \( 1 \) in \( \mathbb{R}P^2 \) as follows: take any path \( \alpha \) in \( S^2 \) from 1 to \( x \). Then \( \Gamma(x) = q \alpha \) is a loop in \( \mathbb{R}P^2 \). Note that this is well-defined because \( S^2 \) is simply-connected, so that any two paths between 1 and \( x \) are homotopic. When \( x = 1 \), this of course gives the class of the constant loop, but when \( x = -1 \), this gives a nontrivial loop in \( \mathbb{R}P^2 \). We claim that this is a bijection. So there is only one nontrivial loop!

To see this, we construct an inverse \( w : \pi_1(\mathbb{R}P^2) \to \{-1,1\} \). We need some lemmas:

**Lemma 1.15.** Given any loop in \( \mathbb{R}P^2 \), there is a unique lift to a path in \( S^2 \) starting at 1.

The proof of this lemma is exactly the same as that of the first lemma in the proof for the circle.
Lemma 1.16. Let \( h : \gamma \simeq_p \delta \) be a path-homotopy between loops at \( 1 \) in \( \mathbb{R}P^2 \). Then there is a unique lift \( \tilde{h} : I \times I \to S^2 \) such that \( \tilde{h}(0,0) = 1 \).

Again, the proof here is identical to that for the sphere. Let’s see how we can use the lemmas to define \( w \). Given any loop \( \gamma \) in \( \mathbb{R}P^2 \), there is a unique lift \( \tilde{\gamma} \) in \( S^2 \) starting at 1. Since it is a lift of a loop, we must have \( \tilde{\gamma}(1) \in \{-1,1\} \). So we define \( w(\gamma) = \tilde{\gamma}(1) \). That this is well-defined follows from the second lemma.

It remains to show that \( w \) really is the inverse. Let \( x \in \{-1,1\} \). Then \( \Gamma(x) = q \circ \alpha \) for some path \( \alpha \) in \( S^2 \) from 1 to \( x \). To compute \( w(\Gamma(x)) \), we must find a lift of \( \Gamma(x) \), but we already know that \( \alpha \) is the lift. Thus \( w(\Gamma(x)) = \alpha(1) = x \).

Similarly, suppose \( \gamma \) is any loop in \( \mathbb{R}P^2 \). Let \( \tilde{\gamma} \) be a lift. Then \( \Gamma(w(\gamma)) = \Gamma(\tilde{\gamma}(1)) = q\alpha \), where \( \alpha \) is any path from 1 to \( \tilde{\gamma}(1) \). But of course \( \tilde{\gamma} \) is such a path and \( \gamma = q\tilde{\gamma} \).

Note that we have given a bijection between \( \pi_1(\mathbb{R}P^2) \) and \( \{-1,1\} \), but we have not talked about a group structure. That’s because we don’t need to: there is only one group of order two! We have shown that

\[
\pi_1(\mathbb{R}P^2) \cong C_2.
\]

In fact, the same proof (replacing \( S^2 \) by \( S^n \)) shows that, for \( n \geq 2 \), we have \( \pi_1(\mathbb{R}P^n) \cong C_2 \).

Wed, Jan. 24

1.5. Fundamental group of \( S^1 \lor S^1 \). We will do one more example before describing the repeated phenomena we have seen in these examples. First, recall from last semester that given based spaces \((X,x_0)\) and \((Y,y_0)\), their wedge sum, or one-point union, is \( X \lor Y = X \amalg Y/\sim \), where \( x_0 \sim y_0 \).

Today, we want to study the fundamental group of \( S^1 \lor S^1 \) following the same approach as in the previous examples. We want to once again find a nice map \( p : X \to S^1 \lor S^1 \) for some \( X \). What we really want is an example of the following:

Definition 1.17. A surjective map \( p : E \to B \) is called a covering map if every \( b \in B \) has a neighborhood \( U \) such that \( p^{-1}(U) \) is a disjoint union \( p^{-1}(U) = \amalg_i V_i \) and such that, for each \( i \), the map \( p \) restricts to a homeomorphism \( p : V_i \xrightarrow{\cong} U \). We say that the neighborhood \( U \) is evenly covered by \( p \).

Remark 1.18. It is common to assume that \( E \) is connected and locally path-connected. We will assume this from now on, as it simplifies the theory. So as to avoid repeatedly saying (or writing) “connected and locally path-connected”, I will simply call these spaces very connected.

It is important to note that the neighborhood condition is local in \( B \), not \( E \). This contrasts with the following definition.

Definition 1.19. A map \( f : X \times Y \) is said to be a local homeomorphism if every \( x \in X \) has a neighborhood \( U \) such that \( f(U) \subseteq Y \) is open and \( f|_U : U \xrightarrow{\cong} f(U) \) is a homeomorphism.

Every covering map is a local homeomorphism: given \( e \in E \), take an evenly covered neighborhood \( U \) of \( p(e) \). Then \( e \) is contained in one of the \( V_j \)'s, which is the desired neighborhood. The converse is not true, as the next example shows.

Example 1.20. Consider the usual exponential map \( p : \mathbb{R} \to S^1 \), but now restrict it to \((0,8.123876)\). This is a local homeomorphism but not a covering map. For instance, the standard basepoint of \( S^1 \) has no evenly covered neighborhood under this map.
Ok, now back to $S^1 \lor S^1$. It is tempting to take $X = \mathbb{R}$ since $S^1 \lor S^1$ looks locally like a line, but there is a problem spot at the crossing of the figure eight. To fix this, we might try to take $X$ to be the union of the coordinate axes inside of $\mathbb{R}^2$. This space is really just $\mathbb{R} \lor \mathbb{R}$, and so we have the map $p \lor p : \mathbb{R} \lor \mathbb{R} \to S^1 \lor S^1$. We want a cover of $S^1 \lor S^1$ which is nicely compatible with our map from $X$. Suppose we consider the cover $U_1$, $U_2$, and $U_3$, where $U_1$ is the complement of the basepoint in one circle, $U_2$ is the complement of the basepoint in the other, and finally $U_3$ is some small neighborhood of the basepoint. Well, $U_1$ and $U_2$ are good neighborhoods for $p \lor p$, but $U_3$ is not. The map $p \lor p$ does not give a homeomorphism from each component of the preimage of $U_3$ to $U_3$. To fix this, we would want to add infinitely many cross-sections to each of the axes. Instead, we take $X$ to be the fractal space given in the picture (see also page 59 of Hatcher). We define $p : X \to S^1 \lor S^1$ as follows. On horizontal segments, use the exponential map to the right branch of $S^1 \lor S^1$. On vertical segments, use the left branch. Then the cover $U_1$, $U_2$, and $U_3$ from above is compatible with this new map $p$, and we see that $p$ is a covering map.

**Lemma 1.21.** The space $X$ is simply-connected.

**Proof.** The main point is that any loop in $X$ is compact and therefore contained in a finite union of edges. Consider the edge furthest from the basepoint that contains part of the loop. The loop is homotopic to one constant on this furthest edge. This furthest edge is now no longer needed, and we have a new furthest edge. We can repeat until the loop is completely contracted. ■

Let $F = p^{-1}(\ast)$ be the fiber. Any point in this fiber may be uniquely described as a “word” in the letters $u, r, d, l$. Define

$$
\Gamma : F \to \pi_1(S^1 \lor S^1)
$$

as follows: given $y \in F$, let $\alpha_y$ be any path in $X$ from the basepoint to $y$. Then $\Gamma(y) = p \circ \alpha$. It does not matter which $\alpha_y$ we choose since $X$ is simply-connected. We will define an inverse to $\Gamma$, but we now state the needed lemmas in the generality of coverings.

**Lemma 1.22.** Let $p : E \to B$ be a covering and suppose $p(e) = b$. Given any path starting at $b$ in $B$, there is a unique lift to a path in $E$ starting at $e$.

The proof of this lemma is exactly the same as that of Lemma 1.3, for the circle.

**Lemma 1.23.** Let $p : E \to B$ be a covering and suppose $p(e) = b$. Let $h : \gamma \simeq_p \delta$ be a path-homotopy between paths starting at $b$ in $B$. Then there is a unique lift $\tilde{h} : I \times I \to E$ such that $\tilde{h}(0,0) = e$.

Just as in the previous examples, the above lemmas allow us to define $w : \pi_1(S^1 \lor S^1) \to F$ by the formula $w(\gamma) = \tilde{\gamma}(1)$. We will skip the verification that $\Gamma$ and $w$ are inverse, as this really follows the same script.
We have established a bijection between $\pi_1(S^1 \vee S^1)$ and the set of “words” in the letters $u, r, d, \text{ and } l$. It remains to describe the group structure. For this, we will back up a little.

**Definition 1.24.** Let $p : E \rightarrow B$ and $q : E' \rightarrow B$ be covers of a space $B$. A **map of covers** from $E$ to $E'$ is simply a map of spaces $\varphi : E \rightarrow E'$ such that $q \circ f = p$. These are also sometimes called **covering homomorphisms**.

The special case in which the two covers are the *same* cover and $f$ is a homeomorphism is referred to as a **deck transformation**. We write $\text{Aut}(E)$ for the set of all deck transformations of $E$. This is a group under composition.

Keeping our notation from earlier, let $b \in B$ be a basepoint and write $F = p^{-1}(b)$ for the fiber.

Note that any deck transformation $\varphi : E \rightarrow E'$ must take $F$ to $F$. Let us pick a basepoint $e$ for $E$. Since we want the covering map $q$ to be based, this means that $e$ lies in the fiber $F$. We may now define a map $A : \text{Aut}(E) \rightarrow F$ by $A(\varphi) = \varphi(e)$.

**Theorem 1.25.** Let $p : X \rightarrow B$ be a covering such that $X$ is simply connected. Then the map $A : \text{Aut}(X) \rightarrow F$ is a bijection and the composition $\Gamma \circ A$ is an isomorphism of groups $\text{Aut}(X) \cong \pi_1(B)$.

**Proof.** Let us first show that $A$ is injective. Thus let $\varphi_1$ and $\varphi_2$ be deck transformations which agree at $e$. Let $x \in X$ be any point and let $\alpha$ be any path in $X$ from $e$ to $x$. Then the paths $\varphi_1 \circ \alpha$ and $\varphi_2 \circ \alpha$ are both lifts of $p \circ \alpha$ starting at the common point $\varphi_1(e) = \varphi_2(e)$. By the uniqueness of lifts, these must be the same path. It follows that their endpoints, $\varphi_1(x)$ and $\varphi_2(x)$ agree.

It remains to show that $A$ is surjective. Let $f \in F$ be any point in the fiber. We wish to produce a deck transformation $\varphi : X \rightarrow X$ such that $\varphi(e) = f$. We build the map $\varphi$ locally and patch together. Let $x \in X$ and pick any path $\alpha : e \leadsto x$. Then $p\alpha$ is a path in $B$ starting at $b$ and ending at $px$. By the path-lifting lemma, there is a unique lift $\tilde{p}\alpha$ in $X$ starting at $f$. We define $\varphi(x) = \tilde{p}\alpha(1)$. From this definition, continuity is not at all clear. But the point is that since $p$ is a covering, we can choose an evenly-covered neighborhood $U$ of $p(x)$. Let $V$ be the slice of $p^{-1}(U)$ containing $x$ and $V'$ the slice containing $\varphi(x) = \tilde{p}\alpha(1)$. Then the restriction of $\varphi$ to $V$ is the composition of homeomorphisms

$$V \xrightarrow{\tilde{p}} U \xleftarrow{p} V'.$$

By the local criterion for continuity (Prop 2.19 in Lee), it follows that $\varphi$ is continuous.

By construction, $\varphi$ will be a map of covers, as long as we can verify that it is well-defined. But if $\delta : e \leadsto x$ is another choice of path, we know that $\alpha \simeq_p \delta$ because $X$ is simply-connected. It follows that $p\alpha \simeq_p p\delta$, and by lifting the path-homotopy, it follows that $\tilde{p}\alpha \simeq_p \tilde{p}\delta$, so that their right endpoints agree.

