CLASS NOTES
CHROMATIC HOMOTOPY THEORY
MATH 751 (SPRING 2024)

BERTRAND GUILLOU

CONTENTS

Date: Updated April 23, 2024.

Mon, Jan. 8

Last summer, the team of Robert Burklund, Jeremy Hahn, Ishan Levy, and Tomer Schlank announced [BHLS] a disproof of Ravenel's Telescope conjecture. This course will **not** cover their work. Rather, this will be an introduction to the area, discussing the Telescope conjecture and its ingredients and motivation.

The subject of chromatic homotopy theory crystallized in a series of conjectures [Ra] by Doug Ravenel which appeared in print in the 1980's. Amazingly, most of Ravenel's conjectures were proved shortly thereafter by Ethan Devinatz, Mike Hopkins, and Jeff Smith [DHS, HS]. The one conjecture that resisted all attempts until this time is the Telescope conjecture. We will give precise forms of the statement later, but here is a very rough idea:

In algebra, given a commutative ring R and an element $f \in R$, we can consider the "localization of R away from f", which is

$$R_f = R[1/f] = \operatorname{colim}\left(R \xrightarrow{f} R \xrightarrow{f} R \xrightarrow{f} \dots \right).$$

If M is an R-module, we can equally well consider the localization

$$M_f = R_F \otimes_R M = \operatorname{colim}\left(M \xrightarrow{f} M \xrightarrow{f} M \xrightarrow{f} \dots \right).$$

This construction plays a prominent role in algebraic geometry.

In homotopy theory, given a self-map $f\colon \Sigma^n X \longrightarrow X$ from (some suspension of) a finite complex X to itself, one can similarly consider the "telescope"

$$X[1/f] = \operatorname{hocolim}\left(X \xrightarrow{f} \Sigma^{-n} X \xrightarrow{f} \Sigma^{-2n} X \dots \right).$$

The negative suspensions only make sense in a "stable" world where suspension is invertible. The homotopy colimit can be built as a kind of infinite mapping cylinder, which resembles a telescope. The homotopy of the localization $X[1/f]$ tells us about "f-periodic" elements in the homotopy of X.

The trouble is that the homotopy of $X[1/f]$ is difficult to compute directly. The Telescope conjecture asserts that the localization $X[1/f]$ is equivalent to another, more computable, localization of X.

Wed, Jan. 10

The main application of these ideas is to the (stable) homotopy groups of spheres. Write $\mathbb{S}/2$ for the cofiber of the degree 2 map $\mathbb{S} \longrightarrow \mathbb{S}$, where $\mathbb{S}$ is the "sphere spectrum" (a stable analogue of S^0). Then Adams showed [A2] that there is a non-nilpotent self-map $v_1^4\colon \Sigma^8\mathbb{S}/2 \longrightarrow \mathbb{S}/2$, meaning that none of the iterates of this map are null-homotopic. Even better, for each $k > 0$, the composition

$$\mathbb{S}^{8k} \hookrightarrow \Sigma^{8k}\mathbb{S}/2 \xrightarrow{v_1^4} \Sigma^{8(k-1)}\mathbb{S}/2 \xrightarrow{v_1^4} \dots \xrightarrow{v_1^4} \mathbb{S}/2 \xrightarrow{j} \mathbb{S}^1$$

is not null-homotopic, where $j\colon \mathbb{S}/2 \longrightarrow \mathbb{S}^1$ is the "projection onto the top cell" of the complex $\mathbb{S}/2$. Thus this construction gives an infinite family of nontrivial elements in the stable homotopy groups of spheres! And this motivates the search for other such non-nilpotent self-maps of finite complexes.

Remark 0.1. This course may be viewed as a much abbreviated version of the course [Ro] of John Rognes given in Spring 2023. We have certainly relied on those notes, along with other resources such as [Ra, Ra2], in preparing these notes.

⋆⋆⋆⋆⋆⋆⋆⋆
⇝⇝⇝⇝⇝⇝⇝⇝⇝⇝⇝⇝⇝⇝⇝⇝⇝⇝⇝⇝⇝⇝⇝⇝⇝⇝⇝
⋆⋆⋆⋆⋆⋆⋆⋆

1. A REVIEW OF SPECTRA AND COHOMOLOGY THEORIES

We will be working in the stable homotopy category of spectra, and so we start with a quick survey of same basic properties that we will need.

1.1. Spectra.

Spectra can be thought of as the "stabilization" of based spaces with respect to the suspension functor $\Sigma\colon \mathbf{Top}_* \longrightarrow \mathbf{Top}_*$ given by $\Sigma X = S^1 \wedge X$.

Definition 1.1. A spectrum E is a collection $\{E_n\}_{n\in\mathbb{Z}_{\geq 0}}$ of based spaces together with structure maps $\sigma_n\colon \Sigma E_n \longrightarrow E_{n+1}$ for each $n \geq 0$. A map of spectra $\varphi\colon E \longrightarrow F$ is simply a collection of based maps $\varphi_n\colon E_n \longrightarrow F_n$ which are compatible with the structure maps, meaning that each diagram of the form

$$\begin{array}{ccc} \Sigma E_n & \xrightarrow{\Sigma\varphi_n} & \Sigma F_n \\ \downarrow{\scriptstyle \sigma_n^E} & & \downarrow{\scriptstyle \sigma_n^F} \\ E_{n+1} & \xrightarrow{\varphi_{n+1}} & F_{n+1} \end{array}$$

commutes. We write **Sp** for the category of spectra.

The simplest kind of example to write down is a "suspension spectrum",

Definition 1.2. Let X be a based space. The **suspension spectrum** of X is the spectrum $\Sigma^\infty X$ defined as the collection $\{\Sigma^n X\}$ and all of whose structure maps are identity maps.

Definition 1.3. The **sphere spectrum** is the suspension spectrum $\mathbb{S} = \Sigma^\infty S^0$.

We similarly write $\mathbb{S}^n = \Sigma^\infty S^n$ for $n \geq 0$. It turns out that we can also make sense of $\mathbb{S}^n$ when n is negative. For example, $\mathbb{S}^{-1}$ is the sequence of based spaces $\{*, S^0, S^1, \dots\}$, where the first structure map is (necessarily) constant. Similarly, $\mathbb{S}^{-n}$ will have the first n spaces being $*$ (for $n > 0$).

Example 1.4. (Eilenberg-Mac Lane spectra) Let A be an abelian group. Recall that an Eilenberg-Mac Lane space of type $K(A,n)$ is a CW complex whose only nontrivial homotopy group is $\pi_n K(A,n) \cong A$. One of the amazing facts about Eilenberg-Mac Lane spaces is that there is a natural isomorphism

$$\tilde{\mathrm{H}}^n(X;A) \cong [X, K(A,n)]_*, \tag{1.1}$$

natural in based spaces X, between reduced cohomology and based homotopy classes of maps.

For any abelian group A, the **Eilenberg-Mac Lane spectrum** HA has $HA_n = K(A,n)$. The structure map $\Sigma K(A,n) \longrightarrow K(A,n+1)$ corresponds under (1.1) to an element of $\tilde{\mathrm{H}}^{n+1}(\Sigma K(A,n);A)$. The suspension isomorphism identifies this with $\tilde{\mathrm{H}}^n(K(A,n);A)$, and there is a canonical element corresponding under (1.1) to the identity map of the based space $K(A,n)$.

For the next example, recalled the **based loop space** functor $\Omega\colon \mathbf{Top}_* \to \mathbf{Top}_*$ defined by $\Omega X = \mathrm{Map}_*(S^1, X)$. This functor is right adjoint to Σ. This functor is also related to the previous example, in that the adjunction allows one to see that $\Omega K(A,n+1)$ is a model for $K(A,n)$. So then the adjoint of a choice of homotopy equivalence $K(A,n) \xrightarrow{\sim} \Omega K(A,n+1)$ can be taken for the structure map of HA.

Example 1.5. (Complex K-theory) Let $U = \bigcup_n U(n)$ be the infinite unitary group, where the inclusion $U(n) \hookrightarrow U(n+1)$ includes $U(n)$ as the $(n+1) \times (n+1)$-unitary matrices with a 1 in the top left corner.

Write $BU(n)$ for the Grassmannian $BU(n) = \mathrm{Gr}_n(\mathbb{C}^\infty)$ of n-dimensional complex subspaces of C^∞. There is an inclusion $BU(n) \to BU(n+1)$ which takes the subspace $V \subset \mathbb{C}^\infty$ to the subspace $\mathbb{C} \oplus V \subset \mathbb{C} \oplus \mathbb{C}^\infty \cong \mathbb{C}^\infty$. Now define $BU = \bigcup BU(n)$.

Then define the (periodic) **complex K-theory spectrum** KU by

$$KU_n = \begin{cases} BU \times \mathbb{Z} & n \text{ even} \\ U & n \text{ odd.} \end{cases}$$

To define the structure maps we use that we have homotopy equivalences $\Omega(BU \times \mathbb{Z}) \simeq U$ (this is not difficult) and $\Omega U \simeq BU \times \mathbb{Z}$ (this is a major theorem, the **Bott Periodicity** theorem). The adjoints to these equivalences give maps

$$\sigma_{\text{odd}} \colon \Sigma U \longrightarrow BU \times \mathbb{Z}, \qquad \sigma_{\text{even}} \colon \Sigma(BU \times \mathbb{Z}) \longrightarrow U,$$

which give the structure maps for KU.

Fri, Jan. 12

Definition 1.6. Let $E \in \mathbf{Sp}$ and $X \in \mathbf{Top}_*$. We then define a spectrum $E \wedge X$ by setting $(E \wedge X)_n = E_n \wedge X$ and with structure map

$$\Sigma(E \wedge X)_n = S^1 \wedge E_n \wedge X \xrightarrow{\sigma_n \wedge \text{id}} E_{n+1} \wedge X = (E \wedge X)_{n+1}.$$

Note that if X and Y are based spaces, then we have an isomorphism

$$(\Sigma^\infty X) \wedge Y \cong \Sigma^\infty(X \wedge Y).$$

Definition 1.7. Given a spectrum E, we define the homotopy groups of E as

$$\pi_n(E) = \operatorname*{colim}_k \pi_{n+k}(E_k) \qquad \text{for } n \in \mathbb{Z},$$

where the colimit is along the maps $\pi_{n+k}(E_k) \xrightarrow{\Sigma} \pi_{n+k+1}(\Sigma E_k) \xrightarrow{\sigma_*} \pi_{n+k+1}(E_{k+1})$.

Note, in particular, that we now have homotopy groups in negative dimensions. Furthermore, in contrast to homotopy groups of spaces, these are abelian groups in all dimensions.

Example 1.8. Let X be a based space. Then the homotopy groups $\pi_*(\Sigma^\infty X)$ are the stable homotopy groups of X, which vanish for n negative.

In particular, for $X = S^0$, these are the stable homotopy groups of spheres, the computation of which is one of the main driving problems in homotopy theory!

Example 1.9. In contrast, an Eilenberg-Mac Lane spectrum has easy homotopy groups. We have

$$\pi_n(HA) = \begin{cases} A & n = 0 \\ 0 & \text{else.} \end{cases}$$

Example 1.10. By Worksheet 1, the homotopy groups of KU are the homotopy groups of $BU \times \mathbb{Z}$ (suitably reindexed for negative homotopy groups). The Bott periodicity theorem states that $BU \times \mathbb{Z}$ is homotopy equivalent to $\Omega^2(BU \times \mathbb{Z})$, so that the homotopy groups are 2-periodic. Now BU is path-connected, so we have that the even homotopy groups of $BU \times \mathbb{Z}$, which agree with $\pi_0(BU \times \mathbb{Z})$, are $\mathbb{Z}$. The odd homotopy groups of $BU \times \mathbb{Z}$ are the same as the even homotopy groups of U. Since U is path-connected, these groups vanish. In other words,

$$\pi_n KU = \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

Wed, Jan. 17

1.2. **The stable homotopy category and homology/cohomology theories.** We do not typically work with the category of spectra. Rather, we usually work with spectra only up to homotopy. We need one definition before we state our main theorem about the stable homotopy category.