We will finish the proof next time.
So given \( f \in F \), we have built a map of covers \( \varphi : X \longrightarrow X \), but we wanted this to be an isomorphism. From the construction of \( \varphi \), we see that it is a local homeomorphism, which implies that it is open. Suppose \( \varphi(x_1) = \varphi(x_2) \). Note that since \( \varphi \) is a map of covers, this implies that \( x_1 \) and \( x_2 \) are in the same fiber. Let \( \alpha_1 : e \rightsquigarrow x_1 \) and \( \alpha_2 : e \rightsquigarrow x_2 \) be paths. By hypothesis, \( \tilde{p}_1 \sim p_1 \alpha_1 \) and \( \tilde{p}_2 \sim p_2 \alpha_2 \) have the same endpoints. Since \( X \) is simply-connected, we know that \( \tilde{p}_1 \sim p_1 \alpha_1 \sim p_2 \alpha_2 \). It follows that \( p_1 \alpha_1 \simeq p_2 \alpha_2 \), and it then follows, by lifting the homotopy, that \( \alpha_1 \simeq p_1 \alpha_2 \). In particular, \( \alpha_1(1) = \alpha_2(1) \), so \( x_1 = x_2 \). This shows that \( \varphi \) is injective.

To see that \( \varphi \) is surjective, let \( x \in X \). We can then pick a path \( \gamma : f \rightsquigarrow x \). Then \( p\gamma \) is a path in \( B \) from \( b \) to \( p(x) \), which lifts uniquely to a path \( \tilde{\gamma} \) from \( e \) to some point \( x' \). But then \( \varphi(x') = x \) by the definition of \( \varphi \).

We have now established that

\[
A : \text{Aut}(X) \longrightarrow F
\]

is a bijection. We also wanted to show that the resulting bijection \( \Gamma \circ A : \text{Aut}(X) \longrightarrow \pi_1(B) \) is a group isomorphism. It remains only to show that this is a group homomorphism.

Let \( \varphi_1, \varphi_2 \in \text{Aut}(X) \). Recall that \( \Gamma(A(\varphi_1)) \) is defined as follows: pick any path \( \alpha_1 \) in \( X \) from \( e \) to \( f_1 = \varphi_1(e) \). Then \( \Gamma(A(\varphi_1)) = p \circ \alpha_1 \). Similarly \( \Gamma(A(\varphi_2)) = p \circ \alpha_2 \). Now \( A(\varphi_2 \circ \varphi_1) = \varphi_2 \circ \varphi_1(e) = \varphi_2(f_1) \). To compute \( \Gamma \) of this point, we need a path in \( X \) from \( e \) to \( \varphi_2(f_1) \). But \( \alpha_2 \ast \varphi_2(\alpha_1) \) is such a path. Then

\[
\Gamma(A(\varphi_2 \circ \varphi_1)) = \Gamma(\varphi_2(f_1)) = p \circ (\alpha_2 \ast \varphi_2(\alpha_1)) = (p \circ \alpha_2) \ast (p \circ \varphi_2 \circ \alpha_1)
\]

\[
= (p \circ \alpha_2) \ast (p \circ \alpha_1) = \Gamma(A(\varphi_2)) \ast \Gamma(A(\varphi_1)).
\]

Returning now to our example \( X \longrightarrow S^1 \vee S^1 \), we have identified \( \pi_1(S^1 \vee S^1) \) with the group of deck transformations \( X \cong X \), and we know we have one such deck transformation for each point in the fiber \( F \). Any transformation can be thought of as a sequence of horizontal and vertical “moves”. Writing \( u \) for an upwards shift and \( r \) for a shift to the right, any element of the group can be described by a sequence of \( u \)'s, \( r \)'s, and their inverses.

**Definition 1.26.** A word in letters \( u, r \), and their inverses is simply a sequence of these letters. We say the word is reduced if no \( u^{-1} \) is adjacent to a \( u \), and similarly for the \( r \)'s. The free group \( F_2 \) or \( F(u, r) \) on the letters \( u \) and \( r \) is the set of reduced (including empty) words, where the group operation is concatenation. The inverse of any word is the same word in reversed order and with the sign of each letter reversed.

We have shown that \( \pi_1(S^1 \vee S^1) \) is the free group on two letters. In particular, this is our first example of a nonabelian fundamental group.

**Wed, Jan. 31**

### 2. The theory of covering spaces

#### 2.1. Lifting Lemmas

So far, the only kind of coverings we have studied have been those in which the covering space is simply connected. Now we will relax this condition and discuss the more general theory.

**Proposition 2.1.** Let \( p : E \longrightarrow B \) be a covering. Then the induced map \( p_* : \pi_1(E) \longrightarrow \pi_1(B) \) is injective.
Proof. Let \( \gamma \in \pi_1(E) \) and suppose \( p_*\gamma = 0 \). In other words, the loop \( p \circ \gamma \) in \( B \) is null. Let \( h : I \times I \to B \) be a null-homotopy. Then this lifts to a homotopy \( \hat{h} : I \times I \to E \) from \( \gamma \) (the unique lift of \( p \circ \gamma \)) to a lift \( \hat{c} \) of the constant loop. Since the constant loop at \( e \) is a lift of the constant loop at \( b \), uniqueness of lifts implies that \( \hat{c} \) is the constant loop. So \( \hat{h} \) is a null-homotopy for \( \gamma \). \( \blacksquare \)

**Example 2.2.** The only example of a covering we have discussed thus far in which the covering space is not simply connected is the \( n \)-fold cover \( S^1 \to S^1 \). In this case, the cover sends the generator of \( \pi_1(S^1) \cong \mathbb{Z} \) to \( n \) times the generator, and the image of \( p_* \) is the subgroup \( n\mathbb{Z} < \mathbb{Z} \).

Given the above result, any covering of \( B \) gives rise to a subgroup of \( \pi_1(B) \). One might wonder what subgroups can arise in this way. We will see that, under mild hypotheses on \( B \), every subgroup arises in this way.

Previously, we have studied lifting paths and path-homotopies against a covering. We can also generalize this to consider lifting arbitrary maps \( f : Z \to B \). As in Remark 1.18, whenever we discuss a covering map \( E \to B \), we assume that \( E \) is “very connected”, which implies the same for \( B \). In particular, this is assumed for the following results.

**Proposition 2.3.** (Homotopy lifting) Let \( Z \) be a locally connected space. Let \( p : E \to B \) be a covering and \( h : Z \times I \to B \) be a homotopy between maps \( f, g : Z \to B \). Let \( \hat{f} \) be a lift of \( f \). Then there is a unique lift of \( h \) to \( E \) with \( \hat{h}_0 = f \).

**Proposition 2.4.** (Unique lifting) Let \( p : E \to B \) be a covering and \( f : Z \to B \) a map, with \( Z \) connected. If \( \hat{f} \) and \( \tilde{f} \) are both lifts of \( f \) that agree at some point of \( Z \), then they are the same lift.

Note that in the second result, we are not asserting that a lift exists! See Theorems 8.3 and 8.4 of [Lee] for complete proofs.

Here is a sketch of Proposition 2.4.

**Sketch.** The idea is to show that the subset of \( Z \) on which the lifts agree is both open and closed; it is already given to be nonempty. For any \( z \in Z \), pick an evenly-covered neighborhood \( U \) of \( f(z) \). On the one hand, suppose \( \hat{f}(z) = \tilde{f}(z) \). Then let \( V \) be the component of \( p^{-1}(U) \) containing this point. Then \( \hat{f}^{-1}(V) \cap \tilde{f}^{-1}(V) \) is a neighborhood of \( z \) on which the lifts agree (since \( q : V \to U \) is a homeomorphism).

On the other hand, if \( \hat{f}(z) \neq \tilde{f}(z) \), then let \( \tilde{V} \) and \( \hat{V} \) be the components of \( \hat{f}(z) \) and \( \tilde{f}(z) \) in \( p^{-1}(U) \). It follows that \( \hat{f}^{-1}(V) \cap \tilde{f}^{-1}(V) \) is a neighborhood of \( z \) on which \( \hat{f} \) and \( \tilde{f} \) disagree (they land in different components of \( p^{-1}(U) \)). \( \blacksquare \)

**Fri, Feb. 2**

The interesting, new result here concerns the existence of lifts.

**Proposition 2.5.** (Lifting Criterion) Let \( p : E \to B \) be a covering and let \( f : Z \to B \), with \( Z \) very connected. Given points \( z_0 \in Z \) and \( e_0 \in E \) with \( f(z_0) = p(e_0) \), there is a lift \( \tilde{f} \) with \( \tilde{f}(z_0) = e_0 \) if and only if \( f_*\pi_1(Z, z_0) \subseteq p_*\pi_1(E, e_0) \).

**Proof.** \((\Rightarrow)\) Since \( f = p \circ \tilde{f} \), we have \( f_* = p_* \circ \tilde{f}_* \).

\((\Leftarrow)\) Here is the more interesting direction. Suppose that \( f_*\pi_1(Z, z_0) \subseteq p_*\pi_1(E, e_0) \). Let \( z \in Z \). We wish to define \( \tilde{f}(z) \). Pick any path \( \alpha \) in \( Z \) from \( z_0 \) to \( z \). Then \( f \circ \alpha \) is a path in \( B \), which therefore lifts uniquely to a path \( \tilde{\alpha} \) in \( E \) starting at, say \( e_0 \). We define \( \tilde{f}(z) = \tilde{\alpha}(1) \). Then \( \tilde{f} \) is a lift of \( f \).

Why is the lift \( \tilde{f} \) well-defined? Suppose \( \beta \) is another path in \( Z \) from \( z_0 \) to \( z \). Then \( f \circ (\alpha \cdot \beta) \) is a loop in \( B \) at \( b_0 = f(z_0) \). By assumption, this means that for some loop \( \delta \) in \( E \), we have \( p \circ \delta \simeq_p f \circ (\alpha \cdot \beta) = f(\alpha) \cdot \tilde{f}(\beta) \).
in $B$. Since path-composition behaves well with respect to path-homotopy, we have a path-homotopy
\[ h: (p \circ \delta) \cdot f(\beta) \simeq_p f(\alpha) \]
of paths in $B$. Note that the path $(p \circ \delta) \cdot f(\beta)$ lifts to the path $\delta \cdot \tilde{\beta}$. The homotopy $h$ then lifts (uniquely) to a path-homotopy in $E$
\[ \tilde{h}: \delta \cdot \tilde{\beta} \simeq_p \tilde{\alpha}. \]
In particular, these have the same endpoints. Of course, the endpoint of $\delta \cdot \tilde{\beta}$ is simply the endpoint of $\tilde{\beta}$. It follows that $\tilde{f}$ is well-defined at $z$.

Just for emphasis, let’s go through the proof that $\tilde{f}$ is continuous. Let $z \in Z$ and let $U$ be an evenly covered neighborhood $U$ of $f(z)$, and let $V$ be the component of $p^{-1}(U)$ containing the lift $\tilde{f}(z)$. Let $W \subseteq Z$ be the path-component of $f^{-1}(U)$ containing $z$. Since $Z$ is locally path-connected, $W$ is open. Moreover, since $W$ is path-connected and $\tilde{f}(W) \cap V \neq \emptyset$, we must have $\tilde{f}(W) \subseteq V$. Then on the neighborhood $W$ of $z$, the lift $\tilde{f}$ may be described as the composition $p|_W \circ f$. It follows that $\tilde{f}$ is continuous on the neighborhood $W$ of $z$. Since $z$ was arbitrary, $\tilde{f}$ is continuous.

This implies what we already know: $S^1$ is not a retract of $\mathbb{R}$. More generally, and less trivially, we have that the identity map $S^1 \to S^1$ does not lift against the $n$-fold cover $p_n: S^1 \to S^1$. Even more generally, we might ask about lifting some $p_k: S^1 \to S^1$ against the cover $p_n: S^1 \to S^1$. By the result above, this happens if and only if $k \mathbb{Z} \subseteq n\mathbb{Z}$. In other words, this happens if and only if $n \mid k$.

More interestingly, we have

**Corollary 2.6.** Suppose that the covering space $E$ is simply-connected. Then a map $f: Z \to B$ lifts to some $\tilde{f}: Z \to E$ if and only if $f$ induces the trivial map on fundamental groups.

**Corollary 2.7.** Suppose that $Z$ is simply-connected and $p: E \to B$ is a covering map. Then any map $f: Z \to B$ lifts against $p$.

Thus if $X \to B$ is a simply connected covering and $E \to B$ is any covering, we automatically get a map of covers $X \to E$. For this reason, simply connected covers are referred to as **universal covers**.
Proposition 2.8. Suppose \( \varphi : E_1 \to E_2 \) is a map of covers. Then \( \varphi \) is a covering map.

Proof. We start by showing that \( \varphi \) is surjective. Let \( e \in E_2 \). Let \( b = p_2(e) \), and pick any \( e' \in p_1^{-1}(b) \). Since \( E_2 \) is very connected, we can find a path \( \alpha : \varphi(e') \sim e \) in \( E_2 \). We can push this path \( \alpha \) down to a loop \( p_2 \alpha \) in \( B \) and then lift this uniquely to a path \( \tilde{\alpha} \) in \( E_1 \) starting at \( e' \). Now \( \varphi(\tilde{\alpha}) \) is a lift of \( p_2 \alpha \) in \( E_2 \) starting at \( \varphi(e') \), so by uniqueness of lifts, we must have \( \varphi(\tilde{\alpha}) = \alpha \). In particular, \( \varphi(\tilde{\alpha}(1)) = e \).

Now we show that \( e \) has an evenly-covered neighborhood of \( e \). We know that the point \( p_2(e) \in B \) has an evenly covered neighborhood \( U_2 \) (with respect to \( p_2 \)). Let \( U_1 \) be an evenly covered neighborhood, with respect to \( p_1 \), of \( p_2(e) \). Write \( U \) for the component of \( U_1 \cap U_2 \) containing \( p_2(e) \). Then \( p_2^{-1}(U) \cong \Pi V_i \). Let \( V_0 \) be the component containing \( e \). Write \( p_1^{-1}(U) \cong \Pi W_j \). Then, since \( U \) is connected, each \( V_i \) and \( W_j \) must be connected. It follows that \( \varphi \) takes each \( W_j \) into a single \( V_i \), so that \( \varphi^{-1}(V_0) \subseteq p_1^{-1}(U) \) is a disjoint union of some of the \( W_j \)'s, and it follows that \( \varphi \) restricts to a homeomorphism on each component because both \( p_1 \) and \( p_2 \) do so.

It follows that any universal cover \( X \to B \) covers every other covering \( E \to B \).

Remark 2.9. Recall that in the proof of Theorem 1.25, we ended up building a map of covers \( \varphi : X \to X \) corresponding to any point in the fiber \( F \), but we wanted to know it was in fact a homeomorphism. Prop 2.8 now gives us that it is a covering map, so that according to the homework, it suffices to show that the \( \varphi \) we constructed was injective. This can be seen by verifying that it is injective on each fiber.

2.2. The monodromy action. Our next goal is to completely understand the possible covers of a given space \( B \). There are two avenues of approach. On the one hand, Prop. 2.1 tells us that covering spaces give rise to subgroups of \( \pi_1(B) \), so we can try to understand the collection of subgroups. Another approach, which we will look at next, focuses on the fiber \( F = p^{-1}(b_0) \).

It will be convenient in what follows to write \( G = \pi_1(B, b_0) \) and \( F = p^{-1}(b_0) \subseteq E \). Given a loop \( \gamma \) based at \( b_0 \) and a point \( f \in F \), we will write \( \tilde{\gamma}_f \) for the lift of \( \gamma \) which starts at \( f \).

Theorem 2.10. Let \( p : E \to B \) be a covering and let \( F = p^{-1}(b) \) be the fiber over the basepoint. Then the function

\[
a : F \times \pi_1(B) \to F; \quad (f, [\gamma]) \mapsto \tilde{\gamma}_f(1)
\]

specifies a transitive right action of \( \pi_1(B) \) on the fiber \( F \). This is called the monodromy action.

Proof. Recall that we have already showed this to be well-defined.

Let \( c_{b_0} \) be the constant loop at \( b_0 \). Then the constant loop \( c_f \) at \( f \) in \( E \) is a lift of \( c_{b_0} \) starting at \( f \), so by uniqueness it must be the only lift. Thus \( f \cdot [c_{b_0}] = f \).

Now let \( \alpha \) and \( \beta \) be loops at \( b \). We wish to show that \( (f \cdot \alpha) \cdot \beta = f \cdot (\alpha \cdot \beta) \). Let \( f_2 = \tilde{\alpha}_f(1) \).

Then \( \tilde{\alpha}_f \cdot \tilde{\beta}_f \) is a (= the) lift of \( \alpha \cdot \beta \) starting at \( f \), so

\[
f \cdot (\alpha \cdot \beta) = \tilde{\alpha}_f \cdot \tilde{\beta}_f(1).
\]
On the other hand, \( f \cdot \alpha = \tilde{\alpha}_f(1) = f_2 \), so
\[
(f \cdot \alpha) \cdot \beta = f_2 \cdot \beta = \tilde{\beta}f_2(1)
\]

Finally, to see that this action is transitive, let \( f_1 \) and \( f_2 \) be points in the fiber \( F \). Let \( \gamma \) be a path in \( E \) from \( f_1 \) to \( f_2 \). Then \( \alpha = \rho \circ \gamma \) is a loop at \( b_0 \). Furthermore \( \tilde{\alpha}_{f_1} = \gamma \), so \( f_1 \cdot \alpha = \gamma(1) = f_2 \). □

Note that if we instead wrote path-composition in the “correct” order (i.e. in the same order as function composition), this would give a left action of \( \pi_1(B) \) on \( F \).

By the Orbit-Stabilizer theorem, since \( G \) acts transitively on \( F \), there is an isomorphism of right \( G \)-sets \( F \cong \bigsqcup_{e \in G_{e_0}} G_{e_0} \setminus G \), where \( G_{e_0} \leq G \) is the stabilizer of \( e_0 \).

**Proposition 2.11.** The stabilizer of \( e \in F \) under the monodromy action is the subgroup \( p_*(\pi_1(E, e)) \leq \pi_1(B, b_0) \).

**Proof.** Let \([\gamma] \in \pi_1(E, e)\). Then \( \gamma \) is a lift of \( p \circ \gamma \) starting at \( e \), so \( e \cdot p_*(\gamma) = \gamma(1) = e \). Thus \( p_*(\gamma) \) stabilizes \( e \).

On the other hand, let \([\alpha] \in \pi_1(B, b_0)\) and suppose that \( e \cdot [\alpha] = e \). This means that \( \alpha \) lifts to a loop \( \tilde{\alpha} \) in \( E \). Thus \( \alpha = \rho \circ \tilde{\alpha} \) and \([\alpha] \in p_*(\pi_1(E, e))\). □

**Corollary 2.12.** Let \( p : E \to B \) be a covering. Then, writing \( H = p_*(\pi_1(E, e)) \) the map
\[
H \setminus \pi_1(B, b) \xrightarrow{\sim} F.
\]
\[
H \gamma \mapsto f \cdot \gamma
\]
is an identification of right \( \pi_1(B) \)-sets

We have seen that any covering gives rise to a transitive \( G \)-set. We would also like to understand maps of coverings.

**Definition 2.13.** Let \( X \) and \( Y \) be (right) \( G \)-sets. A function \( f : X \to Y \) is said to be \( G \)-equivariant (or a map of \( G \)-sets) if \( f(xg) = f(x) \cdot g \) for all \( x \).

**Proposition 2.14.** Let \( \varphi : E_1 \to E_2 \) be a map of covers of \( B \). The induced map on fibers \( F_1 \to F_2 \) is \( \pi_1(B) \)-equivariant.

**Proof.** Let \([\gamma] \in \pi_1(B)\) and \( f \in F_1 \). We have \( f \cdot [\gamma] = \tilde{\gamma}_f(1) \), where \( \tilde{\gamma}_f \) is the lift of \( \gamma \) starting at \( f \). Similarly, we have \( \varphi(f) \cdot [\gamma] = \tilde{\gamma}_{\varphi(f)}(1) \). But \( \varphi(\tilde{\gamma}) \) is a lift of \( \gamma \) starting at \( \varphi(\gamma(0)) = \varphi(f) \), so \( \tilde{\gamma}_{\varphi(f)} = \varphi(\tilde{\gamma}_f) \).

Thus
\[
\varphi(f) \cdot [\gamma] = \tilde{\gamma}_{\varphi(f)}(1) = \varphi(\tilde{\gamma}_f)(1) = \varphi(\tilde{\gamma}_f(1)) = \varphi(f \cdot [\gamma]).
\]

□

**Proposition 2.15.** Let \( H, K \leq G \). Then every \( G \)-equivariant map \( \varphi : H \setminus G \to K \setminus G \) is of the form \( Hg \mapsto K\gamma g \) for some \( \gamma \in G \) satisfying \( \gamma H \gamma^{-1} \leq K \).

**Proof.** Since \( H \setminus G \) is a transitive \( G \)-set, an equivariant map out of it is determined by the value at any point. Suppose we stipulate
\[
He \mapsto K\gamma.
\]
Then equivariance would force
\[
Hg \mapsto K\gamma g.
\]
Is this well-defined? Since \( Hg = H\gamma h \) for any \( h \in H \), we would need \( K \gamma g = K \gamma h g \). Multiplying by \( g^{-1} \gamma^{-1} \) gives \( K = K \gamma h \gamma^{-1} \). Since \( h \in H \) is arbitrary, this says that \( \gamma H \gamma^{-1} \leq K \). □
Corollary 2.16. A $G$-equivariant map $\phi : H \backslash G \rightarrow K \backslash G$ exists if and only if $H$ is conjugate in $G$ to a subgroup of $K$. The two orbits are isomorphic (as right $G$-sets) if and only if $H$ is conjugate to $K$.

Notation. Given covers $(E_1, p_1)$ and $(E_2, p_2)$ of $B$, we denote by $\text{Map}_B(E_1, E_2)$ the set of covering homomorphisms $\varphi : E_1 \rightarrow E_2$. Given two right $G$-sets $X$ and $Y$, we denote by $\text{Hom}_G(X, Y)$ the set of $G$-equivariant maps $X \rightarrow Y$.

The following theorem classifies covering homomorphisms.

Theorem 2.17. Let $E_1$ and $E_2$ be coverings of $B$. Then Proposition 2.14 induces a bijection

$$\text{Map}_B(E_1, E_2) \cong \text{Hom}_G(F_1, F_2).$$

Proof. The key is that a covering homomorphism is a lift in the diagram to the right. Uniqueness of lifts gives injectivity in the theorem. For surjectivity, we use the lifting criterion Prop 2.5. Thus suppose given a $G$-equivariant map $\lambda : F_1 \rightarrow F_2$ and fix a point $e_1 \in F_1$. Let $e_2 = \lambda(e_1) \in F_2$. The lifting criterion will provide a lift if we can verify that

$$(p_1)_*(\pi_1(E_1, e_1)) \leq (p_2)_*(\pi_1(E_2, e_2)).$$

But remember that according to Prop 2.11, these are precisely the stabilizers of $e_1$ and $e_2$, respectively. Writing $H_1$ and $H_2$ for these groups, the map $\lambda : F_1 \rightarrow F_2$ corresponds to a map

$$\hat{\lambda} : H_1 \backslash G \rightarrow H_2 \backslash G.$$

According to Prop 2.15, this means that $\gamma H_1 \gamma^{-1} \leq H_2$, where $\hat{\lambda}(H_1 e) = H_2 \gamma$. The fact that $\lambda(e_1) = e_2$ means that $\gamma = e$. So $H_1 \leq H_2$ as desired. $\blacksquare$

Corollary 2.18. If $E$ is a cover of $B$, then we have group isomorphisms

$$\text{Aut}_B(E) \cong \text{Aut}_G(H \backslash G, H \backslash G) \cong N_G(H)/H,$$

where $N_G(H)$ is the normalizer of $H$ in $G$, consisting of those elements of $G$ which conjugate $H$ to itself.

Proof. Theorem 2.17 gives the first bijection. By Corollary 2.15, we have a surjective group homomorphism $N_G(H) \rightarrow \text{Aut}_G(H \backslash G, H \backslash G)$, and it remains only to identify the kernel. But $\gamma \in N_G(H)$ lies in the kernel if $Hg \rightarrow H\gamma g$ is the identity map of $H \backslash G$, which happens just if $\gamma \in H$. So we conclude that the kernel is $H$. $\blacksquare$

The quotient group $N_G(H)/H$ is known as the Weyl group of $H$ in $G$ and is sometimes denoted $W_G(H)$.