Definition 1.11. A map $\varphi\colon E \longrightarrow F$ of spectra is called a **stable equivalence** if it induces an isomorphism on homotopy groups π_n for all $n \in \mathbb{Z}$.

We then define the **stable homotopy category** by formally inverting the stable equivalences.

Theorem 1.12. *There is a* **stable homotopy category** *of spectra, denoted* $\mathrm{Ho}\mathbf{Sp}$*, together with a functor* $\mathbf{Sp} \longrightarrow \mathrm{Ho}\mathbf{Sp}$*, such that*

1. *The functor* $\mathbf{Sp} \longrightarrow \mathrm{Ho}\mathbf{Sp}$ *is the universal example of a functor with source* $\mathbf{Sp}$ *and which takes the stable equivalences to isomorphisms, meaning that any other such functor factors through* $\mathrm{Ho}\mathbf{Sp}$ *up to isomorphism*
2. *The functor* $(-) \wedge S^1\colon \mathbf{Sp} \to \mathbf{Sp}$ *becomes an equivalence on* $\mathrm{Ho}\mathbf{Sp}$*.*
3. *The stable homotopy category* $\mathrm{Ho}\mathbf{Sp}$ *has a closed symmetric monoidal structure, where the monoidal product is a smash product, and the closed structure is a function spectrum construction. The unit object is* $\mathbb{S}$*. The function spectrum is written* $F(-,-)$*. In particular, the dual* $DE = F(E, \mathbb{S})$ *is known as the* **Spanier-Whitehead dual** *of* E*.*
4. *The dualizable objects in* $\mathrm{Ho}\mathbf{Sp}$ *are the retracts of finite cell complexes.*
5. $\mathrm{Ho}\mathbf{Sp}$ *is an additive category.*
6. *The suspension spectrum functor induces a strong symmetric monoidal functor*

$$\Sigma^\infty\colon \mathrm{Ho}\mathbf{Top}_* \longrightarrow \mathrm{Ho}\mathbf{Sp}$$

with right adjoint $\Omega^\infty\colon \mathrm{Ho}\mathbf{Sp} \longrightarrow \mathrm{Ho}\mathbf{Top}_*$ *given by the telescope* $\Omega^\infty E = \mathrm{hocolim}_n\, \Omega^n E_n$*.*

Notation 1.13. Given spectra E and F, we write $[E, F]$ for the set of maps $E \longrightarrow F$ in $\mathrm{Ho}\mathbf{Sp}$. By the above, this is an abelian group.

There are many more important properties, some of which will arise later in the course. A proof of the above theorem goes by building a "model" for the stable homotopy category. There are many models of the stable homotopy category, though for many purposes, the choice of model does not have significant impact.

One of the reasons to study spectra is that they give rise to, and in fact correspond to, generalized homology and cohomology theories on spaces.

Definition 1.14. Let X be a based CW complex and let E be a spectrum. We then define the (reduced) E**-homology** and E**-cohomology** of X by

$$E_n(X) = \pi_n(E \wedge X), \qquad E^n(X) = \pi_{-n}F(\Sigma^\infty X, E) = [\Sigma^{-n}\Sigma^\infty X, E].$$

We similarly talk of the E-homology or cohomology of X when X is a spectrum.

The E-homology of $\mathbb{S}$ is the same as the homotopy of E. This is typically abbreviated to E_*, and is called the **coefficients** of the theory E.

Example 1.15. For A an abelian group, the above says that the ordinary reduced homology group $\tilde{\mathrm{H}}_n(X; A)$ is isomorphic to $\pi_n(HA \wedge X)$.

1.3. **Cofiber and fiber sequences, connective covers.** Many constructions for spectra happen "levelwise".

Definition 1.16. (Wedge sums) Given E and F in **Sp**, their wedge sum is $(E \vee F)_n = E_n \vee F_n$, with structure map

$$S^1 \wedge (E_n \vee F_n) \cong (S^1 \wedge E_n) \vee (S^1 \wedge F_n) \xrightarrow{\sigma_n \vee \sigma_n} E_{n+1} \vee F_{n+1}.$$

One of the most useful tools in the stable homotopy category is a cofiber sequence.

Definition 1.17. Given a map $\varphi\colon E \longrightarrow F$ of spectra, the **cofiber** of φ, written $C\varphi$, is defined to be levelwise the cofiber of φ_n. This can be extended to a spectrum, using that $\Sigma C(\varphi_n) \cong C(\Sigma\varphi_n)$.

A **cofiber sequence** is a sequence $X \to Y \to Z$ that is stably equivalent to one of the form $E \xrightarrow{\varphi} F \to C\varphi$.

Proposition 1.18. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a cofiber sequence of spectra, and let $D \in \mathrm{Ho}\mathbf{Sp}$.*
(1) *$D \wedge A \longrightarrow D \wedge B \longrightarrow D \wedge C$ is again a cofiber sequence in $\mathrm{Ho}\mathbf{Sp}$.*
(2) *$[C, D] \to [B, D] \to [A, D]$ is exact.*

Moreover, the cofiber of $B \xrightarrow{g} C(f)$ is ΣA, and the cofiber sequence $A \xrightarrow{f} B \xrightarrow{g} C$ can be extended to the Puppe sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \xrightarrow{-\Sigma f} \Sigma B \xrightarrow{-\Sigma g} \Sigma C \xrightarrow{-\Sigma h} \Sigma^2 A \dots$$

This yields the following.

Corollary 1.19. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a cofiber sequence of spectra, and let $D \in \mathrm{Ho}\mathbf{Sp}$. Then the Puppe sequence induces a long exact sequence*

$$\cdots \to [\Sigma^2 A, D] \to [\Sigma C, D] \to [\Sigma B, D] \to [\Sigma A, D] \to [C, D] \to [B, D] \to [A, D] \to \cdots$$

Example 1.20. Let us write $\mathbb{S}/2$ for the cofiber of the degree 2 map on $\mathbb{S}$. Then we get a long exact sequence

$$\cdots \to \pi_n D \xrightarrow{2} \pi_n D \to [\mathbb{S}^n/2, D] \to \pi_{n-1} D \xrightarrow{2} \pi_{n-1} D \to \dots$$

The previous few results also hold unstably. However, the next result, which can be summarized as "fiber sequences and cofiber sequences agree stably", is a purely stable result.

Proposition 1.21. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a cofiber sequence of spectra, and let $D \in \mathrm{Ho}\mathbf{Sp}$. Then the Puppe sequence induces a long exact sequence*

$$\cdots \to [D, A] \to [D, B] \to [D, C] \to [D, \Sigma A] \to [D, \Sigma B] \to [D, \Sigma C] \to [D, \Sigma^2 A] \to \cdots$$

Fri, Jan. 18

We can use these ideas to build "connective covers". Let X be a spectrum, and let G_0 denote a set of generators for $\pi_0(X)$. Then define W_1 to be the cofiber of $\bigvee_{G_0} \mathbb{S} \to X$. Since $\mathbb{S}$ has no negative homotopy groups, it follows that $\pi_0(W_1)$ vanishes and that the negative homotopy groups of W_1 are those of X. Next, write G_1 for a set of generators of W_1 and define W_2 to be the cofiber of $\bigvee_{G_1} \mathbb{S}^1 \to W_1$. Then both π_0 and π_1 of W_2 vanish, while the negative homotopy groups still agree with those of X. Now set $P^{-1}X = \mathrm{hocolim}_n\, W_n$. This is a spectrum whose negative homotopy groups agree with those of X, and whose homotopy groups in non-negative degrees all vanish. It is called the (-1)**-Postnikov approximation**.

On the other hand, define a spectrum P_0X to be the (homotopy) fiber of $X \longrightarrow P^{-1}X$. Then the long exact sequence shows that the negative homotopy groups of P_0X vanish, while the rest agree with those of X. The spectrum P_0X is called the **connective cover** of X. Notation for this in the literature is very inconsistent. It is also written $P_0^{\infty}X$, or $\tau_{\geq 0}X$, or $X\langle -1\rangle$, among many others.

Definition 1.22. A spectrum X is said to be **connective**, or 0-connective, if all of its negative homotopy vanishes. More generally, X is n**-connective** if all of its homotopy vanishes in degree below n.

Example 1.23. Both $\mathbb{S}$ and any Eilenberg-Mac Lane spectrum HA are connective.

Example 1.24. The connective cover of KU is denoted ku.

Mon, Jan. 22

2. v_0-PERIODICITY AND RATIONALIZATION

We are interested in computing stable homotopy groups, in particular the stable homotopy groups of spheres. We first note that these (abelian) groups are not so bad.

Theorem 2.1 ([Ser]). *Let X be simply connected. Then all of the homotopy groups of X are finitely generated if and only if all of the homology groups of X are finitely generated.*

In particular, this implies that all of the stable homotopy groups of spheres are finitely generated abelian groups. By the Structure Theorem for finitely generated abelian groups, such a group is of the form

$$\mathbb{Z}^{r_0} \oplus \mathbb{Z}/p_1^{r_1} \oplus \cdots \oplus \mathbb{Z}/p_k^{r_k}$$

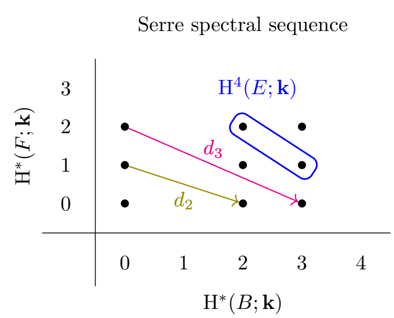
for primes p_i and nonnegative integers r_i. In practice, it is convenient to focus on one prime at a time. This is accomplished through localization (we will return to this later).

Theorem 2.2. [Ser] *The stable homotopy groups of spheres are finite in positive degrees.*

This follows from

Theorem 2.3. [Ser] *The homotopy group $\pi_k(S^n)$ is finite unless either $k = n$ or if n is even and $k = 2n - 1$.*

The proof of Theorem 2.3 uses the **Serre spectral sequence**. Here is a quick intro to the Serre spectral sequence. The context is a fibration $F \to E \to B$, and for simplicity assume that B is simply connected. A (cohomological) spectral sequence is a sequence $E_r^{*,*}$ of bigraded dga's, where $E_{r+1}^{*,*}$ is the cohomology of $E_r^{*,*}$ for each $r \geq 1$. In the case of the Serre spectral sequence and working with coefficients in a field $\mathbf{k}$, the E_2 page of the spectral sequence is $\mathrm{H}^*(B;\mathbf{k}) \otimes_{\mathbf{k}} \mathrm{H}^*(F;\mathbf{k})$. The differential d_r on E_r goes right r and down $r-1$. In the end, the vector space $\mathrm{H}^n(E;\mathbf{k})$ is the direct sum of the vector spaces on the diagonal $x + y = n$ of the E_∞ page.



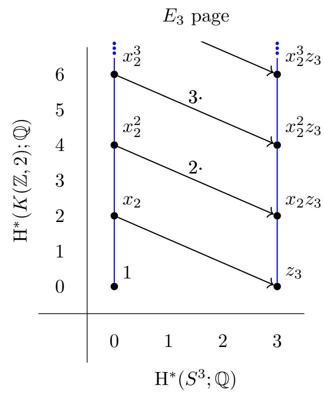
We will also rely on the following lemma, which can be proved by induction using the Serre spectral sequence.

Lemma 2.4. *The rational cohomology of $K(\mathbb{Z}, n)$ is $\mathbb{Q}[x_n]$ if n is even and $E(x_n)$ (an exterior algebra) if n is odd.*

Wed, Jan. 24

Proof of Theorem 2.3. By the Hurewicz theorem, we know that $\pi_n(S^n) \cong \mathbb{Z}$. We can then equivalently describe the rational homotopy $\pi_k(P_{\geq n+1}S^n) \otimes \mathbb{Q}$ of the $(n+1)$-connective cover. Since this cover is defined as the fiber of $S^n \longrightarrow K(\mathbb{Z}, n)$, it follows that the fiber of $P_{\geq n+1}S^n \to S^n$ is $\Omega K(\mathbb{Z}, n) \simeq K(\mathbb{Z}, n-1)$.

Case 1: n odd. Here we want to show that $P_{\geq n+1}S^n$ has no rational homotopy. By the rational Hurewicz theorem, it remains to show that $P_{\geq n+1}S^n$ has no (reduced) rational homology, or equivalently no (reduced) rational cohomology.



This is an easy exercise with the Serre spectral sequence, for the fiber sequence $K(\mathbb{Z}, n-1) \to P_{\geq n+1}S^n \to S^n$. The spectral sequence is displayed to the right in the case $n = 3$. It converges to $\mathrm{H}^*(P_{\geq 4}S^3; \mathbb{Q})$. Since $P_{\geq 4}S^3$ is 3-connected, we know that the rational homology and cohomology must vanish in degrees 1, 2, and 3. Thus the classes x_2 and z_3 cannot survive to the E_∞ page of the spectral sequence. This implies that there must be a differential from x_2 to z_3 and therefore also on powers of x_2 by the Leibniz (product) rule.

Case 2: n even. Let's write $n = 2k$. Then the Serre spectral sequence for the fiber sequence $K(\mathbb{Z}, 2k-1) \to P_{\geq 2k+1}S^{2k} \to S^{2k}$ only has two rows, according to Lemma 2.4. It follows that the reduced rational cohomology of $P_{\geq 2k+1}S^{2k}$ is $\mathbb{Q}$ in degree $4k-1$. In other words, the rational cohomology of $P_{\geq 2k+1}S^{2k}$ is that of S^{4k-1}. By the rational Hurewicz theorem, we conclude that the homotopy groups of $P_{\geq 2k+1}S^{2k}$ are finite below degree $4k-1$ and that $\pi_{4k-1}(P_{\geq 2k+1}S^{2k})$ is the direct sum of $\mathbb{Z}$ and something finite. It follows that any map $S^{4k-1} \to P_{\geq 2k+1}S^{2k}$ picking out a non-torsion element of homotopy will induce an isomorphism on rational homology and therefore an isomorphism on rational homotopy. But we have already shown that S^{4k-1} has no higher rational homotopy groups. ■

Fri, Jan. 26

2.1. **Rationalization.** We can **rationalize** a space or spectrum by inverting all primes. In other words, for $X \in \mathbf{Sp}$, we write

$$X_{\mathbb{Q}} = \operatorname{hocolim}\left(X \xrightarrow{2} X \xrightarrow{2\cdot 3} X \xrightarrow{2\cdot 3\cdot 5} X \xrightarrow{2\cdot 3\cdot 5\cdot 7} X \xrightarrow{2\cdot 3\cdot 5\cdot 7\cdot 11} \dots\right).$$

Proposition 2.5. *For $X \in \mathbf{Sp}$, the homotopy of $X_{\mathbb{Q}}$ is $\pi_n(X_{\mathbb{Q}}) \cong \pi_n(X) \otimes \mathbb{Q}$.*

Proof. The key point is that homotopy interacts well with homotopy colimits, as was mentioned on Worksheet 2.

$$\pi_n\left(\operatorname{hocolim} X \xrightarrow{2} X \xrightarrow{2\cdot 3} X \xrightarrow{2\cdot 3\cdot 5} \dots\right) \cong \operatorname{colim}\left(\pi_n X \xrightarrow{2} \pi_n X \xrightarrow{2\cdot 3} \pi_n X \xrightarrow{2\cdot 3\cdot 5} \dots\right)$$
$$\cong \pi_n(X) \otimes \mathbb{Q}.$$
■

Then Theorem 2.2 can be summarized as

Corollary 2.6. *The rationalization of $\mathbb{S}$ is $\mathbb{S}_{\mathbb{Q}} \simeq H\mathbb{Q}$.*

Rationalization is a type of localization. We will see other examples, so let's describe rationalization in the language that we will use for other examples. We first observe that, since smashing with a fixed spectrum commutes with sequential homotopy colimits, we have an equivalence

$$X_{\mathbb{Q}} \simeq \mathbb{S}_{\mathbb{Q}} \wedge X. \tag{2.1}$$

By Corollary 2.6, it follows that the (stable) homotopy of $X_{\mathbb{Q}}$ is the rational homology of X.

Definition 2.7. We say a spectrum X is **rationally acyclic** if $X_{\mathbb{Q}} \simeq \mathbb{S}_{\mathbb{Q}} \wedge X$ is null (stably equivalent to the one-point spectrum).

Corollary 2.8. *A spectrum X is rationally acyclic if and only if $\pi_n(X) \otimes \mathbb{Q} \cong \mathrm{H}_n(X; \mathbb{Q})$ is zero for all n.*

Mon, Jan. 29

Definition 2.9. Say a spectrum Y is $\mathbb{Q}$**-local**, or **rational**, if $[X, Y] = 0$ for all $\mathbb{Q}$-acyclic X.

Proposition 2.10. *A spectrum Y is $\mathbb{Q}$-local if and only if $\pi_n(Y)$ is a rational vector space for all n.*

In the course of this proof, we will use that $\mathbb{S}_{\mathbb{Q}} \simeq H\mathbb{Q}$ is a **ring spectrum**. There are many interpretations of this term. For now, all we mean is that we have a monoid in Ho**Sp**. Thus we require maps

$$\eta\colon \mathbb{S} \to R, \qquad \mu\colon R \wedge R \to R$$

that satisfy appropriate unit and associativity laws. This is now often called a **homotopy ring spectrum**, to distinguish it from a point-set level monoid in **Sp**. We will typically refer to these as h**-ring spectra** in this course. If R is a h-ring spectrum, an R-module will mean a spectrum M with a map $R \wedge M \longrightarrow M$ in Ho**Sp** that satisfies appropriate unit and associativity laws.

Proof. Suppose that Y is $\mathbb{Q}$-local. For any n, we want to show that $\pi_n(Y)$ is a rational vector space. It suffices to show that multiplication by any integer k induces an isomorphism on $\pi_n(Y)$. This follows from the long exact sequence arising from mapping the cofiber sequence $\mathbb{S}^n \xrightarrow{k} \mathbb{S}^n \to \mathbb{S}^n/k$ into Y, since $\mathbb{S}^n/k$ is $\mathbb{Q}$-acyclic.

On the other hand, suppose that $\pi_n(Y)$ is a rational vector space for all n. Then every map in the system defining the homotopy colimit $Y_{\mathbb{Q}}$ is an equivalence, so that $Y \to Y_{\mathbb{Q}}$ is an equivalence. Now we use that $Y_{\mathbb{Q}} \simeq \mathbb{S}_{\mathbb{Q}} \wedge Y$ is an $\mathbb{S}_{\mathbb{Q}} \simeq H\mathbb{Q}$-module. Then we have

$$[X, Y] \cong [X, Y_{\mathbb{Q}}] \cong [X, \mathbb{S}_{\mathbb{Q}} \wedge Y] \cong [\mathbb{S}_{\mathbb{Q}} \wedge X, \mathbb{S}_{\mathbb{Q}} \wedge Y]_{\mathbb{S}_{\mathbb{Q}}-\mathrm{Mod}}.$$

However, $\mathbb{S}_{\mathbb{Q}} \wedge X$ is contractible by assumption, so we conclude that this hom set vanishes. ■

The argument we just used also shows the following.

Example 2.11. For any spectrum X, the spectrum $X_{\mathbb{Q}}$ is $\mathbb{Q}$-local.

Definition 2.12. A map of spectra $f\colon X \to Y$ is a **rational equivalence** if the induced map $\pi_n(X) \otimes \mathbb{Q} \to \pi_n(Y) \otimes \mathbb{Q}$ is an isomorphism for all n. Equivalently, f is a rational equivalence if $\mathrm{H}_*(f; \mathbb{Q})$ is an isomorphism.

Example 2.13. For any spectrum X, the map $X \to X_{\mathbb{Q}}$ is a rational equivalence.

Proposition 2.14. *Let $W \to X$ be a rational equivalence, and suppose that Y is $\mathbb{Q}$-local. Then the induced map $[X, Y] \to [W, Y]$ is an isomorphism.*

Proof. Since $W \to X$ is a rational equivalence, it follows that the fiber F is rationally acyclic (this is actually equivalent to the map being a rational equivalence). The long exact sequence in $[-, D]$ from the cofiber sequence $F \to W \to X$ then gives the result. ■

Remark 2.15. Proposition 2.14 is in fact an if-and-only if. Suppose that Y thinks that every $\mathbb{Q}$-local equivalence is a stable equivalence, and let X be $\mathbb{Q}$-acyclic. The map $X \to X_{\mathbb{Q}}$ is a $\mathbb{Q}$-local equivalence, so $[X, Y] \cong [X_{\mathbb{Q}}, Y] \cong 0$ since $X_{\mathbb{Q}} \simeq *$ by the assumption that X is $\mathbb{Q}$-acyclic.

One way to think about this is that if Y is $\mathbb{Q}$-local, then as a cohomology theory, it sees no more than $H\mathbb{Q}$. Another important point is that if we consider the subcategory of Ho**Sp** consisting of $\mathbb{Q}$-local spectra, then the rational equivalences coincide with the stable equivalences.

Definition 2.16. A map $X \xrightarrow{f} Y$ is said to be a $\mathbb{Q}$**-localization**, or **rationalization**, if (1) Y is $\mathbb{Q}$-local and (2) f is a rational equivalence.

It follows from Example 2.11 and Example 2.13 that $X \to X_{\mathbb{Q}}$ is a rationalization in the sense of Definition 2.16.

Wed, Jan. 31

3. Towards v_1-periodicity, or what K-theory knows

We discussed the rational part of the homotopy groups of spheres last week. This is known as the "height 0" information. The higher height information is studied one prime at a time. From now on we will be working p-locally.

The theory for this is nearly identical to what we studied in the recent lectures. The p-localization of X is a spectrum $X_{(p)}$ with $\pi_n X_{(p)} = \pi_n X \otimes \mathbb{Z}_{(p)}$, where $\mathbb{Z}_{(p)} \subset \mathbb{Q}$ consists of fractions whose denominators are coprime to p. On both the algebraic and the homotopical sides, this is obtained by inverting all primes **except for** the chosen prime p. In effect, this allows us to ignore all torsion except for p-torsion.

Definition 3.1. The p-local stable homotopy category $\mathrm{Ho}\mathbf{Sp}_{(p)}$ is obtained from $\mathrm{Ho}\mathbf{Sp}$ by inverting the p-local equivalences. It is equivalent to the full subcategory of $\mathrm{Ho}\mathbf{Sp}$ on the p-local spectra.

The height 1 part of the homotopy groups of spheres is called v_1-periodic. Rational information is also called v_0-periodic. The element v_0 simply means the prime p. The point is that for a p-local spectrum, then rationalization is the same as just inverting $v_0 = p$.

The element v_1 comes from KU. We previously described the homotopy groups of KU in Example 1.10. But in fact there is more structure. KU is a commutative h-ring spectrum, so that its homotopy is a commutative ring. Indeed, the homotopy of KU is a ring of Laurent polynomials in the Bott element:

$$\pi_*(KU) \cong \mathbb{Z}[\beta^{\pm 1}],$$

where $\beta \in \pi_2(KU)$.

Convention 3.2. The element v_1 will denote the power β^{p-1} of the Bott element.

In particular, if p is 2, then v_1 is simply the Bott element β. One reason for focusing on v_1 rather than β is the following.