Fri, Feb. 9

2.3. The classification of covers. We have almost shown that working with covers of $B$ is the same as working with transitive right $G$-sets (technically, we are heading to an “equivalence of categories”). All that is left is to show that for every $G$-orbit $F$, there is a cover $p : E \rightarrow B$ whose fiber is $F$ as a $G$-set.

We assume that $B$ has a universal cover $q : X \rightarrow B$. Recall that we showed in Theorem 1.25 that the group of deck transformations of $X$ is isomorphic to $G$.

Proposition 2.19. The (left) action of $G$ on $X$ via deck transformations is free and properly discontinuous.
Let \( x \in X \) and suppose \( gx = x \) for some \( g \in G \). Recall that here \( g \) is a covering homomorphism \( X \to X \) and thus a lift of \( q : X \to B \). By the uniqueness of lifts, since \( g \) looks like the identity at the point \( x \), it must be the identity. This shows the action is free.

Again, let \( x \in X \). We want to find a neighborhood \( V \) of \( x \) such that only finitely many translates \( gV \) meet \( V \). Consider \( b = q(x) \). Let \( U \) be an evenly-covered neighborhood of \( b \). Then \( q^{-1}(U) \cong \coprod V_i \), and \( x \in V_j \) for some \( j \). Recall that \( G \) freely permutes the pancakes \( V_i \). In particular, the only translate of \( V_j \) that meets \( V_j \) is the identity translate \( eV_j \).

According to Homework IV.2, this means that the quotient map \( X \to G\backslash X \) is a cover. Actually, the cover \( X \xrightarrow{\pi} B \) factors through a homeomorphism \( G\backslash X \cong B \). If we consider the action of a subgroup \( H \leq G \), it is still free and properly discontinuous. So we get a covering \( q_H : X \to H\backslash X = X_H \) for every \( H \). Moreover, the universal property of quotients gives an induced map \( p_H : H\backslash X \to B \).

**Proposition 2.20.** The map \( p_H : H\backslash X \to B \) is a covering map, and the fiber \( F \) is isomorphic to \( H\backslash G \) as a \( G \)-set.

**Proof.** Let \( b \in B \). Then we have a neighborhood \( U \) which is evenly-covered by \( q \). Recall again that the \( G \)-action, and therefore also the \( H \)-action, simply permutes the pancakes in \( p^{-1}(U) \). We thus get an action of \( H \) on the indexing set \( I \) for the pancakes in \( p^{-1}(U) \). If we write \( W_i = q_H(V_i) \), we thus have the diagram

\[
\begin{array}{cccc}
q^{-1}(U) & \xrightarrow{q_H} & p_H^{-1}(U) & \xrightarrow{p_H} & U \\
\cong & \uparrow & \cong & \uparrow \\
\coprod V_i & \xrightarrow{} & \coprod W_j & \to & U
\end{array}
\]

To see that the restriction of \( p_H \) to a single \( W_j \) gives a homeomorphism, we use the fact that \( q_H : V_j \to W_j \) is a homeomorphism, since \( q_H : X \to X_H \) is a covering, and that \( q : V_j \to U \) is a homeomorphism. It follows that \( p_H = q \circ q_H^{-1} \) is a homeomorphism.

For the identification of the fiber \( F \subseteq X_H \), notice that the \( H \)-action on \( X \) acts on each fiber separately, and the quotient of this action on the fiber of \( X \) gives precisely \( H\backslash G \).

**Example 2.21.** Suppose that \( G = \Sigma_3 \), the symmetric group on 3 letters, and let \( H = \{e, (12)\} \leq G \). If we take an evenly-covered neighborhood \( U \) in \( B \), then the situation described in the proof above is given in the picture to the right.

As an aside, note that \( X_H \) here is an example of a covering in which the deck transformations do not act transitively on the fibers.

To sum up, we have shown that if \( B \) has a universal cover, then the assignment \((E,p) \mapsto F \) gives an “equivalence of categories” between coverings of \( B \) (\( \text{Cov}_B \)) and \( G \)-orbits (\( \text{Orb}_G \)).
Mon, Feb. 12

We can form a category Cov\(_B\) whose objects are the covers of \(B\) and whose morphisms are the maps of covers. We can also form a category Orb\(_G\) whose objects are the transitive (right) \(G\)-sets. Our recent discussion has shown that the assignment (technically 'functor')

\[
\text{Cov}_B \rightarrow \text{Orb}_G, \quad (E,p) \mapsto F := p^{-1}(b_0)
\]

is an equivalence of categories. This means that

1. (fully faithful) We have a bijection Cov\(_B\)(\(E, E'\)) \(\cong\) Orb\(_G\)(\(F, F'\))
2. (essentially surjective) Every \(G\)-orbit arises in this way, meaning that any \(G\)-orbit is isomorphic to \(p^{-1}(b_0)\) for some cover of \(B\).

One consequence of having an equivalence of categories is that this produces a bijection between isomorphism classes of objects.

**Corollary 2.22.** The fiber functor Cov\(_B\) \(\rightarrow\) Orb\(_G\) induces a bijection

\[
\{\text{isomorphism classes of covers}\} \cong \{\text{isomorphism classes of orbits}\} \\
\cong \{\text{conjugacy classes of subgroups of } G\}
\]

Note that there is no obvious choice of functor in the other direction. Given a \(G\)-orbit \(X\), picking a point in the orbit produces an isomorphism to some \(H \setminus G\), and then Proposition 2.20 produces a cover whose fiber is isomorphic to \(X\). But this really does involve making a choice. This is a pretty typical situation: a functor that is essentially surjective and fully faithful is called an equivalence of categories, but to produce a functor that looks like an inverse, choices need to be made.

2.4. **Existence of universal covers.** The last result we need to tie this story together is the existence of universal covers.

**Definition 2.23.** Let \(B\) be any space. A subset \(U \subseteq B\) is relatively simply connected (in \(B\)) if every loop in \(U\) is contractible in \(B\). We say that \(B\) is semilocally simply connected if every point has a relatively simply connected neighborhood.

**Remark 2.24.** Note that if \(B\) is very connected and semilocally simply connected, then every point has a path-connected, relatively simply connected neighborhood. This is because if \(b \in U\) is relatively simply connected, then the path component of \(b\) in \(U\) is open (\(B\) is locally path-connected) and also relatively simply-connected (true of any subset of a relatively simply connected subset).

**Theorem 2.25.** Let \(B\) be very connected. Then there exists a universal cover \(X \rightarrow B\) if and only if \(B\) is semilocally simply connected.

**Proof.** The forward implication is left as an exercise. For convenience, we fix a basepoint \(b_0 \in B\).

We start by working backwards. That is, suppose that \(q : X \rightarrow B\) exists. Given a point \(b \in B\), what can we say about the fiber \(q^{-1}(b)\)? Pick a basepoint \(x_0 \in q^{-1}(b_0)\). Then, for each \(f \in q^{-1}(b)\), we get a (unique) path-homotopy class of paths \(\alpha : x_0 \leadsto f\). Composing with the covering map \(q\) gives a (unique) path-homotopy class of paths \(q \circ \alpha : b_0 \leadsto b\). This now gives a description of the fiber \(q^{-1}(b)\) purely in terms of \(B\).

We now take this as a starting point. As a set, we take \(X\) to be the set of path-homotopy classes of paths starting at \(b_0\). The map \(q : X \rightarrow B\) takes a class \([\gamma]\) to the endpoint \(\gamma(1)\). It remains to (1) topologize \(X\), (2) show that \(q\) is a covering map, and (3) show that \(X\) is simply-connected.

We specify the topology on \(X\) by giving a basis. Let \(\gamma\) be a path in \(B\) starting at \(b_0\). Let \(U\) be any path-connected, relatively simply-connected neighborhood of the endpoint \(\gamma(1)\). Define a...
subset $U[\gamma] \subseteq X$ to be the set of equivalence classes of paths of the form $[\gamma \delta]$, where $\delta : I \rightarrow U$ is a path in $U$. These cover $X$ since each $[\gamma]$ is contained in some $U[\gamma]$ by Remark 2.24. Now suppose that $\gamma \in U_1[\gamma_1] \cap U_2[\gamma_2]$. Then the path-component of $\gamma(1)$ in $U_1 \cap U_2$ is again path-connected and relatively simply connected. Thus

$$\gamma \in U[\gamma] \subseteq U_1[\gamma_1] \cap U_2[\gamma_2].$$

We have shown that the $U[\gamma]$ give a basis for a topology on $X$.

Next, we show that $q$ is continuous. Let $V \subseteq B$ be open and let $q([\gamma]) \in V$, so that $\gamma(1) \in V$. Then we can find a path-connected, relatively simply connected $U$ satisfying $\gamma(1) \in U \subseteq V$. So $U[\gamma]$ is a neighborhood of $[\gamma]$ in $q^{-1}(V)$, as desired.

Since $B$ is path-connected, it follows that $q$ is surjective. Let $b \in B$ and let $b \in U$ be a path-connected, relatively simply-connected neighborhood. We claim that $U$ is evenly covered by $q$. First, we claim that

$$q^{-1}(U) = \bigcup_{[\gamma] \in q^{-1}(b)} U[\gamma].$$

It is clear that the RHS is contained in the LHS. Suppose that $q([\alpha]) \subseteq U$. Then $\alpha(1) \in U$ and we may pick a path $\delta : \alpha(1) \sim b$ in $U$. Then $\alpha \in U[\alpha \delta]$.

(This will be continued on Monday . . .)

\[\hfill\]

**Fri, Feb. 16**

Exam day!!
Mon, Feb. 19

Proof. (Continued . . . ) Finally, we wish to show that this is a disjoint union. By the definition of the topology on $X$, each $U[\gamma]$ is open. Thus suppose that $[\alpha_1] \in U[\gamma_1] \cap U[\gamma_2]$. This means that

$$[\alpha] = [\gamma_1 \delta_1] = [\gamma_2 \delta_2].$$

In other words,

$$[\gamma_1 \delta_1 \delta_2] = [\gamma_2].$$

Since $U$ is relatively simply-connected, this implies that $[\gamma_1] = [\gamma_2]$. So any two overlapping $U[\gamma]$ are in fact the same. To finish the proof that $q$ is a covering, we need to show that $q$ restricts to a homeomorphism $q : U[\gamma] \cong U$. Surjectivity follows from the assumption that $U$ is path-connected. Injectivity is the relatively simply-connected hypothesis. Finally, $q$ takes any basis $V[\lambda]$ to the open set $V$ (since $V$ is path-connected), so it is open. We have shown that $q$ is a covering map.

The final step is to show that $X$ is very connected and simply connected. Since $X$ is locally homeomorphic to $B$ and $B$ is locally path-connected, it follows that the same is true of $X$. Next, we show that $X$ is path-connected (and therefore connected). Let $[\gamma] \in X$. We define a path $h$ in $X$ from the constant path $[c_0]$ to $[\gamma]$ by $h(s) = [\gamma][0,s]$. In the interest of time, we skip the verification that $h$ is continuous (but see Lee, proof of Theorem 11.43).

To see that $X$ is simply connected, let $\Gamma$ be a loop in $X$ at the basepoint $[c_b]$. Write $\gamma = q \circ \Gamma$. Then $\Gamma$ is a lift of $\gamma$, but so is the loop $s \mapsto [\gamma][0,s]$. By uniqueness of lifts, $[\Gamma(s)] = [\gamma][0,s]$. Then, since $\Gamma$ is a loop, we have

$$[\gamma] = [\gamma][0,1] = [\Gamma(1)] = [\Gamma(0)] = [\gamma][0,0] = [c_b].$$

In other words, $\gamma$ is null. Since $q$ is a covering, this implies that $\Gamma$ is null as well.

We have shown that if a space is semilocally simply-connected, then it has a universal cover. So to provide an example of a space without a universal cover, it suffices to give an example of a space with a point which has no relatively simply connected neighborhood.

Example 2.26 (The Hawaiian earring). Let $C_n \subseteq \mathbb{R}^2$ be the circle of radius $1/n$ centered at $(1/n, 0)$. So each such circle is tangent to the $y$-axis at the origin. Let $C = \cup_n C_n$. We claim that the origin has no relatively simply connected neighborhood. Indeed, let $U$ be any neighborhood of the origin. Then for large enough $n$, the circle $C_n$ is contained in $U$. A loop $\alpha$ that goes once around the circle $C_n$ is not contractible in $C$. To see this, note that the map $r_n : C \rightarrow S^1$ which collapses every circle except for $C_n$ is a retraction. The loop $r \circ \alpha$ is not null, so $\alpha$ can’t be null.

This example looks like an infinite wedge of circles, but it is not just a wedge. For instance, in each $C_n$ consider an open interval $U_n$ of radian length $1/n$ centered at the origin (or the open left semicircle, if you prefer). The union $U = \cup_n U_n$ of the $U_n$’s is open in the infinite wedge of circles but not in $C$, since no $\epsilon$-neighborhood of the origin is contained in $U$.

Wed, Feb. 21

3. The van Kampen Theorem

The focus of the next unit of the course will be on computation of fundamental groups.

One example we have already studied is the fundamental group of $S^1 \vee S^1$. We saw that this is the free group on two generators. We will see similarly that the fundamental group of $S^1 \vee S^1 \vee S^1$ is a free group on three generators. We will also want to compute the fundamental group of the two-holed torus (genre two surface), the Klein bottle, and more.

The main idea will be to decompose a space $X$ into smaller pieces whose fundamental groups are easier to understand. For instance, if $X = U \cup V$ and we understand $\pi_1(U)$, $\pi_1(V)$, and $\pi_1(U \cap V)$, we might hope to recover $\pi_1(X)$. 

20
Proposition 3.1. Suppose that $X = U \cup V$, were $U$ and $V$ are path-connected open subsets and both contain the basepoint $x_0$. If $U \cap V$ is also path-connected, then the smallest subgroup of $\pi_1(X)$ containing the images of both $\pi_1(U)$ and $\pi_1(V)$ is $\pi_1(X)$ itself.

In group theory, we would say $\pi_1(X) = \pi_1(U) \pi_1(V)$.

Note that we really do need the assumption that $U \cap V$ is path-connected. If we consider $U$ and $V$ to be open arcs that together cover $S^1$, then both $U$ and $V$ are simply-connected, but their intersection is not path-connected. Note that here that the product of two trivial subgroups is not $\pi_1(S^1) \cong \mathbb{Z}!$

Proof. Let $\gamma : I \rightarrow X$ be a loop at $x_0$. By the Lebesgue number lemma, we can subdivide the interval $I$ into smaller intervals $[s_i, s_{i+1}]$ such that each subinterval is taken by $\gamma$ into either $U$ or $V$. We write $\gamma_1$ for the restriction of $\gamma$ to the first subinterval. Suppose, for the sake of argument, that $\gamma_1$ is a path in $U$ and that $\gamma_2$ is a path in $V$. Since $U \cap V$ is path-connected, there is a path $\delta_1$ from $\gamma_1(1)$ to $x_0$. We may do this for each $\gamma_i$. Then we have

$$[\gamma] = [\gamma_1] * [\gamma_2] * [\gamma_3] * \cdots * [\gamma_n] = [\gamma_1 * \delta_1] * [\delta_1^{-1} * \gamma_2 * \delta_2] * \cdots * [\delta_n^{-1} * \gamma_n].$$

This expresses the loop $\gamma$ as a product of loops in $U$ and loops in $V$. ■

This is a start, but it is not the most convenient formulation. In particular, if we would like to use this to calculate $\pi_1(X)$, then thinking of the product of $\pi_1(U)$ and $\pi_1(V)$ inside of $\pi_1(X)$ is not so helpful. Rather, we would like to express this in terms of some external group defined in terms of $\pi_1(U)$ and $\pi_1(V).$ We have homomorphisms $\pi_1(U) \rightarrow \pi_1(X)$, $\pi_1(V) \rightarrow \pi_1(X)$, and we would like to put these together to produce a map from some sort of product of $\pi_1(U)$ and $\pi_1(V)$ to $\pi_1(X).$ Could this be the direct product $\pi_1(U) \times \pi_1(V)?$. No. Elements of $\pi_1(U)$ commute with elements of $\pi_1(V)$ in the product $\pi_1(U) \times \pi_1(V)$, so this would also be true in the image of any homomorphism $\pi_1(U) \times \pi_1(V) \rightarrow \pi_1(X)$.

What we want instead is a group freely built out of $\pi_1(U)$ and $\pi_1(V)$. The answer is the free product $\pi_1(U) * \pi_1(V)$ of $\pi_1(U)$ and $\pi_1(V)$. Its elements are finite length words $g_1 g_2 g_3 g_4 \cdots g_n$, where each $g_i$ is in either $\pi_1(U)$ or in $\pi_1(V)$. Really, we use the reduced words, where none of the $g_i$ is allowed to be an identity element and where if $g_i \in \pi_1(U)$ then $g_{i+1} \in \pi_1(V)$.

Example 3.2. We have already seen an example of a free product. The free group $F_2$ is the free product $\mathbb{Z} * \mathbb{Z}$.

Example 3.3. Similarly, the free group $F_3$ on three letters is the free product $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.

Example 3.4. Let $C_2$ be the cyclic group of order two. Then the free product $C_2 * C_2$ is an infinite group. If we denote the nonidentity elements of the two copies of $C_2$ by $a$ and $b$, then elements of $C_2 * C_2$ look like $a, ab, abababa, bababa$, etc.

Note that there is a homomorphism $C_2 * C_2 \rightarrow C_2$ that sends both $a$ and $b$ to the nontrivial element. The kernel of this map is all words of even length. This is the (infinite) subgroup generated by the word $ab$ (note that $ba = (ab)^{-1}$). In other words, $C_2 * C_2$ is an extension of $C_2$ by the infinite cyclic group $\mathbb{Z}$. Another way to say this is that $C_2 * C_2$ is a semidirect product of $C_2$ with $\mathbb{Z}$.

The free product has a universal property, which should remind you of the property of the disjoint union of spaces $X \amalg Y$. First, for any groups $H$ and $K$, there are inclusion homomorphisms $H \rightarrow H \rtimes K$ and $K \rightarrow H \rtimes K$.

Proposition 3.5. Suppose that $G$ is any group with homomorphisms $\varphi_H : H \rightarrow G$ and $\varphi_K : K \rightarrow G$. Then there is a (unique) homomorphism $\Phi : H \rtimes K \rightarrow G$ which restricts to the given homomorphisms from $H$ and $K$.
In other words, the free product is the coproduct in the world of groups.

So Proposition 3.1 can be restated as follows:

**Proposition 3.6** (weak van Kampen). Suppose that \( X = U \cup V \), where \( U \) and \( V \) are path-connected open subsets and both contain the basepoint \( x_0 \). If \( U \cap V \) is also path-connected, then the natural homomorphism

\[
\Phi : \pi_1(U) * \pi_1(V) \to \pi_1(X)
\]

is surjective.

**Fri, Feb. 23**

Now that we have a surjective homomorphism to \( \pi_1(X) \), the next step is to understand the kernel \( N \). Indeed, then the First Isomorphism Theorem will tell us that \( \pi_1(X) \cong (\pi_1(U) * \pi_1(V))/N \). Here is one way to produce an element of the kernel. Consider a loop \( \alpha \) in \( U \cap V \). We can then consider its image \( \alpha_U \in \pi_1(U) \) and \( \alpha_V \in \pi_1(V) \). Certainly these map to the same element of \( \pi_1(X) \), so \( \alpha_U\alpha_V^{-1} \) is in the kernel.

**Proposition 3.7.** With the same assumptions as above, the kernel \( K \) of \( \pi_1(U) * \pi_1(V) \to \pi_1(X) \) is the normal subgroup \( N \) generated by elements of the form \( \alpha_U\alpha_V^{-1} \).

Recall that the normal subgroup generated by the elements \( \alpha_U\alpha_V^{-1} \) can be characterized either as (1) the intersection of all normal subgroups containing the \( \alpha_U\alpha_V^{-1} \) or (2) the subgroup generated by all conjugates \( g\alpha U\alpha V^{-1}g^{-1} \).

We will put off the proof of Proposition 3.7 for the moment. Assembling these recent results gives the van Kampen theorem:

**Theorem 3.8** (Van Kampen). Suppose that \( X = U \cup V \), where \( U \) and \( V \) are path-connected open subsets and both contain the basepoint \( x_0 \). If \( U \cap V \) is also path-connected, then

\[
\pi_1(X, x_0) \cong (\pi_1(U, x_0) * \pi_1(V, x_0))/N,
\]

where \( N \subseteq \pi_1(U, x_0) * \pi_1(V, x_0) \) is the normal subgroup generated by elements of the form \( \omega_U(\alpha)\omega_V(\alpha)^{-1} \), for \( \alpha \in \pi_1(U \cap V, x_0) \).

There is another, more elegant, way to state the Van Kampen theorem.

**Definition 3.9.** Suppose given a pair of group homomorphisms \( \varphi_G : H \to G \) and \( \varphi_K : H \to K \). We define the **amalgamated free product** (or simply amalgamated product) to be the quotient

\[
G *_H K = (G * K)/N,
\]

where \( N \subseteq G * K \) is the normal subgroup generated by elements of the form \( \varphi_G(h)\varphi_K(h)^{-1} \).

It is easy to check that the amalgamated free product satisfies the universal property of the pushout in the category of groups.

**Theorem 3.10** (Van Kampen, restated). Let \( X \) be given as a union of two open, path-connected subsets \( U \) and \( V \) with path-connected intersection \( U \cap V \). Then the inclusions of \( U \) and \( V \) into \( X \) induce an isomorphism

\[
\pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \cong \pi_1(X).
\]

Since the pasting lemma tells us that in this situation, \( X \) can itself be written as a pushout, the Van Kampen theorem can be interpreted as the statement that, under the given assumptions, the fundamental group construction takes a pushout of spaces to a pushout of groups.

One important special case of this result is when \( U \cap V \) is simply connected.
Example 3.11. Take $X = S^1 \vee S^1$. Take $U$ to be an open set containing one of the circles, plus an $\epsilon$-ball around the basepoint in the other circle, and similarly for $V$ with regard to the other circle. Then the intersection $U \cap V$ looks like an ‘X’ and is contractible, and $U$ and $V$ are both equivalent to $S^1$. We conclude from this that
\[
\pi_1(S^1 \vee S^1) \cong \pi_1(S^1) \ast \pi_1(S^1) \cong \mathbb{Z} \ast \mathbb{Z} \cong F_2.
\]

Example 3.12. Take $X = S^1 \vee S^1 \vee S^1$. We can take $U$ to be a neighborhood of $S^1 \vee S^1$ and $V$ to be a neighborhood of the remaining $S^1$. Then
\[
\pi_1(S^1 \vee S^1 \vee S^1) \cong (\mathbb{Z} \ast \mathbb{Z}) \ast \mathbb{Z} \cong F_3.
\]

Example 3.13. Take $X = S^1 \vee S^2$. Take $U$ to be a neighborhood of $S^1$ and $V$ to be a neighborhood of $S^2$. We conclude from this that
\[
\pi_1(S^1 \vee S^2) \cong \pi_1(S^1) \ast \pi_1(S^2) \cong \mathbb{Z}.
\]

A natural question now is whether $\pi_1(X \vee Y)$ is always the free product of the $\pi_1(X)$ and $\pi_1(Y)$. Not quite, but a mild assumption allows us to make the conclusion. Note that in the $S^1 \vee S^1$ example, we needed to know that the neighborhoods $U$ and $V$ were homotopy equivalent to $S^1$ (and that the intersection was contractible).

Definition 3.14. We say that $x_0 \in X$ is a nondegenerate basepoint for $X$ if $x_0$ has a neighborhood $U$ such that $x_0$ is a deformation retract of $U$.