Theorem 3.3. [A1] *Fix a prime p. There is an h-ring spectrum L with $\pi_* L \cong \mathbb{Z}_{(p)}[v_1^{\pm 1}]$ and v_1 in degree $2(p-1)$ and with*

$$KU_{(p)} \simeq L \vee \Sigma^2 L \vee \cdots \vee \Sigma^{2(p-2)} L.$$

We will sketch this in the case $p = 3$ (it is vacuous in the case $p = 2$). We will use the Adams operation $\psi^{-1} \colon KU \to KU$. This is a map of h-ring spectra such that $\psi^{-1}(\beta^n) = (-1)^n \beta^n$. If we invert 2, then we can define an endomorphism of $KU\left[\frac{1}{2}\right]$ by the formula $e = \frac{\mathrm{id}+\psi^{-1}}{2}$. This is an idempotent, meaning that $e^2 = e$. It follows that $\mathrm{id} - e$ is also an idempotent.

The idea now is that we want to decompose $KU\left[\frac{1}{2}\right]$ into two spectra: the image of e and the image of $\mathrm{id} - e$ (the latter is the kernel of e). This follows from the following more general construction.

Proposition 3.4. *Let $e \colon X \longrightarrow X$ be an idempotent endomorphism, meaning that $e \circ e = e$. Then there are spectra X_e and X_{1-e} and an isomorphism $X \simeq X_e \vee X_{1-e}$ in* $\mathrm{Ho}\mathbf{Sp}$.

Proof. We define the spectra X_e and X_{1-e} as homotopy colimits:

$$X_e = \mathrm{hocolim}\left(X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} \dots\right)$$

and

$$X_{1-e} = \mathrm{hocolim}\left(X \xrightarrow{1-e} X \xrightarrow{1-e} X \xrightarrow{1-e} \dots\right)$$

Then, since the homotopy of a directed homotopy colimit is the colimit of the homotopy, it follows that $\pi_*(X_e)$ is the image of $e_* \colon \pi_* X \to \pi_* X$ and similarly for $\pi_*(X_{1-e})$.

Now the maps e_* and $(1-e)_*$ are idempotents of the groups $\pi_n X$ that give rise to an algebraic splitting. Moreover, the map $X \longrightarrow X_e \vee X_{1-e}$ induces the projection from $\pi_n X$ to each of the summands and is therefore an isomorphism. ■

The summand $KU\left[\frac{1}{2}\right]_e$ is traditionally called L.

Proposition 3.5. *The homotopy of L is given by* $\pi_* L \cong \mathbb{Z}\left[\frac{1}{2}\right][\beta^{\pm 2}]$.

Proof. As we indicated in the proof of Proposition 3.4, this is merely a matter of computing e_* on $KU\left[\frac{1}{2}\right]$. Since $\psi^{-1}\beta = -\beta$, it follows that e wipes out the odd powers of β and is the identity on even powers. ■

Fri, Feb. 2

It turns out that the other summand is also a (shifted) copy of L.

Proposition 3.6. *The summand* $KU\left[\frac{1}{2}\right]_{1-e}$ *is isomorphic to* $\Sigma^2 L$.

Proof. Recall that $\Sigma^2 KU \simeq KU$. Moreover, as ψ^{-1} multiplies the odd powers of β by -1 and the even powers by 1, the opposite is true of $\Sigma^2\psi^{-1}$. In other words, $\Sigma^2\psi^{-1} \simeq -\psi^{-1}$, so that the roles of e and $1-e$ are interchanged by Σ^2. The result follows. ■

Mon, Feb. 5

Our goal now is to use $KU_{(p)}$, or equivalently the Adams summand L, to learn about $\mathbb{S}_{(p)}$. One way to do this is to again use the technology of localization.

4. Bousfield localization

Previously, we considered localization at $\mathbb{S}_{\mathbb{Q}} \simeq H\mathbb{Q}$. Bousfield showed [Bo] that one can more generally localize at any spectrum E. Let us now fixed a choice of $E \in \mathrm{Ho}\mathbf{Sp}$.

Definition 4.1. We say a spectrum X is E**-acyclic** if $E \wedge X \simeq *$. A map of spectra $X \xrightarrow{f} Y$ is an E**-equivalence** if $E \wedge X \xrightarrow{\mathrm{id} \wedge F} E \wedge Y$ is a stable equivalence. A spectrum Z is E**-local** if $[X, Z] = 0$ for all E-acyclic spectra X.

In parallel to Proposition 2.14, we have:

Proposition 4.2. *Let $W \to X$ be an E-equivalence, and suppose that Y is E-local. Then the induced map $[X, Y] \to [W, Y]$ is an isomorphism.*

Definition 4.3. A map $X \xrightarrow{f} Y$ is said to be an E**-localization** if (1) Y is E-local and (2) f is an E-equivalence.

We often write $L_E X$ for the E-localization of X.

Definition 4.4. We define the E-local stable homotopy category $\mathrm{Ho}\mathbf{Sp}_E$ to be the full subcategory of $\mathrm{Ho}\mathbf{Sp}$ on the E-local spectra.

Theorem 4.5 ([Bo])**.** *Let $E \in \mathrm{Ho}\mathbf{Sp}$. There is an adjunction*

$$L_E \colon \mathrm{Ho}\mathbf{Sp} \rightleftarrows \mathrm{Ho}\mathbf{Sp}_E \colon \iota$$

such that each component $\eta \colon X \longrightarrow \iota L_E X$ of the unit for the adjunction is an E-localization.

Much of what we discussed for rationalization works as well for E-localization. For example, the proof of Proposition 2.10 applies to show

Proposition 4.6. *Suppose that E is an h-ring spectrum. Then any E-module is E-local.*

Another way to establish that a spectrum is E-local is to exhibit it as the (co)fiber of a map between E-local spectra:

Proposition 4.7. *Suppose that $W \to X \to Y$ is a cofiber sequence in* $\mathrm{Ho}\mathbf{Sp}$*. Then if two of W, X, and Y are E-local, then so is the third.*

There is one important way in which rationalization differs from most localizations.

Definition 4.8. Let $E \in \mathrm{Ho}\mathbf{Sp}$. We say that L_E is a **smashing localization** if

$$X \cong \mathbb{S} \wedge X \xrightarrow{\eta \wedge \mathrm{id}} L_E \mathbb{S} \wedge X$$

is an E-localization, so that $L_E X \simeq L_E(\mathbb{S}) \wedge X$.

Thus rationalization is an example of a smashing localization, but we will see that this does not hold for many localizations.

Wed, Feb. 7

One important example of localization is the case of $E = \mathbb{S}/p$. Recall that the cokernel of the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[1/p]$ is sometimes denoted $\mathbb{Z}/p^\infty$. The reason is that it can be written as

$$\mathbb{Z}/p^\infty \cong \operatorname*{colim}_n \mathbb{Z}/p^n,$$

where each map $\mathbb{Z}/p^n \to \mathbb{Z}/p^{n+1}$ is multiplication by p. We similarly have a cofiber sequence

$$\mathbb{S} \to \mathbb{S}[1/p] \to \mathbb{S}/p^\infty,$$

where $\mathbb{S}/p^\infty = \text{hocolim}_n \mathbb{S}/p^n$. Rotating this gives a cofiber sequence

$$\mathbb{S}^{-1}/p^\infty \to \mathbb{S} \to \mathbb{S}[1/p].$$

Now, for any X, applying $F(-, X)$ to this cofiber sequence produces a fiber sequence

$$F(\mathbb{S}[1/p], X) \to F(\mathbb{S}, X) \to F(\mathbb{S}^{-1}/p^\infty, X).$$

This middle term is isomorphic to X.

Proposition 4.9. *The spectrum $F(\mathbb{S}^{-1}/p^\infty, X)$ is a localization of X at $\mathbb{S}/p$. In other words,*

$$L_{\mathbb{S}/p}X \simeq F(\mathbb{S}^{-1}/p^\infty, X) \simeq \underset{n}{\text{holim}}\, X/p^n.$$

Proof. First, we show that $F(\mathbb{S}[1/p], X)$ is $\mathbb{S}/p$-acyclic. This just means that multiplication by p is a stable equivalence. But that is true for $\mathbb{S}[1/p]$, and so the same is true for the mapping spectrum $F(\mathbb{S}[1/p], X)$.

Next, we wish to see that $F(\mathbb{S}^{-1}/p^\infty, X)$ is $\mathbb{S}/p$-local. Thus let W be $\mathbb{S}/p$-acyclic, in other words p-periodic. We wish to know that

$$[W, F(\mathbb{S}^{-1}/p^\infty, X)] \cong [W \wedge \mathbb{S}^{-1}/p^\infty, X] \cong [W \wedge \underset{n}{\text{hocolim}}\, \mathbb{S}^{-1}/p^n, X]$$

vanishes. But smash commutes with sequential homotopy colimits, and $W \wedge \mathbb{S}^{-1}/p^n$ is contractible for all n, since W is p^n-periodic, being already p-periodic. This establishes $F(\mathbb{S}^{-1}/p^\infty, X)$ as a localization of X.

In order to get the alternative description, we use that the mapping spectrum construction converts a homotopy colimit in the first variable into a homotopy limit. Thus $F(\mathbb{S}^{-1}/p^\infty, X)$ is equivalent to $\text{holim}_n F(\mathbb{S}^{-1}/p^n, X)$. But now the cofiber sequence $\mathbb{S}^{-1}/p^n \to \mathbb{S} \xrightarrow{p^n} \mathbb{S}$ gives $F(\mathbb{S}^{-1}/p^n, X) \simeq X/p^n$. ∎

The alternative description suggests that $\mathbb{S}/p$-localization has to do with the p-adics, which are the inverse limit of the groups $\mathbb{Z}/p^n$.

Proposition 4.10. [Bo, Proposition 2.5] *If the homotopy groups of X are finitely generated, then $\pi_n L_{\mathbb{S}/p}X \cong \pi_n X \otimes \mathbb{Z}_p^\wedge$.*

We therefore will write $X_p^\wedge := L_{\mathbb{S}/p}X$. This is related to rationalization via the following **fracture square**.

Proposition 4.11. *Let X be a spectrum. Then the square*

$$\begin{array}{ccc} X_{(p)} & \longrightarrow & X_{(p)}[1/p] \simeq X_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ X_p^\wedge & \longrightarrow & (X_p^\wedge)_{\mathbb{Q}} \end{array}$$

is a homotopy pullback square.

This means that we get a Mayer-Vietoris style long exact sequence in homotopy. We will deduce the fracture square from a more general result.

Fri, Feb. 9

We can use the fracture square to show

Proposition 4.12. *If X is connective (or bounded below), then $L_{H\mathbb{F}_p}X \simeq L_{\mathbb{S}/p}X$.*

Proof. First, note that $H\mathbb{Z} \wedge \mathbb{S}/p \simeq H\mathbb{F}_p$. Thus any $\mathbb{S}/p$-acyclic W spectrum is also $H\mathbb{F}_p$-acyclic. It follows that $L_{H\mathbb{F}_p}X$ is $\mathbb{S}/p$-local.

It remains to show that $\eta_{H\mathbb{F}_p}\colon X \longrightarrow L_{H\mathbb{F}_p}X$ is an $\mathbb{S}/p$-equivalance. First, we claim that since X is connective, it follows that X is $H\mathbb{Z}$-local. The point is to use the dual Postnikov tower. Any Eilenberg-Mac Lane spectrum HA is an $H\mathbb{Z}$-module and is therefore $H\mathbb{Z}$-local. Then we can write each Postnikov section in a fiber sequence $\Sigma^n H\pi_n X \to P_0^n X \to P_0^{n-1}X$, and by induction it follows that $P_0^n X$ is $H\mathbb{Z}$-local. Then X is the homotopy inverse limit of the $P_0^n X$, and it follows that X is $H\mathbb{Z}$-local.