Proposition 3.15. Let $x_0$ and $y_0$ be nondegenerate basepoints for $X$ and $Y$, respectively. Then
\[
\pi_1(X \vee Y) \cong \pi_1(X) \ast \pi_1(Y).
\]

Proof. Suppose that $x_0$ is a deformation retract of the neighborhood $N_X \subseteq X$ and that $y_0$ is a deformation retract of the neighborhood $N_Y \subseteq Y$. Let $U = X \vee N_Y$ and $V = N_X \vee Y$. Then $U \cap V = N_X \vee N_Y$. The retracting homotopies for $N_X$ and $N_Y$ give $U \simeq X$, $V \simeq Y$, and $U \cap V \simeq \ast$. The van Kampen theorem then gives the conclusion. \qed

23
Lemma 3.16 (Square Lemma). Let $\alpha$, $\beta$, $\gamma$, and $\delta$ be paths in $X$ with
\[ \alpha(0) = \gamma(0), \quad \alpha(1) = \beta(0), \quad \gamma(1) = \delta(0), \quad \beta(1) = \delta(1). \]
Then path-homotopies $h : \alpha \ast \beta \simeq_p \gamma \ast \delta$ correspond bijectively to maps $H : I^2 \rightarrow X$ as in the figure.

Proof of Proposition 3.7. Again, it is clear that the kernel $K$ must contain the subgroup $N$. It remains to show that $K \leq N$. Consider an element of $K$. For simplicity, we assume it is $\alpha_1 \cdot \beta_1 \cdot \alpha_2$, where $\alpha_i \in \pi_1(U)$ and $\beta_1 \in \pi_1(V)$. The assumption that this is in $K$ means that there exists a homotopy $H : I \times I \rightarrow X$ from the path composition $\alpha_1 \ast \beta_1 \ast \alpha_2$ in $X$ to the constant loop.

By the Lebesgue lemma, we may subdivide the square into smaller squares such that each small square is taken by $H$ into either $U$ or $V$. Again, we suppose for simplicity that this divides $\alpha_1$ into $\alpha_{11}$ and $\alpha_{12}$ and $\beta_1$ into $\beta_{11}$ and $\beta_{12}$ (and $\alpha_2$ is not subdivided).

Note that we cannot write
\[ \alpha_1 \cdot \beta_1 \cdot \alpha_2 = \alpha_{11} \cdot \alpha_{12} \cdot \beta_{11} \cdot \beta_{12} \cdot \alpha_2 \]
in $\pi_1(U) \ast \pi_1(V)$ since these are not all loops. But we can fix this, using the same technique as in the proof of Prop 3.1. In other words, we append a path $\delta$ back to $x_0$ at the end of every path on an edge of a square. If that path is in $U$ (or $V$ or $U \cap V$), we take $\delta$ in $U$ (or $V$ or $U \cap V$). Also, if the path already begins or ends at $x_0$, we do not append a $\delta$. For convenience, we keep the same notation, but remember that we have really converted all of these paths to loops.

Let us turn our attention now to the homotopy $H$ on the first (lower-left) square. Either $H$ takes this into $U$ or into $V$. If it is $U$, then we get a path homotopy in $U$ $\alpha_{11} \simeq_p \gamma_1 \cdot v_{11}^{-1}$. If, on the other hand, $H$ takes this into $V$, then it follows that $\alpha_{11}$ is really in $U \cap V$. This gives us a path homotopy in $V$ $\alpha_{11} \simeq_p \gamma_1 \cdot v_{11}^{-1}$. But the group element $\alpha_{11}$ comes from $\pi_1(U)$ in the free product $\pi_1(U) \ast \pi_1(V)$. We would like to replace this with the element $\alpha_{11}$ from $\pi_1(V)$.

Lemma 3.17. Let $\gamma$ be any loop in $U \cap V$. Then, in the quotient group $Q = (\pi_1(U) \ast \pi_1(V))/N$, the elements $\gamma_U$ and $\gamma_V$ are equivalent.

Proof. The point is that
\[ \gamma_V N = \gamma_U \gamma_U^{-1} \gamma_V N = \gamma_U \cdot \left( (\gamma^{-1})U(\gamma^{-1})_V^{-1} \right) N = \gamma_U N. \]

From here on out, we work in the quotient group $Q$. The goal is to show that the original element $\alpha_1 \cdot \beta_1 \cdot \alpha_2$ is trivial in $Q$. According to the above, we can replace $(\alpha_1)_U(\beta_1)_V(\alpha_2)_U$ with either
\[ (\gamma_1)_U(\gamma^{-1}_1)_U(\alpha_{12})_U(\beta_{11})_V(\beta_{12})_V(\alpha_2)_U \]
or
\[ (\gamma_1)_V(\gamma^{-1}_1)_V(\alpha_{12})_U(\beta_{11})_V(\beta_{12})_V(\alpha_2)_U. \]
We then do the same with each of $\alpha_{12}, \ldots, \alpha_2$. The resulting expression will have adjacent terms $v_i$ and $v_i^{-1}$. For the same $i$, these two loops may have the same label $(U$ or $V$) or different labels. But by the lemma, we can always change the label if the loop lies in the intersection. So we get
the path-composition of the paths along the top edges of the bottom squares. We then repeat
the procedure, moving up rows until we get to the very top. But of course the top edges of the
top squares are all constant loops. It follows that we end up with the trivial element (of $Q$). So
$K = N$. ■

The next application is the computation of the fundamental group of any graph. We start
by specifying what we mean by a graph. Recall that $S^0 \subseteq \mathbb{R}$ is usually defined to be the set
$S^0 = \{-1, 1\}$. For the moment, we take it to mean instead $S^0 = \{0, 1\}$ for convenience.

Definition 3.18. A graph is a 1-dimensional CW complex.

Of special importance will be the following type of graph.

Definition 3.19. A tree is a connected graph such that it is not possible to start at a vertex $v_0$,
travel along successive edges, and arrive back at $v_0$ without using the same edge twice.
(Give examples and nonexamples)

Proposition 3.20. Any tree is contractible. Even better, if $v_0$ is a vertex of the tree $T$, then $v_0$ is
a deformation retract of $T$.

Proof. We give the proof in the case of a finite tree. Use induction on the number of edges. If $T$ has
one edge, it is homeomorphic to $I$. Assume then that any tree with $n$ edges deformation retracts
onto any vertex and let $T$ be a tree with $n + 1$ edges. Let $v_0 \in T$. Now let $v_1 \in T$ be a vertex that
is maximally far away from $v_0$ in terms of number of edges traversed. Then $v_1$ is the endpoint of a
unique edge $e$, which we can deformation retract onto its other endpoint. The result is then a tree
with $n$ edges, which deformation retracts onto $v_0$. ■

Wed, Feb. 28

Corollary 3.21. Any tree is simply connected.

Definition 3.22. If $X$ is a graph and $T \subseteq X$ is a tree, we say that $T$ is a maximal tree if it is
not contained in any other (larger) tree.

By Zorn’s Lemma, any tree is contained in some maximal tree.

Theorem 3.23. Let $X$ be a connected graph and let $T \subseteq X$ be a maximal tree. The quotient space
$X/T$ is a wedge of circles, one for each edge not in the tree. The quotient map $q : X \rightarrow X/T$ is a
homotopy equivalence.

Proof. Since $T$ contains every vertex, all edges in the quotient become loops, or circles. To see
that $q$ is a homotopy equivalence, we first define a map $b : X/T \cong \bigvee S^1 \rightarrow X$. Recall that to
define a continuous map out of a wedge, it suffices to specify the map out of each wedge summand.
Fix a vertex $v_0 \in T \subseteq X$. Pick a deformation retraction $T$ down to $v_0$ as in Proposition 3.20.
Then, for each vertex $v$, the homotopy provides a path $\alpha_{v_0} : v_0 \rightarrow v$. Now suppose we have a circle
corresponding to the edge $e$ in $X$ from $v_1$ to $v_2$. We then send our circle to the loop $\alpha_{v_1} e \alpha_{v_2}^{-1}$.

The composition $q \circ b$ on a wedge summand $S^1$ looks like $c \ast \text{id} \ast c$ and is therefore
homotopic to the identity. For the other composition, we want to extend the given homotopy on $T$ to a homotopy on $X$. For simplicity, we give the argument in the
case that $X = T \cup e$ has a single edge not in a maximal tree. We wish to define
a homotopy $h : X \times I \rightarrow X$, but we already have the homotopy on the subspace
$T \times I$. It remains to specify the homotopy on $e \times I$, where we already have the
homotopy on the edges $e_0 \times I$ and $e_1 \times I$. At time 0, the map $b \circ q$ takes $e$ to
$\alpha_1 \ast e \ast \alpha_2^{-1}$, whereas at time 1, the identity map takes $e$ to $e$. We are now done by
the Square Lemma (3.16). ■
Corollary 3.24. The fundamental group of any graph is a free group.

We will use this to deduce an algebraic result about free groups. But first, a result about coverings of graphs.

Theorem 3.25. Let \( p : E \to B \) be a covering, where \( B \) is a connected graph. Then \( E \) is also a connected graph.

Proof. Recall our definition of a graph. It is a space obtained by glueing a set of edges to a set of vertices. Let \( B \) be the vertices of \( B \) and \( B_1 \) be the set of edges. Let \( E_0 \subseteq E \) be \( p^{-1}(B_0) \) and define

\[
E_1 \subseteq B_1 \times E_0
\]

to be the set of pairs \((\alpha : \{0,1\} \to B_0,v)\) such that \( \alpha(0) = p(v) \). We then have compatible maps \( E_0 \to E \) and \( \Pi E_1 I \to E \). The second map is given by the unique path-lifting property. These assemble to give a continuous map from the pushout

\[
f : \hat{E} = E_0 \cup \bigsqcup_{\epsilon_1} \Pi E_1 I \to E.
\]

This pushout is a 1-dimensional CW complex, which is our definition of a graph.

Fri, Mar. 2

To see that \( f \) is surjective, let \( x \in E \). Then \( p(x) \) lies in some 1-cell \( \beta \) of \( B \). Pick a path \( \gamma \) in \( B \), lying entirely in \( \beta \), from \( p(x) \) to a vertex \( b_0 \). Then \( \gamma \) lifts uniquely to a path \( \hat{\gamma} \) starting at \( x \) in \( \hat{E} \). Write \( v = \hat{\gamma}(1) \). Then \( x \) lives in the 1-cell \((\beta,v)\), so \( f \) is surjective.

The restriction of \( f \) to \( E_0 \) is injective, since this is the inclusion of the subset \( E_0 \to E \). If \( y \) and \( z \) are two points of \( \hat{E} \), where \( z \) is not a zero-cell and \( f(y) = f(z) \), then \( pf(y) = pf(z) \) in \( B \). Since \( pf(z) \) is not a 0-cell of \( B \), we conclude that \( y \) is also not a 0-cell in \( \hat{E} \). Now \( pf(y) \) and \( pf(z) \) live in the same 1-cell of \( B \), and since \( f(y) = f(z) \) in \( E \), uniqueness of lifts tells us that \( y \) and \( z \) live in the same 1-cell of \( \hat{E} \). But the restriction of \( pf \) to the interior of this 1-cell is a homeomorphism onto the 1-cell of \( B \). Since \( pf(y) = pf(z) \), we conclude that \( y = z \).

There are now several arguments for why this must be a homeomorphism. If \( B \) is a finite graph and \( E \) is a finite covering, we are done since \( E' \) is compact and \( E \) is Hausdorff (since \( B \) is Hausdorff). More generally, the map \( E \to E \) is a map of covers which induces a bijection on fibers, so it must be an isomorphism of covers. ■

Now here is a purely algebraic statement, which we can prove by covering theory.

Theorem 3.26. Any subgroup \( H \) of a free group \( G \) is free. If \( G \) is free on \( n \) generators and the index of \( H \) in \( G \) is \( k \), then \( H \) is free on \( 1 - k + nk \) generators.

Proof. Define \( B \) to be a wedge of circles, one circle for each generator of \( G \). Then \( \pi_1(B) \cong G \). Let \( H \leq G \) and let \( p : E \to B \) be a covering such that \( p_*(\pi_1(E)) = H \). By the previous result, \( E \) is a graph and so \( \pi_1(E) \) is a free group by the result from last time.

For the second statement, we introduce the Euler characteristic of a graph, which is defined as \( \chi(B) = \# \text{ vertices} - \# \text{ edges} \). In this case, we have \( \chi(B) = 1 - n \). Since \( H \) has index \( k \) in \( G \), this means that \( G/H \) has cardinality \( k \). But this is the fiber of \( p : E \to B \). So \( E \) has \( k \) vertices, and each edge of \( B \) lifts to \( k \) edges in \( E \). So \( \chi(E) = k - kn \).

On the other hand, we know from last time that \( E \) is homotopy equivalent to \( E/T \), where \( T \subseteq E \) is a maximal tree. Note that collapsing any edge in a tree does not change the Euler characteristic. The number of generators, say \( m \) of \( H \), is then the number of edges in \( E/T \), so we find that \( \chi(E) = 1 - m \). Setting these equal gives

\[
k - kn = 1 - m, \quad \text{or} \quad m = 1 - k + kn.
\]

■
We encountered an important idea in this proof, which can be defined more generally.

**Definition 3.27.** Let $X$ be a CW complex having finitely many cells in each dimension (we saw that $X$ is finite type). Then the **Euler characteristic** of $X$ is

$$\chi(X) := \sum_{n \geq 0} (-1)^n \# \{ \text{n-cells of } X \}.$$  

In fact, the number $\chi(X)$ does not depend on the choice of CW structure on $X$, though this is far from obvious from the definition. We will see Euler characteristics again later in the course.

3.1. **The effect of attaching cells.** The van Kampen Theorem also gives an effective means of computing the fundamental group of a CW complex.

Given a space $X$ and a map $\alpha : S^1 \rightarrow X$, we may attach a disc along the map $\alpha$ to form a new space

$$X' = X \cup_\alpha D^2.$$  

Since the inclusion of the boundary $S^1 \hookrightarrow D^2$ is null, it follows that the composition

$$\alpha : S^1 \rightarrow X \rightarrow X'$$

is also null. So we have effectively killed off the class $[\alpha] \in \pi_1(X)$.

We can use the van Kampen theorem to show that this is all that we have done.

**Proposition 3.28.** Let $X$ be path-connected and let $\alpha : S^1 \rightarrow X$ be a loop in $X$, based at $x_0$. Write $X' = X \cup_\alpha D^2$. Then

$$\pi_1(X', \iota(x_0)) \cong \pi_1(X)/[\alpha].$$

Of course, we really mean the normal subgroup generated by $\alpha$.

**Proof.** Consider the open subsets $U$ and $V$ of $D^2$, where $U = D^2 - \overline{B_{1/3}}$ and $V = B_{2/3}$. The map $\iota_{D^2} : D^2 \rightarrow X'$ restricts to a homeomorphism (with open image) on the interior of $D^2$, so the image of $V$ in $X'$ is open and path-connected. Now let $U' = X \cup U$. Since this is the image under the quotient map $X \amalg D^2 \rightarrow X'$ of the saturated open set $X \amalg U$, $U'$ is open in $X'$. It is easy to see that $U'$ is also path-connected.

Now $U'$ and $V$ cover $X'$. Since $U$ deformation retracts onto the boundary, it follows that $U'$ deformation retracts onto $X$. The open set $V$ is contractible. Finally, the path-connected subset $U' \cap V$ deformation retracts onto the circle of radius $1/2$. Moreover, the map

$$\mathbb{Z} \cong \pi_1(U' \cap V) \rightarrow \pi_1(U') \cong \pi_1(X)$$

sends the generator to $[\alpha]$. The van Kampen theorem then implies that

$$\pi_1(X') \cong \pi_1(X)/[\alpha].$$

Actually, we cheated a little bit in this proof, since in order to apply the van Kampen theorem, we needed to work with a basepoint in $U' \cap V$. A more careful proof would include the necessary change-of-basepoint discussion.

What about attaching higher-dimensional cells?

**Proposition 3.29.** Let $X$ be path-connected and let $\alpha : S^{n-1} \rightarrow X$ be an attaching map for an $n$-cell in $X$, based at $x_0$. Write $X' = X \cup_\alpha D^n$. Then, if $n \geq 3$,

$$\pi_1(X', \iota(x_0)) \cong \pi_1(X).$$
Then pushout is the torus, presented as a quotient of group, so that \( \pi_morphic to \) \( \RP \). This presents \( \gamma \) covering \( \gamma^2 \).

**Example 3.30.** If we attach a 2-cell to \( S^1 \) along the identity map \( \text{id} : S^1 \to S^1 \), we obtain \( D^2 \). We have killed all of the fundamental group. If we attach another 2-cell, we get \( S^2 \). Then \( \chi(S^2) = 2 - 2 + 2 = 2 \).

Attaching a 3-cell to \( S^2 \) via \( \text{id} : S^2 \to S^2 \) gives \( D^3 \). Attaching a second 3-cell gives \( S^3 \). The previous results tells us that all spaces obtained in this way \( (D^n \text{ and } S^n) \) will be simply connected. Here we get \( \chi(S^3) = 2 - 2 + 2 - 2 = 0 \). More generally, we get

\[
\chi(S^n) = \begin{cases} 
2 & \text{n even} \\
0 & \text{n odd.}
\end{cases}
\]

**Example 3.31.** (\( \RP^n \)) A more interesting example is attaching a 2-cell to \( S^1 \) along the double covering \( \gamma_2 : S^1 \to S^1 \). Since this loop in \( S^1 \) corresponds to the element 2 in \( \pi_1(S^1) \cong \Z \), the resulting space \( X' \) has \( \pi_1(X') \cong \Z/2 \). We have previously seen (last semester) that this is just the space \( \RP^2 \), since \( \RP^2 \) can be realized as the quotient of \( D^2 \) by the relation \( x \sim -x \) on the boundary. This presents \( \RP^2 \) as a cell complex with a single 0-cell (vertex), a single 1-cell, and a single 2-cell. Then\( \chi(\RP^2) = 1 - 1 + 1 = 1 \).

We can next attach a 3-cell to \( \RP^2 \) along the double cover \( S^2 \to \RP^2 \). The result is homeomorphic to \( \RP^3 \) by an analogous argument. By the above, this does not change the fundamental group, so that \( \pi_1(\RP^3) \cong \Z/2 \), and we count \( \chi(\RP^3) = 1 - 1 + 1 - 1 = 0 \). In general, we have \( \RP^n \) given as a cell complex with a single cell in each dimension. We have \( \pi_1(\RP^n) \cong \Z/2 \) for all \( n \geq 2 \), and

\[
\chi(\RP^n) = \begin{cases} 
1 & \text{n even} \\
0 & \text{n odd.}
\end{cases}
\]

**Example 3.32.** (\( \CP^n \)) Recall that \( \CP^1 \cong S^2 \) is simply connected. Last semester, we showed that \( \CP^n \) has a CW structure with a single cell in every even dimension. For example, \( \CP^2 \) is obtained from \( \CP^1 \) by attaching a 4-cell. It follows that every \( \CP^n \) is simply-connected, and \( \chi(\CP^n) = n + 1 \).

Let’s look at a few more examples of CW complexes.

**Example 3.33.** (Torus) Attach a 2-cell to \( S^1 \vee S^1 \) along the map \( S^1 \to S^1 \vee S^1 \) given by \( aba^{-1}b^{-1} \), where \( a \) and \( b \) are the standard inclusions \( S^1 \hookrightarrow S^1 \vee S^1 \). We saw last semester that the resulting pushout is the torus, presented as a quotient of \( D^2 \cong I^2 \).

We claim that \( \pi_1(T^2) \cong F_2/aba^{-1}b^{-1} \cong \Z^2 \).

**Proposition 3.34.** The natural map \( \varphi : F(a, b) \to \Z^2 \) defined by \( \varphi(a) = (1, 0) \) and \( \varphi(b) = (0, 1) \) induces an isomorphism

\[
F(a, b)/\langle aba^{-1}b^{-1} \rangle \cong \Z^2.
\]

**Proof.** Let \( K = \ker(\varphi) \) and let \( N \leq F(a, b) \) be the normal subgroup generated by \( aba^{-1}b^{-1} \). By the First Isomorphism Theorem, \( F(a, b)/K \cong \Z^2 \), so it suffices to show that \( N = K \). Since \( aba^{-1}b^{-1} \in K \), it follows that \( N \leq K \). Since \( N \leq K \), we wish to show that the quotient group \( K/N \) is trivial. Let \( g = a^{n_1}b^{k_1}a^{n_2}b^{k_2}a^{n_3} \in K/N \). In \( K/N \), we have \( \overline{ab} = \overline{ba} \), so

\[
\overline{a^{n_1}b^{k_1}a^{n_2}b^{k_2}a^{n_3}} = \overline{a^{n_1+n_2+n_3}b^{k_1+k_2}}.
\]

Since \( g \in K \), we have \( n_1 + n_2 + n_3 = 0 \) and \( k_1 + k_2 = 0 \), so \( \overline{g} = e \) in \( K/N \).
So the answer coming from the van Kampen theorem matches our previous computation of $\pi_1(T^2)$.

In this cell structure on the torus, there is a single 0-cell (a vertex), two 1-cells (the two circles in $S^1 \vee S^1$), and a single 2-cell, so that

$$\chi(T^2) = 1 - 2 + 1 = 0.$$ 

Fri, Mar. 9

**Example 3.35.** (Klein bottle) One definition of the Klein bottle $K$ is as the quotient of $I^2$ in which one opposite pair of edges is identified with a flip, while the other pair is identified without a flip. This leads to the computation

$$\pi_1(K) \cong F(a, b)/(aba^{-1}b).$$

For certain purposes, this is not the most convenient description. Cut the square along a diagonal and repaste the triangles along the previously flip-identified edges. The resulting square leads to the computation

$$\pi_1(K) \cong F(a, c)/(a^2c^2).$$

The equation $c = a^{-1}b$ allows you to go back and forth between these two descriptions.

Like the torus, the resulting cell complex has a single 0-cell, two 1-cells, and a single 2-cell, so

$$\chi(K) = 1 - 2 + 1 = 0.$$ 

The next example is not obtained by attaching a cell to $S^1 \vee S^1$.

**Example 3.36.** If we glue the boundary of $I^2$ according to the relation $abab$, the resulting space can be identified with $\mathbb{R}P^2$. Notice in this case that the four vertices do not all become identified. Rather they are identified in pairs, and we are left with two vertices after making the quotient. This example can be visualized by thinking of identifying the two halves of $\partial D^2$ via a twist. Using this cell structure, we get

$$\chi(\mathbb{R}P^2) = 2 - 2 + 1 = 1.$$ 

3.2. The classification of surfaces. These 2-dimensional cell complexes are all examples of surfaces (compact, connected 2-dimensional manifolds).

There is an important construction for surfaces called the connected sum.

**Definition 3.37.** Suppose $M$ and $N$ are surfaces. Pick subsets $D_M \subseteq M$ and $D_N \subseteq N$ that are homeomorphic to $D^2$ and remove their interiors from $M$ and $N$. Write $M' = M - \text{Int}(D_M)$ and $N' = N - \text{Int}(D_N)$. Then the connected sum of $M$ and $N$ is defined to be

$$M \# N = M' \cup_{S^1} N',$$

where the maps $S^1 \to M'$ and $S^1 \to N'$ are the inclusions of the boundaries of the removed discs.

**Example 3.38.** If $M$ is a surface, then the connect sum $M \# S^2$ is again homeomorphic to $M$. 

29
Proposition 3.39. The connected sum $\mathbb{RP}^2 \# \mathbb{RP}^2$ is homeomorphic to the Klein bottle, $K$.

Proof. See the figure to the right. ■

Example 3.40. If $M$ is a surface, then the connect sum $M \# T^2$ can be viewed as $M$ with a “handle” glued on.

For example, consider $M = T^2$. Then $T^2 \# T^2$ looks like a “two-holed torus”. This is called $M_2$, the (orientable) surface of genus two. From the cell structure resulting from the picture, we see a wedge of four circles (let’s call the generators of the circles $a_1, b_1, a_2, b_2$) with a two-cell attached along the element $[a_1, b_1][a_2, b_2]$. It follows that the fundamental group of $M_2$ is

$$F(a_1, b_1, a_2, b_2)/[a_1, b_1][a_2, b_2].$$

We also find that $\chi(M_2) = 1 - 4 + 1 = -2$. 

30
We now have $\chi(M_g) = 1 - 2g + 1 = 2 - 2g$.

We are headed towards a “classification theorem” for compact surfaces, so let us now show that if $g_1 \neq g_2$ then $M_{g_1}$ is not homeomorphic to $M_{g_2}$. We show this by showing they have different fundamental groups. As we have said already, understanding a group given by a list of generators and relations is not always easy, so we make life easier by considering the abelianizations of the fundamental groups.