So now we can replace X with $L_{H\mathbb{Z}}X$. For simplicity, we will assume that X is p-local, in which case we can replace X with $L_{H\mathbb{Z}_{(p)}}X$. There is a fracture square

$$\begin{array}{ccc} L_{H\mathbb{Z}_{(p)}}X & \longrightarrow & X_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ L_{H\mathbb{F}_p}X & \longrightarrow & (L_{H\mathbb{F}_p}X)_{\mathbb{Q}} \end{array}$$

that is similar to the one mentioned last time. But now the right vertical map is an $\mathbb{S}/p$-equivalence, since any rational spectrum is $\mathbb{S}/p$-acyclic. It follows that the left vertical map is an $\mathbb{S}/p$-equivalence. ■

Mon, Feb. 12

We have seen a couple of instances of fracture squares. Many of them can be derived from the following general result.

Proposition 4.13. [DFHH, Chapter 6, Proposition 2.2] *Let D, and E be spectra. Suppose that for all spectra Z, if $D_*(Z)$ vanishes, then so does $D_*(L_E Z)$. Then for all spectra X, the square*

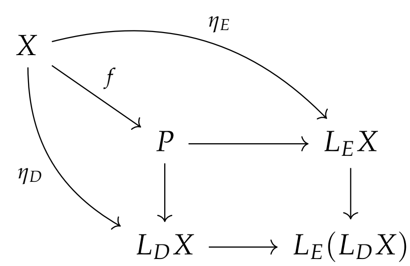
$$\begin{array}{ccc} L_{D\vee E}X & \xrightarrow{i} & L_E X \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle L_E(\eta_D)} \\ L_D X & \xrightarrow{\eta_E} & L_E(L_D X) \end{array}$$

is a homotopy pullback square.

Note that the maps i and j exist becuase both $L_D X$ and $L_E X$ are $D \vee E$-local.

Proof. Let $f\colon X \longrightarrow P$ be the map to the homotopy pullback. We will show that this is the localization with respect to $D \vee E$.

First, let W be $D \vee E$ acyclic, meaning D-acyclic and E-acyclic. Then there are no maps from W to the other three vertices of the square, so by the Mayer-Vietoris sequence, there are no maps from W to P. Thus P is $D \vee E$-local.



It remains to show that the map f is a $D \vee E$-equivalence, which means precisely that it is a D-equivalence and an E-equivalence. We start with E, since that argument is slightly simpler. First, notice that since the E-equivalence $X \longrightarrow L_E X$ factors through P, it suffices to show that $P \longrightarrow L_E(X)$ is an E-equivalence. By the Mayer-Vietoris sequence in E-homology, this is equivalent to the map $L_D X \longrightarrow L_E(L_D X)$ being an E-equivalence, which is true by the definition of E-localization.

Similarly, the map f being a D-equivalence reduces to the map $L_E X \longrightarrow L_E(L_D X)$ being a D-equivalence. But the fiber $\mathrm{Nil}_D X$ of $\eta\colon X \longrightarrow L_D X$ is D-acyclic. It follows, by the assumption, that the fiber $L_E(\mathrm{Nil}_D X)$ of $L_E X \longrightarrow L_E(L_D X)$ is D-acyclic. ■

We will now use this to prove Proposition 4.11

Proof of Proposition 4.11. We take $D = \mathbb{S}/p$ and $E = H\mathbb{Q}$. Here we get that $(\mathbb{S}/p)(Z_{\mathbb{Q}})$ vanishes, even without the assumption that $(\mathbb{S}/p)_*(Z)$ vanishes. It remains to show that localization at $\mathbb{S}/p \vee H\mathbb{Q}$ is localization at p. For this it suffices to see that the class of acyclics for $\mathbb{S}/p \vee H\mathbb{Q}$ is the same as the acyclics for $\mathbb{S}_{(p)}$. But acyclics for $H\mathbb{Q}$ are spectra with torsion homotopy groups, whereas acyclics for $\mathbb{S}/p$ have p-periodic homotopy. Thus acyclics for $\mathbb{S}/p \vee H\mathbb{Q}$ have homotopy that is torsion but with no p-torsion. These are precisely the acyclics for $\mathbb{S}_{(p)}$. ■

Wed, Feb. 14

5. Bousfield localization at $KU_{(p)}$

Again, our goal is to use KU, or $KU_{(p)}$, to learn about $\mathbb{S}_{(p)}$. One way to do this is to use Bousfield localization. Since $KU_{(p)}$ splits into copies of the Adams summand L (by Theorem 3.3), it is equivalent to localize at L.

First, we need one preliminary result.

Lemma 5.1. *Suppose that E is a spectrum such that $E_{\mathbb{Q}}$ is nontrivial. Then $L_{E_{\mathbb{Q}}}$ agrees with $L_{H\mathbb{Q}}$. In other words, localization at* **any** *(nontrivial) rational spectrum is rationalization.*

Proof. The point is that any rational spectrum automatically decomposes into a sum of (suspensions of) $H\mathbb{Q}$. Said differently, the functor $\pi_* \colon \mathrm{Ho}\mathbf{Sp}_{\mathbb{Q}} \longrightarrow \mathrm{gr}\mathbf{Vect}_{\mathbb{Q}}$ is an equivalence of categories.

To see this, note that if $F = E_{\mathbb{Q}}$ is a rational spectrum then for any $n \in \mathbb{Z}$ we can build a map $\Sigma^n H\pi_n(X) \longrightarrow F$ inducing an isomorphism on π_n. The reason is that we have an equivalence $\bigvee \mathbb{S}^n_{\mathbb{Q}} \simeq \Sigma^n H\pi_n(F)$, where the wedge can be indexed over a basis of the $\mathbb{Q}$-vector space $\pi_n(F)$. Now we can assemble these into a stable equivalence $\bigvee_n \Sigma^n H\pi_n(F) \xrightarrow{\sim} F$.

Now the statement about localizations follows because a spectrum will be F-acyclic if and only if it is $H\mathbb{Q}$-acyclic. ∎

Notation 5.2. Write $E(1) = L$ and $K(1) = L \wedge \mathbb{S}/p \simeq L/p$. It is common to abbreviate $L_{E(1)}$ to L_1 and $L_{K(1)}$ to $\widehat{L}_1$.

Proposition 5.3. *For any X, there is a homotopy pullback square*

$$\begin{array}{ccc} L_1 X & \longrightarrow & X_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ \widehat{L}_1 X & \longrightarrow & (\widehat{L}_1 X)_{\mathbb{Q}} \end{array}$$

Proof of Proposition 5.3. We will use Proposition 4.13, with $D = K(1) = L/p$ and $E = L[1/p] = L_{\mathbb{Q}}$. Note that the proposition applies, since $K(1)_*$ vanishes on any rational spectrum (because $\mathbb{S}/p$ annihilates rational spectra). Also, Lemma 5.1 shows that localization at $L_{\mathbb{Q}}$ is the same as rationalization.

Then the main task is to identify $E(1)$-localization with $K(1) \vee L_{\mathbb{Q}}$-localization. Again, it suffices to see that the acyclics for both localizations agree. But now we can use the same argument as in the proof of Proposition 4.11. There we showed that a p-local spectrum is null if and only if it is acyclic for both $\mathbb{S}/p$ and $H\mathbb{Q}$. Since L is p-local, it follows that $L \wedge W$ is null if and only if $L \wedge \mathbb{S}/p \wedge W$ and $L \wedge \mathbb{S}_{\mathbb{Q}} \wedge W$ are null. ∎

Notation 5.4. Given spectra E and D, we write $\langle E \rangle$ for the set of E-acyclics. Thus a way to summarize what we showed in the proof of Proposition 5.3 is that $\langle E(1) \rangle = \langle K(1) \vee H\mathbb{Q} \rangle$. Or, equivalently, $\langle KU_{(p)} \rangle = \langle KU/p \vee H\mathbb{Q} \rangle$.

Remark 5.5. On the worksheet for this week, you will show that $\widehat{L}_1 X$ is in fact $(L_1 X)^\wedge_p$, which partly explains the notation.

Thus, in order to understand $L_1\mathbb{S}$, it remains to first calculate $\widehat{L}_1\mathbb{S}$.

Fri, Feb. 16

We will just mention some notation. As we said previously, $\langle E \rangle$ will denote the set of E-acyclics. We can define an equivalence relation on spectra by saying that $E \sim F$ if $\langle E \rangle = \langle F \rangle$. The equivalence class of E is referred to as the **Bousfield class** of E.

It is also common to write $\langle E \rangle \vee \langle F \rangle$ for $\langle E \vee F \rangle$ and similarly $\langle E \rangle \wedge \langle F \rangle$ for $\langle E \wedge F \rangle$. Thus, in the proof of Proposition 5.3, we showed that

$$\langle E(1) \rangle = \langle K(1) \rangle \vee \langle K(0) \rangle,$$

where $K(0) = E(0)$ means $H\mathbb{Q}$.

Mon, Feb. 19

The $K(1)$-local sphere $\widehat{L}_1\mathbb{S}$ looks a little different at odd primes than at the prime 2. We first state the (easier) odd-primary case.

Proposition 5.6. *If p is odd, then the fiber of $E(1)_p^\wedge \xrightarrow{\psi^{p+1}-\mathrm{id}} E(1)_p^\wedge$ is $\widehat{L}_1\mathbb{S}$.*

We will not prove this. It is easy to see that the fiber is $K(1)$-local. Since $E(1)$ is $E(1)$-local, it follows that the fiber of $\psi^{p+1}-\mathrm{id}$ on $E(1)$ (in other words, before p-completion) is $E(1)$-local. Then passing to p-completion makes it $K(1)$-local. On the other hand, showing that the fiber has the right $K(1)$-homology is not so simple, given what we have covered so far.

Let's now use Proposition 5.6 to calculate $\pi_*\widehat{L}_1\mathbb{S}$ (aka the "$K(1)$-local sphere").

Example 5.7. Consider the case of $p = 3$. Then the homotopy of $E(1)_3^\wedge$ is $\mathbb{Z}_3[v_1^{\pm 1}]$, with v_1 in degree 4. We are left to consider the map $(\psi^4 - \mathrm{id})_*\colon \pi_{4n}E(1)_3^\wedge \longrightarrow \pi_{4n}E(1)_3^\wedge$. Recall that the Adams operation ψ^k acts as multiplication by k on the Bott element β. The case of ψ^{-1} came up in the discussion of Theorem 3.3. As v_1 is $\beta^{p-1} = \beta^2$, it follows that ψ^k acts as multiplication by k^{2n} on v_1^n. Thus the difference $\psi^{p+1} - \mathrm{id} = \psi^4 - \mathrm{id}$ acts on the element v_1^n as multiplication by $4^{2n} - 1$. In degree 0, this is the zero map, so that we conclude

$$\pi_0\widehat{L}_1\mathbb{S} \cong \mathbb{Z}_3 \qquad \text{and} \qquad \pi_{-1}\widehat{L}_1\mathbb{S} \cong \mathbb{Z}_3.$$

In other degrees, the map is injective, so that the cokernel calculates $\pi_{4n-1}\widehat{L}_1\mathbb{S}$, and the other homotopy groups vanish.

We now have an isomorphism

$$\pi_{4n-1}\widehat{L}_1\mathbb{S} \cong \operatorname{coker}\left(\mathbb{Z}_3 \xrightarrow{4^{2n}-1} \mathbb{Z}_3\right), \qquad \text{for } n \neq 0.$$

However, $4^{2n} - 1 = 16^n - 1 = (4^n - 1)(4^n + 1)$, and $4^n + 1$ is a unit in $\mathbb{Z}_3$, since it is not a multiple of 3. So the cokernel will be isomorphic to the cokernel of multiplication by $4^n - 1$. This cokernel will be $\mathbb{Z}/3^{\nu_3(4^n-1)}$, where $\nu_3(4^n - 1)$ is the 3-adic valuation of $4^n - 1$.