The abelianization $G_{ab}$ of $G$ is the group defined by

$$G_{ab} = G/\langle\langle G, G \rangle\rangle,$$

where $\langle\langle G, G \rangle\rangle$ is the (normal) subgroup generated by commutators.

**Lemma 3.42.** The abelianization $F(a_1, \ldots, a_n)_{ab}$ is the free abelian group $\mathbb{Z}^n$.

**Proof.** We already did this in the case $n = 2$ for understanding the fundamental group of the torus, and the proof generalizes.

The abelianization is characterized by a universal property. For a group $G$, let $q : G \to G_{ab}$ be the quotient map. Then the universal property of the quotient gives the following result.

**Proposition 3.43.** Let $G$ be a group and $A$ an abelian group. Then any homomorphism $\varphi : G \to A$ factors uniquely as $G \xrightarrow{q} G_{ab} \xrightarrow{\pi} A$.

When we apply this to the surface $M_g$, we get

**Proposition 3.44.** $\pi_1(M_g)_{ab} \cong \mathbb{Z}^{2g}$.

**Proof.** Let $F = F(a_1, b_1, \ldots, a_n, b_n)$, let $N \leq F$ be the normal subgroup generated by (i.e. the normal closure of) the product $[a_1, b_1] \ldots [a_g, b_g]$, and let $G = \pi_1(M_g) \cong F/N$. Since the quotient map $q : F \to G$ is surjective, it follows that $q([F, F]) = [G, G]$. By the Third Isomorphism Theorem, we get

$$G_{ab} := G/[G, G] = G/q([F, F]) \cong F/[F, F] \cong \mathbb{Z}^{2g}.$$

**Lemma 3.45.** If $H \cong G$ then $H_{ab} \cong G_{ab}$.

As a result, we see that if $g_1 \neq g_2$ then $\pi_1(M_{g_1}) \neq \pi_1(M_{g_2})$ because their abelianizations are not isomorphic.

**Corollary 3.46.** If $g_1 \neq g_2$ then $M_{g_1} \not\cong M_{g_2}$.

Note that we have also distinguished all of these from $S^2$ (which has trivial fundamental group) and from $\mathbb{R}P^2$ (which has abelian fundamental group $\mathbb{Z}/2\mathbb{Z}$).

What about the Klein bottle $K$? We found before the break that $\pi_1(K) \cong F(a, b)/aba^{-1}b$. If we abelianize this fundamental group, we get

**Proposition 3.47.** The abelianized fundamental group of the Klein bottle is

$$\pi_1(K)_{ab} \cong \langle \mathbb{Z}\{a\} \times \mathbb{Z}\{b\}\rangle/(a + b - a + b) = \mathbb{Z}\{a\} \times \mathbb{Z}\{b\}/2b \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$
Proof. The idea is the same as in the previous examples. Under the quotient \( F(a, b) \rightarrow \mathbb{Z}\{a\} \times \mathbb{Z}\{b\} \), the element \( aba^{-1}b \) is sent to \( a + b - a + b \) (this is simply changing from multiplicative notation to additive notation.

This group is different from all of the others, so \( K \) is not homeomorphic to any of the above surfaces. The last main example is

**Example 3.48.** \((\mathbb{RP}^2 \# \mathbb{RP}^2 \# \ldots \# \mathbb{RP}^2)\) Suppose we take a connect sum of \( g \) copies of \( \mathbb{RP}^2 \). We will call this surface \( N_g \). Following the previous examples, we see that we get a fundamental group of

\[
\pi_1(N_g) \cong F(a_1, \ldots, a_g)/a_1^2a_2^2\ldots a_g^2
\]

and \( \chi(N_g) = 1 - g + 1 = 2 - g \). The abelianization is then

\[
\pi_1(N_g)_{ab} \cong \mathbb{Z}^g/(2, 2, \ldots, 2).
\]

Define a homomorphism \( \varphi : \mathbb{Z}^g/(2, \ldots, 2) \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^{g-1} \) by

\[
\varphi(n_1, \ldots, n_g) = (n_1, n_2 - n_1, n_3 - n_1, \ldots, n_g - n_1).
\]

Then it is easily verified that \( \varphi \) is an isomorphism. In other words,

\[
\pi_1(N_g)_{ab} \cong \mathbb{Z}/2 \times \mathbb{Z}^{g-1}.
\]

Ok, so we have argued that the compact surfaces \( S^2 \), \( M_g \ (g \geq 1) \), and \( N_g \ (g \geq 1) \) all have different fundamental groups and thus are not homeomorphic. The remarkable fact is that these are all of the compact (connected) surfaces.

**Theorem 3.49.** Every compact, connected surface is homeomorphic to some \( M_g \), \( g \geq 0 \) or to some \( N_g \), \( g \geq 1 \).

**Corollary 3.50.** If \( \chi(M) = n \) is odd, then \( M \cong N_{2-n} \).

All of these examples are formed by taking connected sums of \( T^2 \)'s or \( \mathbb{RP}^2 \)'s. What happens if we mix them?

**Lemma 3.51.** \( T^2 \# \mathbb{RP}^2 \cong \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2 \).

In other words, one bad apple spoils the whole bunch. The proof is in the picture:
In particular, this implies that \( M_g \# N_k \cong N_{2g+k} \).

**Proof of the theorem.** Let \( M \) be a compact, connected surface. We assume without proof (see Prop 6.14 from Lee) that

- \( M \) is a 2-cell complex with a single 2-cell.
- the attaching map \( \alpha : S^1 \to M^1 \) for the 2-cell has the following property: let \( U \) be the interior of a 1-cell. Then the restriction \( \alpha : \alpha^{-1}(U) \to U \) is a double cover. In other words, if we label \( \partial D^2 \) according to the edge identifications as we have done in the examples, each edge appears exactly twice. Note that this must happen since each interior point on the edge needs to have a half-disk on two sides.

So we can visualize \( M \) as a quotient of a \( 2n \)-sided polygon.

As we said above, each edge appears exactly twice on the boundary of the two-cell. If the two occurrences have opposite orientations (as in the sphere), we say the pair is an oriented pair. If the two occurrences have the same orientation (as in \( \mathbb{RP}^2 \)), we say this is a twisted pair. There will be 4 reductions in the proof!!

1. If \( M \cong S^2 \), we are done, so suppose (for the rest of the proof) this is not the case. Then we can reduce to a cell structure with no adjacent oriented pairs. (Just fold these together.)

2. We can reduce to a cell structure where all twisted pairs are adjacent.

3. We can reduce to a cell structure with a single 0-cell. Suppose \( a \) is an edge from \( x \) to \( y \) and that \( x \neq y \). Let \( b \) be the other edge connecting to \( y \). By (1), \( b \) can’t be \( a^{-1} \). If \( b = a \) then \( x = y \). Suppose \( b \neq a \), and write \( z \) for the other vertex on \( b \). Then the edge \( b \) must occur somewhere else on the boundary. We use the moves in the pictures below, depending on whether the pair \( b \) is oriented or twisted.
This converts a vertex $y$ into a vertex $x$. Note that this procedure does not separate any adjacent twisted pairs, since the adjacent twisted pair $b$ gets replaced by $d$.

(4) Observe that any oriented pair $a$, $a^{-1}$ is interlaced with another oriented pair $b$, $b^{-1}$. If not, we can write the boundary in the form $aW_1a^{-1}W_2$. Now, given our assumption and previous steps, no edge in $W_1$ gets identified with an edge in $W_2$. It follows that if the endpoints of $a$ are $x$ and $y$, then these two vertices never get identified with each other, as the vertex $x$ cannot appear in $W_1$ and similarly $y$ cannot appear in $W_2$.

(5) We can further arrange it so that there is no interference: the oriented pairs of edges occur as $aba^{-1}b^{-1}$ with no other edges in between. The proof is in the picture below, taken from p. 177 of Lee.

![Diagram](image.png)

Fig. 6.22: Bringing intertwined complementary pairs together.

Now we are done by Lemma 3.51. $M$ is homeomorphic either to a connect sum of projective planes or to a connect sum of tori.

Fri, Mar. 23

We saw in Corollary 3.50 that if $\chi(M)$ is odd, we can immediately identify the homeomorphism type of $M$. If $\chi(M)$ is even, this is not the case, as $T^2$ and $K$ both have Euler characteristic equal to 0. To handle the even case, we make a definition.

Say that a surface $M$ is **orientable** if it has a cell structure as above with no twisted pairs of edges.

**Proposition 3.52.** A surface is orientable if and only if it is homeomorphic to some $M_g$.

**Proof.** ($\Leftarrow$) Our standard cell structures for these surfaces have no twisted pairs of edges. ($\Rightarrow$) Apply the algorithm described in the above proof, starting with only oriented pairs of edges. Step 1 does not introduce any new edges. Step 2 can be skipped. Steps 3 cuts-and-pastes along a pair of oriented edges and so does not change the orientation of any edges. Step 4 does not change the surface. Step 5 again only cuts-and-pastes along oriented edges. It follows that in reducing to standard form, we do not introduce any twisted pairs of edges.

In fact, you should be able to convince yourself that a surface is orientable if and only if every cell structure as above has no twisted pairs. The point is that if you start with a cell structure involving some twisted pairs and you perform the reductions described in the proof, you will never get rid of any twisted pairs of edges.

The fact that the $M_g$ can be embedded in $\mathbb{R}^3$ whereas the $N_g$ cannot is precisely related to orientability. In general, you can embed a (smooth) $n$-dimensional manifold in $\mathbb{R}^{2n}$, but you can improve this to $\mathbb{R}^{2n-1}$ if the manifold is orientable. The definition we have given here depends on particular kinds of CW structures, but other definitions of orientability (in terms of homology) apply more widely.

In addition to the $N_g$’s, the Möbius band is a 2-manifold that is famously non-orientable.
4. Higher homotopy groups

We have just been studying surfaces and have determined (well, at least given presentations for) their fundamental groups. We have also seen (on exam 1) that there are higher homotopy groups \( \pi_n(X) \), so we might ask about the groups \( \pi_n(M_g) \) and \( \pi_n(N_k) \).

Recall, again from the exam, that any covering \( E \to B \) induces an isomorphism on all higher homotopy groups. So it suffices to understand the universal covers of these surfaces.

The first example would be \( M_0 = S^2 \), which is simply-connected. Note that this space is also the universal cover of \( N_1 = \mathbb{R}P^2 \), so these will have the same higher homotopy groups. We will come back to these on Monday.

Another example is the componentwise-exponential covering \( q \times q : \mathbb{R}^2 \to T^2 \), which shows that \( T^2 \) has no higher homotopy groups. Note that we also could have deduced this using that \( \pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y) \) and that \( S^1 \) has no higher homotopy groups (also from Exam 1).

What about the Klein bottle \( K \)? Well, consider the relation on \( T^2 \) given by \((x, y) \sim (x + \frac{1}{2}, 1 - y)\). The quotient \( T^2 / \sim \) is \( K \), and the quotient map \( T^2 \to K \) is a double cover. It follows that the universal cover of \( T^2 \), which is \( \mathbb{R}^2 \), is also the universal cover of \( K \). So \( K \) also has no higher homotopy groups!

For the surfaces of higher genus, we start by generalizing the double cover \( T^2 \to K \).

**Proposition 4.1.** If \( g \geq 1 \), then there is a double cover of \( N_g \) by \( M_{g-1} \).

**Lemma 4.2.** Suppose that \( p : E \to B \) is a double cover of a surface \( B \), and let \( W \) be another surface. Then there is a double cover \( E \# W \# W \to B \# W \).

The lemma implies the proposition as follows:

**Proof.** We already know about the double cover \( S^2 \to \mathbb{R}P^2 \), which is the case \( g = 1 \). Recall (Lemma 3.51) that \( N_3 \cong \mathbb{R}P^2 \# T^2 \). By the lemma, we get a double cover \( M_2 \cong S^2 \# T^2 \# T^2 \to \mathbb{R}P^2 \# T^2 \cong N_3 \). By tacking on more copies of \( T^2 \), this handles the case of \( g \) odd.

We also discussed above the double cover \( T^2 \to K \), which is the case \( g = 2 \). By the lemma, we get a double cover \( M_3 \cong T^2 \# T^2 \# T^2 \to K \# T^2 \cong N_4 \). By tacking on more copies of \( T^2 \), this handles the case of \( g \) even.\[\blacksquare\]