We claim that $\nu_3(4^n - 1)$ is equal to $\nu_3(n) + 1$, so that

$$\pi_{4n-1}\widehat{L}_1\mathbb{S} \cong \mathbb{Z}/3^{\nu_3(n)+1}, \qquad \text{for } n \neq 0.$$

Writing $\partial\colon \pi_n E(1)_p^\wedge \longrightarrow \pi_{n-1}\widehat{L}_1\mathbb{S}$, we have

$$\begin{array}{lll} \pi_3\widehat{L}_1\mathbb{S} \cong \mathbb{Z}/3\{\partial v_1\}, & \pi_7\widehat{L}_1\mathbb{S} \cong \mathbb{Z}/3\{\partial v_1^2\}, & \pi_{11}\widehat{L}_1\mathbb{S} \cong \mathbb{Z}/9\{\partial v_1^3\}, \\ \pi_{15}\widehat{L}_1\mathbb{S} \cong \mathbb{Z}/3\{\partial v_1^4\}, & \pi_{19}\widehat{L}_1\mathbb{S} \cong \mathbb{Z}/3\{\partial v_1^5\}, & \pi_{23}\widehat{L}_1\mathbb{S} \cong \mathbb{Z}/9\{\partial v_1^6\}, \\ \pi_{27}\widehat{L}_1\mathbb{S} \cong \mathbb{Z}/3\{\partial v_1^7\}, & \pi_{31}\widehat{L}_1\mathbb{S} \cong \mathbb{Z}/3\{\partial v_1^8\}, & \pi_{35}\widehat{L}_1\mathbb{S} \cong \mathbb{Z}/27\{\partial v_1^9\}, \end{array}$$

and so on.

One way to argue the claim is as follows. We wish to find the largest k for which the congruence $4^n \equiv 1 \pmod{3^k}$ holds. Thus we want the order of 4 in the unit group $(\mathbb{Z}/3^k)^\times$ to divide n. But we have a group isomorphism $(\mathbb{Z}/3^k)^\times \cong C_2 \times C_{3^{k-1}}$. The binomial expansion on $4^{3^{k-2}} = (3+1)^{3^{k-2}}$ shows that it is not congruent to 1 modulo 3^k, so that 4 projects to a generator of $C_{3^{k-1}}$. Since it projects to the identity element of C_2, we conclude that the order of 4 in $(\mathbb{Z}/3^k)^\times$ is exactly 3^{k-1}. To sum up, we have shown that $4^n \equiv 1 \pmod{3^k}$ holds precisely when 3^{k-1} divides n. Thus the 3-adic valuation of n is one less than the 3-adic valuation of $4^n - 1$.

Wed, Feb. 21

The arguments of this example generalize to give:

Proposition 5.8. *Let p be an odd prime. Then the homotopy of the $K(1)$-local sphere is*

$$\pi_k L_{K(1)}\mathbb{S} = \pi_k \widehat{L}_1 \mathbb{S} \cong \begin{cases} \mathbb{Z}_p & k = 0, -1 \\ \mathbb{Z}/p^{\nu_p(n)+1} & k = 2(p-1)n - 1 \quad (\text{for } k \neq -1) \\ 0 & \text{else.} \end{cases}$$

We can then use the fracture square to get the homotopy of the $E(1)$-local sphere.

Corollary 5.9. *Let p be an odd prime. Then the homotopy of the $E(1)$-local sphere is*

$$\pi_k L_{E(1)}\mathbb{S} = \pi_k L_1 \mathbb{S} \cong \begin{cases} \mathbb{Z}_{(p)} & k = 0 \\ \mathbb{Z}/p^{\infty} & k = -2 \\ \mathbb{Z}/p^{\nu_p(n)+1} & k = 2(p-1)n - 1 \quad (\text{for } k \neq -1) \\ 0 & \text{else.} \end{cases}$$

Proof. This mostly follows easily from the fracture square, given that $\mathbb{S}_{\mathbb{Q}} \simeq H\mathbb{Q}$. The answer for π_0 follows from the algebraic antecedent of Proposition 4.11. The answer for π_{-2} follows from the isomorphism $\mathbb{Q}_p/\mathbb{Z}_p \cong \mathbb{Z}/p^{\infty}$. ∎

Proposition 5.6 is not quite right at the prime $p = 2$. Instead of using an Adams operation on $KU_p^{\wedge}$, we instead use $KO_2^{\wedge}$.

Proposition 5.10. *The fiber of $KO_2^{\wedge} \xrightarrow{\psi^3 - \mathrm{id}} KO_2^{\wedge}$ is 2-primary $\widehat{L}_1\mathbb{S}$.*

Recall that KO is periodic real K-theory. Its homotopy ring is

$$\pi_*(KO) \cong \mathbb{Z}[\eta, \alpha, \beta^{\pm 1}]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta)$$

with η, α, and β in degrees 1, 4, and 8, respectively. As in Proposition 5.6, we will not prove that the fiber is $K(1)$-equivalent to the sphere. The fact that it is $K(1)$-local follows from the fact that the fiber of $\psi^3 - \mathrm{id}$ on $KO_{(2)}$ is $KU_{(2)}$-local. This uses the following.

Proposition 5.11. *The spectra KO and KU are Bousfield equivalent.*

Proof. This uses the Wood cofiber sequence

$$\Sigma^1 KO \xrightarrow{\eta} KO \longrightarrow KU.$$

Here $\eta \in \pi_1\mathbb{S}$ is the Hopf map. Thus another way to write this cofiber sequence is to say that $\mathbb{S}/\eta \wedge KO$ is KU. If follows that any KO-acyclic is KU-acyclic. On the other hand, if X is KU-acyclic, it follows that η acts as a stable equivlence on $KO \wedge X$. But η^3 is zero in KO, so it follows that X is also KO-acyclic. ∎

Proposition 5.12. *The homotopy of the 2-primary $K(1)$-local sphere is*

$$\pi_k L_{K(1)}\mathbb{S} = \pi_k \widehat{L}_1 \mathbb{S} \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}/2 & k = 0 \\ \mathbb{Z}_2 & k = -1 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & k \equiv 1 \pmod 8 \\ \mathbb{Z}/2 & k \equiv 0, 2 \pmod 8 \quad (\text{for } k \neq 0) \\ \mathbb{Z}/2^{\nu_2(n)+3} & k = 4n - 1 \quad (\text{for } k \neq -1) \\ 0 & \text{else.} \end{cases}$$

Again, the fracture square gives

Corollary 5.13. *The homotopy of the 2-primary $E(1)$-local sphere is*

$$\pi_k L_{E(1)}\mathbb{S} = \pi_k L_1 \mathbb{S} \cong \begin{cases} \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2 & k = 0 \\ \mathbb{Z}/2^\infty & k = -2 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & k \equiv 1 \pmod 8 \\ \mathbb{Z}/2 & k \equiv 0, 2 \pmod 8 \quad (\text{for } k \neq 0) \\ \mathbb{Z}/2^{\nu_2(n)+3} & k = 4n-1 \quad (\text{for } k \neq -1) \\ 0 & \text{else.} \end{cases}$$

Ok, we have computed $L_1\mathbb{S}$, but what does this tell us?

Proposition 5.14. *If X is $E(1)$-local, then the homotopy of X/p is v_1-periodic.*

What do we mean by this? It turns out that $\mathbb{S}/p$ has a "v_1-self-map" in the following sense.

Proposition 5.15 ([A2]). *(**The v_1-self-map on $\mathbb{S}/p$**) Let p be an odd prime. Then there exists a map $v_1 \colon \Sigma^{2(p-1)}\mathbb{S}/p \longrightarrow \mathbb{S}/p$ which induces multiplication by v_1 on $E(1)$-homology. For $p = 2$, there is a map $v_1^4 \colon \Sigma^8\mathbb{S}/2 \longrightarrow \mathbb{S}/2$ which induces multiplication by v_1^4 on $E(1) = KU_{(2)}$-homology.*

We will discuss this result soon. Now if this self-map induces a stable-equivalence on X/p, we say that the homotopy of X/p is v_1-periodic.

Fri, Feb. 23

Proof of Proposition 5.14. Suppose that X is $E(1)$-local. By the 2-out-of-3 property, it follows that X/p is $E(1)$-local. Notice that the v_1-self-map on $\mathbb{S}/p$ is an $E(1)$-equivalence, so it follows that this gives an $E(1)$-equivalence on $X/p \cong \mathbb{S}/p \wedge X$. But an $E(1)$-equivalence between $E(1)$-local spectra is a stable equivalence. ■

In turn, v_1-periodicity implies locality.

Notation 5.16. Write $\mathrm{Tel}(1) = \mathbb{S}/p[v_1^{-1}]$.

This is also often simply written as $T(1)$.

Proposition 5.17. *Any $K(1)$-local spectrum is* Tel(1)*-local. In other words, we have $\langle \mathrm{Tel}(1) \rangle \geq \langle K(1) \rangle$.*

This applies, for instance, to X/p when X is $E(1)$-local.

Proof. Let W be Tel(1)-acyclic. Thus $W/p[v_1^{-1}] \simeq *$. As in the proof of Proposition 5.14, we know that $\Sigma^{2(p-1)}W/p \xrightarrow{v_1} W/p$ induces an isomorphism $E(1)_*(W/p) \cong E(1)_{*+2(p-1)}(W/p)$, or in other words $K(1)_*W \cong K(1)_{*+2(p-1)}W$. On the other hand, the colimit

$$\begin{aligned} &\operatorname{colim}\left(K(1)_*W \xrightarrow{\cong} K(1)_{*+2(p-1)}W \xrightarrow{\cong} \dots\right) \\ \cong\ &\operatorname{colim}\left(E(1)_*W/p \xrightarrow{\cong} E(1)_{*+2(p-1)}W/p \xrightarrow{\cong} \dots\right) \\ \cong\ &E(1)_*\left(\operatorname{hocolim} W/p \xrightarrow{v_1} \Sigma^{-2(p-1)}W/p \xrightarrow{v_1} \dots\right) \cong 0 \end{aligned}$$

vanishes by the assumption on W. It follows that $K(1)_*W = 0$. ■

In fact, more is true.

Theorem 5.18 ([Bo], [Ma], [Mi]). *(**Height 1 Telescope Conjecture**)* $\mathrm{Tel}(1)$*-localization is* $K(1)$*-localization.*

Bousfield deduced this from calculations of Mahowald (at $p = 2$) [Ma] and later Miller (at odd primes) [Mi].

Mon, Feb. 26

Proposition 5.19. [MRS] *The height 1 telescope conjecture is equivalent to the statement that* $\mathrm{Tel}(1) = \mathbb{S}/p[v_1^{-1}]$ *is* $L_1\mathbb{S}/p = \widehat{L}_1\mathbb{S}/p$.

On the last worksheet, working at $p = 3$ you computed that $\pi_n L_1(\mathbb{S}/3)$ is $\mathbb{Z}/3$ if n is congruent to 0 or -1 modulo 4, and it vanishes otherwise. Thus the homotopy groups have a 4-fold periodicity. But it is not at all obvious that this agrees with $\mathbb{S}/3[v_1^{-1}]$!

6. THE v_1-SELF-MAP ON $\mathbb{S}/p$

We will now discuss a proof of Proposition 5.15. The main tool here is the **Adams spectral sequence**.

Roughly, the idea for the Adams spectral sequence is as follows. Let's say we have spectra X and Y, and we would like to understand the graded abelian group $[X,Y]_*$. For example, X and Y could both be $\mathbb{S}$! A simpler thing is to choose some nice homology theory E and consider the simpler graded abelian group $\mathrm{Hom}(E_*(X), E_*(Y))$. For example, we could take $E = H\mathbb{F}_p$. Then we might ask to what extent we can recover $[X,Y]_*$ from $\mathrm{Hom}(E_*(X), E_*(Y))$.

The answer to this simple question is: not much. But we can ask a better question. First of all, if $f\colon X \longrightarrow Y$ is a map of spectra, then we expect more structure on the map $f_*\colon E_*(X) \longrightarrow E_*(Y)$. For starters, let us choose a nice, multiplicative homology theory E_*. This corresponds to the spectrum E being a commutative h-ring spectrum. Then $E_*(X)$ will be an E_*-module, and the induced map f_* will be a map of E_*-modules, so we should restrict our attention to E_*-linear maps $E_*(X) \longrightarrow E_*(Y)$. But there is more.

Temporarily, let us switch to cohomology. Then $E^*(X)$ also has an action by the ring of stable E-cohomology operations, written E^*E. In the case that E is $H\mathbb{F}_p$, this is the mod p Steenrod algebra $\mathcal{A}_p$. Now the map $f^*\colon E^*(Y) \longrightarrow E^*(X)$ will be both an E^*-module map *and* an E^*E-module map. So we should further restrict attention to the set $\mathrm{Hom}_{E^*E}(E^*Y, E^*X)$ of E^*E-module maps in E^*-modules.

But we still don't get much in the case of $E = H\mathbb{F}_p$ and $X = Y = \mathbb{S}$. $H\mathbb{F}_p^*\mathbb{S}$ is just $\mathbb{F}_p$ in a single degree, and so there are not many maps. Instead, we can consider the **derived** set of maps, which means Ext groups.

In the case of $H\mathbb{F}_p$, this idea turns out to work well. But for a general E, it turns out there are a number of technical hurdles, which make working with cohomology not a great choice. See [A1, pages 51-55] for some good discussion of this. It turns out to be better to go back to homology. Rather than working with the ring E^*E of cohomology operations, it is better to work with the ring $E_*E = \pi_*(E \wedge E)$, sometimes called the ring of homology "co-operations". With some mild assumptions, the ring E_*E acquires a coalgebra structure, and E_*X becomes a "comodule" over this coalgebra. The general Adams spectral sequence will use comodule Ext as the starting point.

Wed, Feb. 28

For our current application of finding the v_1-self-map on $\mathbb{S}/3$, the original Adams spectral sequence, based on $H\mathbb{F}_p$, will suffice. We will write $\mathrm{H}^*(X)$ for $\mathrm{H}^*(X;\mathbb{F}_p)$ and $\mathcal{A}_p$ for the mod p Steenrod algebra.

Theorem 6.1. *Let X be connective (or bounded below). There is a spectral sequence (the* **Adams spectral sequence***) having E_2-term* $\mathrm{Ext}_{\mathcal{A}_p}(\mathrm{H}^*(X), \mathbb{F}_p)$ *and converging to* $\pi_* L_{H\mathbb{F}_p} X = \pi_* X_p^\wedge$.

There are several ways to grade the spectral sequence. First of all, Ext has its won grading (the "homological" grading, traditionally written as s). And since the input module $\mathrm{H}^*(X)$ is graded, there is an additional grading (the "internal grading", traditionally written as t). However, we will use the spectral sequence to tell us about a stable homotopy group π_n, so it is convenient to regrade, so that the part of the grading contributing to the stable "stem" π_n is kept together. It turns out that this is the difference

$$\text{stem} = \text{internal degree} - \text{homological degree}.$$

Also, the homological degree is also called the "Adams filtration". We will then use the index s for stem and f for Adams filtration. With this notation, the Adams spectral sequence looks like

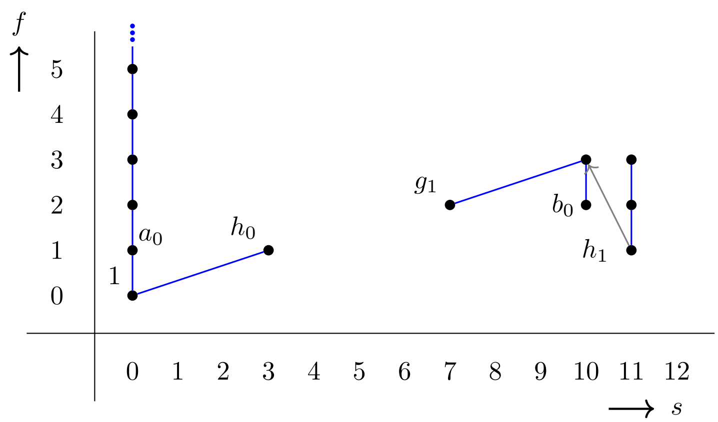
$$E_2^{s,f} = \text{Ext}_{\mathcal{A}_p}^{s,f}(\text{H}^*X, \mathbb{F}_p) \Rightarrow \pi_s(X_p^\wedge).$$

The d_2 differential takes the shape $d_2\colon E_2^{s,f} \longrightarrow E_2^{s-1,f+2}$, and more generally we have

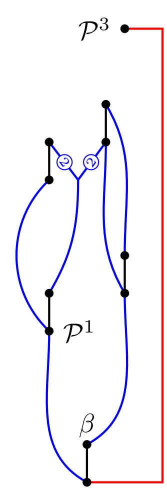
$$d_r\colon E_r^{s,f} \longrightarrow E_r^{s-1,f+r}.$$

In other words, Adams differentials go **left one** and **up** r.

Here is the Adams spectral sequence for $\mathbb{S}$, at $p = 3$, in low degrees:



The vertical lines depict multiplication by the element a_0, which here detects the element 3 in $\pi_0\mathbb{S}$. The slope 1/3 line depicts by the element h_0, which detects a 3-torsion element of $\pi_3(\mathbb{S})$.



Where does this answer come from? First, one must calculate the E_2-term, which is given by the Ext groups $\text{Ext}_{\mathcal{A}_3}(\mathbb{F}_3, \mathbb{F}_3)$. To start off, the mod p Steenrod algebra, for p odd, is generated by elements

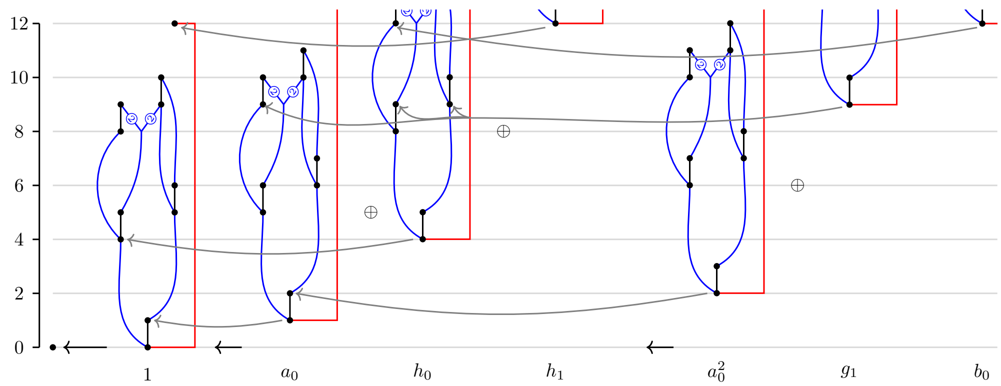
$$\beta \in \mathcal{A}_p^1, \qquad \mathcal{P}^i \in \mathcal{A}_p^{2i(p-1)}.$$

modulo the "Adem relations". In fact, the Adem relations tell you that the algebra is generated by β and the $\mathcal{P}^{p^k}$ as k varies. In particular, for $p = 3$, we have β in degree 1, $\mathcal{P}^1$ in degree 4, $\mathcal{P}^3$ in degree 12, and so on. Some examples of the Adem relations in this case are that

$$\beta\beta = 0, \qquad \mathcal{P}^1\beta\mathcal{P}^1 = 2\beta\mathcal{P}^1\mathcal{P}^1 + 2\mathcal{P}^1\mathcal{P}^1\beta, \qquad \text{and} \qquad \mathcal{P}^1\mathcal{P}^1\mathcal{P}^1 = 0.$$

The portion of $\mathcal{A}_3$ in degrees up to 12 is depicted to the right, where each line corresponds to left multiplication by some element. The element 1 is at the bottom.

Now $\text{Ext}_{\mathcal{A}_3}(\mathbb{F}_3, \mathbb{F}_3)$ can be computed as the cohomology of $\text{Hom}_{\mathcal{A}_3}(P_\bullet, \mathbb{F}_3)$, where $P_\bullet$ is a free (or projective) resolution of $\mathbb{F}_3$ as $\mathcal{A}_3$-modules. We can explicitly build a free resolution as follows:



This is an example of what is called a "minimal" resolution, which means that when we take this resolution and Hom it into $\mathbb{F}_3$, the resulting cochain complex will have all differentials zero. Thus the generators 1, a_0, h_0, etc. that we see in the resolution give the elements of Ext displayed in the above Adams chart. In addition, the above is not **actually** a resolution, in that a true resolution would have a summand h_n for each $n \geq 0$ at stage 1. We have only displayed the part that is relevant to the portion of the Adams E_2-page displayed above.

Fri, Mar. 1

Last time, we saw from the Adams spectral sequence that $\pi_3(\mathbb{S}_3^\wedge) \cong \mathbb{Z}/3$ and $\pi_4(\mathbb{S}_3^\wedge) = 0$. It follows from the long exact sequence that $\pi_4(\mathbb{S}/3) \cong \mathbb{Z}/3$. We may take either of the nonzero elements as the element $v_1 \colon \Sigma^4\mathbb{S} \longrightarrow \mathbb{S}/3$. As it is a 3-torsion element and $\pi_5(\mathbb{S}/3) = 0$, it follows that v_1 extends to a map $v_1 \colon \Sigma^4\mathbb{S}/3 \longrightarrow \mathbb{S}/3$. To see that this induces an isomorphism on $E(1)_*$, it suffices to see that the diagram

$$\begin{array}{ccc} \Sigma^4\mathbb{S} & \xrightarrow{v_1} & \mathbb{S}/3 \\ \downarrow & & \downarrow \\ \Sigma^4 E(1) & \xrightarrow{v_1} & E(1)/3 \simeq K(1) \end{array}$$

commutes, where the vertical maps come from the unit of the ring spectrum $E(1)$. This can be seen from a comparison of Adams spectral sequences for $\mathbb{S}/3$ and $E(1) \wedge \mathbb{S}/3$ if we replace the periodic Adams summand $E(1) = L$ with its connective cover ℓ. The point is that one can identify the E_2-term of the Adams spectral sequence for $\ell_* X$ with $\mathrm{Ext}_{\mathcal{E}(1)}(\mathrm{H}^*(X), \mathbb{F}_p)$, where $\mathcal{E}(1) \subset \mathcal{A}_p$ is the exterior subalgebra generated by $Q_0 = \beta$ and $Q_1 = \mathcal{P}^1\beta - \beta\mathcal{P}^1$.

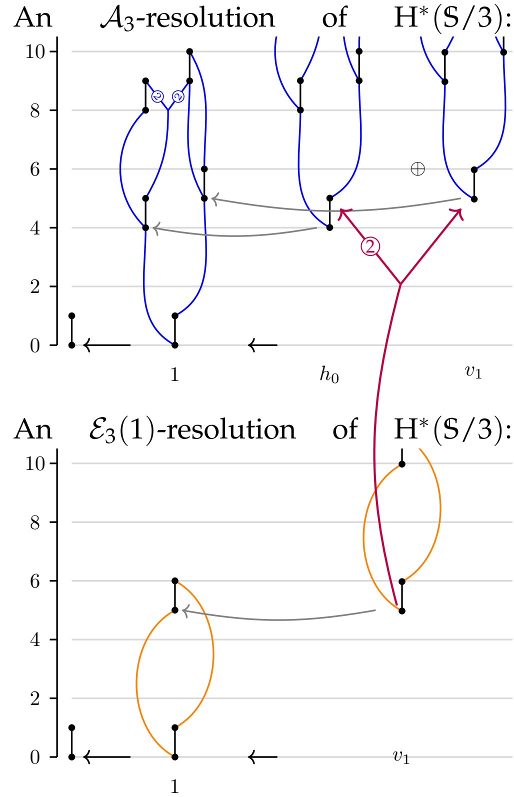
Mon, Mar. 4

Last time, we found a map $v_1 \colon \Sigma^4\mathbb{S}/3 \longrightarrow \mathbb{S}/3$, and we said that one way to verify it induces an isomorphism on $E(1)_*$ is to use a comparison of Adams spectral sequences for $\mathbb{S}/3$ and $\ell \wedge \mathbb{S}/3$, for ℓ the connective cover of $E(1)$. The $\mathcal{A}_3$-module $\mathrm{H}^*(\ell)$ is known to be the quotient $\mathcal{A}_3 \otimes_{\mathcal{E}_3(1)} \mathbb{F}_3$, where $\mathcal{E}_3(1)$ is the (exterior) subalgebra on $Q_0 = \beta$ in degree 1 and $Q_1 = P^1\beta - \beta P^1$ in degree 5. Using a change-of-rings isomorphism, the comparison of Adams spectral sequences takes the form

$$\mathrm{Ext}_{\mathcal{A}_3}(\mathrm{H}^*(\mathbb{S}/3), \mathbb{F}_3) \longrightarrow \mathrm{Ext}_{\mathcal{E}_3(1)}(\mathrm{H}^*(\mathbb{S}/3), \mathbb{F}_3),$$

on E_2-pages. These are computed by taking free $\mathcal{A}_3$ and $\mathcal{E}_3(1)$-module resolutions, respectively, of $\mathrm{H}^*(\mathbb{S}/3)$. Now the point is that if $P_{\mathcal{A}}$ is the $\mathcal{A}_3$-module resolution, it restricts to give a resolution of $\mathcal{E}_3(1)$-modules as well, and we can then find a comparison of resolutions $P_{\mathcal{E}(1)} \to P_{\mathcal{A}}$, which will induce the desired map on Ext.

Since the map will be $1 \mapsto 1$ at the P_0-level, it follows that, at the P_1-level, the map will is given by $v_1 \mapsto v_1 - \beta h_0$ as indicated to the right. After applying $\mathrm{Hom}_{\mathcal{E}(1)}(-, \mathbb{F}_3)$, we conclude that the map $\mathrm{Ext}_{\mathcal{A}_3}(\mathbb{F}_3, \mathbb{F}_3) \to \mathrm{Ext}_{\mathcal{E}(1)}(\mathbb{F}_3, \mathbb{F}_3)$ sends v_1 to v_1.

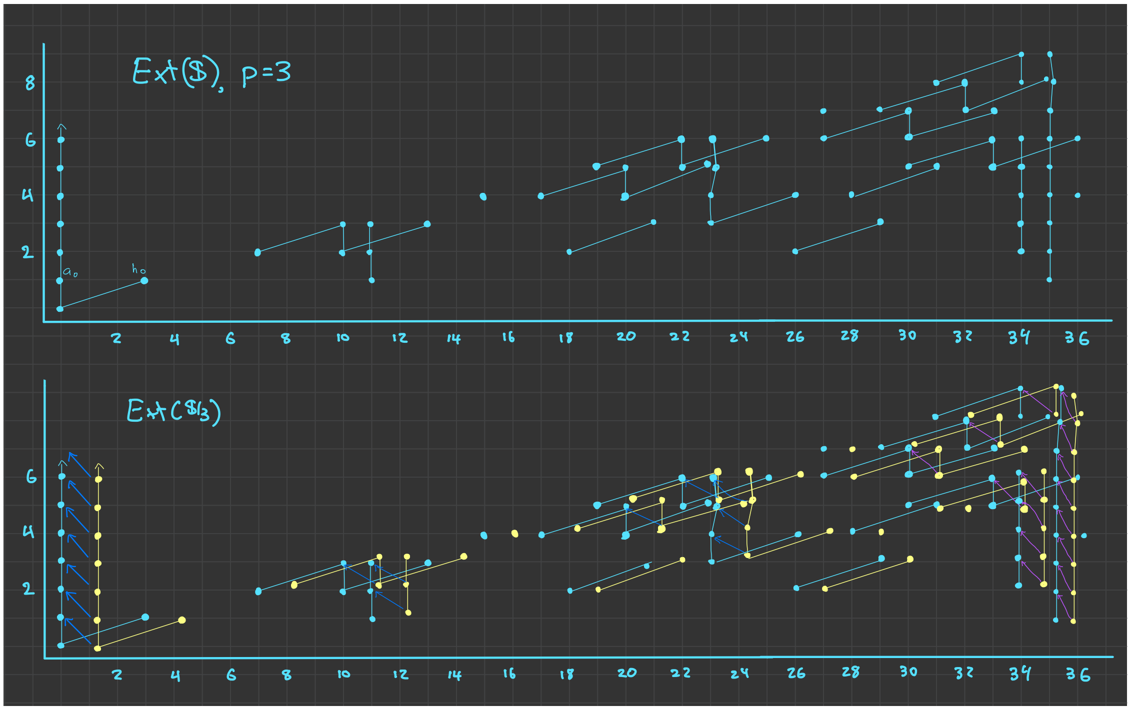


7. v_1-PERIODIC ELEMENTS IN $\pi_*\mathbb{S}$

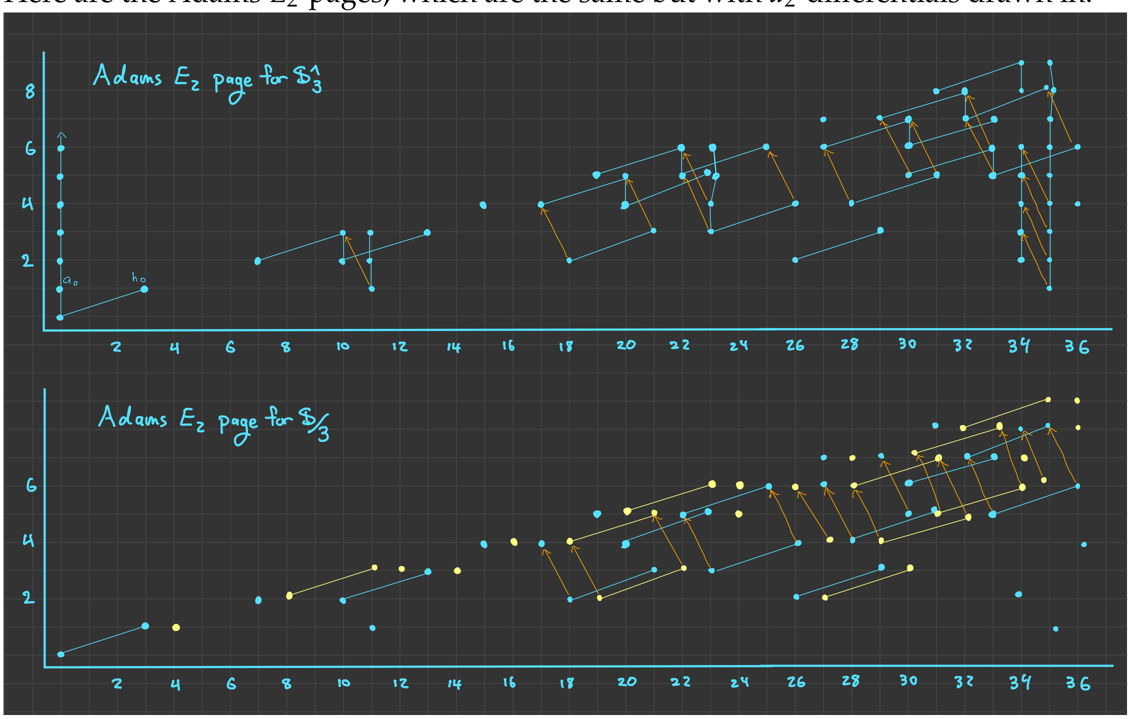
We have described the homotopy of $L_1\mathbb{S}$ as well as the related $L_1\mathbb{S}/p \simeq v_1^{-1}\mathbb{S}/p$. Here, we want to discuss how this relates to the original object of study, namely $\pi_*\mathbb{S}$.

We start with the case of $p = 3$ (which is essentially the same as for any odd primes). We start by displaying $\mathrm{Ext}_{\mathcal{A}_3}(\mathrm{H}^*(\mathbb{S}), \mathbb{F}_3)$ and $\mathrm{Ext}_{\mathcal{A}_3}(\mathrm{H}^*(\mathbb{S}/3), \mathbb{F}_3)$ (on the next page). Here is how we are thinking about $\mathrm{Ext}_{\mathcal{A}_3}(\mathrm{H}^*(\mathbb{S}/3), \mathbb{F}_3)$. The degree 3 map on $\mathbb{S}$ is zero in $\mathrm{H}^*(\mathbb{S})$, since we are taking coefficients in $\mathbb{F}_3$. Thus the rotated cofiber sequence $\mathbb{S} \to \mathbb{S}/3 \to \Sigma^1\mathbb{S}$ induces a short exact sequence on cohomology. This is a short exact sequence of $\mathcal{A}_3$-modules, which therefore gives rise to a long exact sequence in Ext.

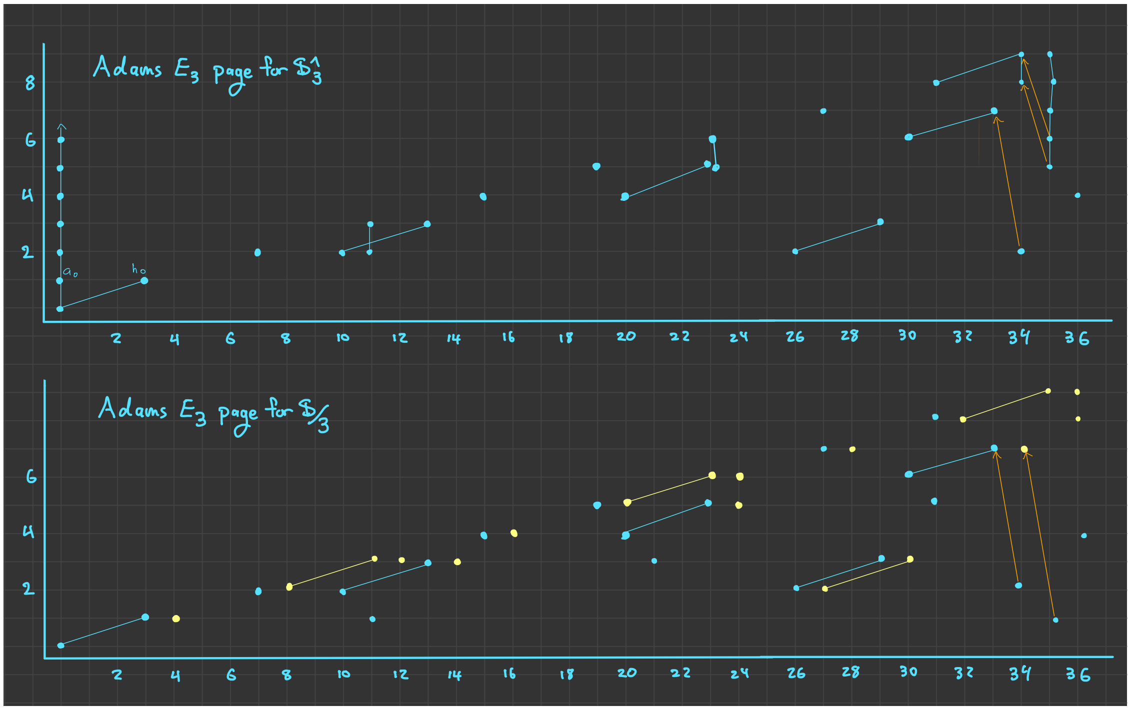
Thus the Ext group that we care about is one term in a long exact sequence in which the other terms are $\mathrm{Ext}_{\mathcal{A}_3}(\mathrm{H}^*(\mathbb{S}), \mathbb{F}_3)$ and a suspended copy of the same. It turns out that the connecting homomorphism in this long exact sequence corresponds to multiplication by the element a_0. Thus we can obtain the Ext group that we want by displaying two copies of $\mathrm{Ext}_{\mathcal{A}_3}(\mathrm{H}^*(\mathbb{S}), \mathbb{F}_3)$, one shifted to the right by 1, and drawing a differential from a shifted copy of a class x to $a_0 \cdot x$. The Ext group we want is essentially the resulting homology of this complex.



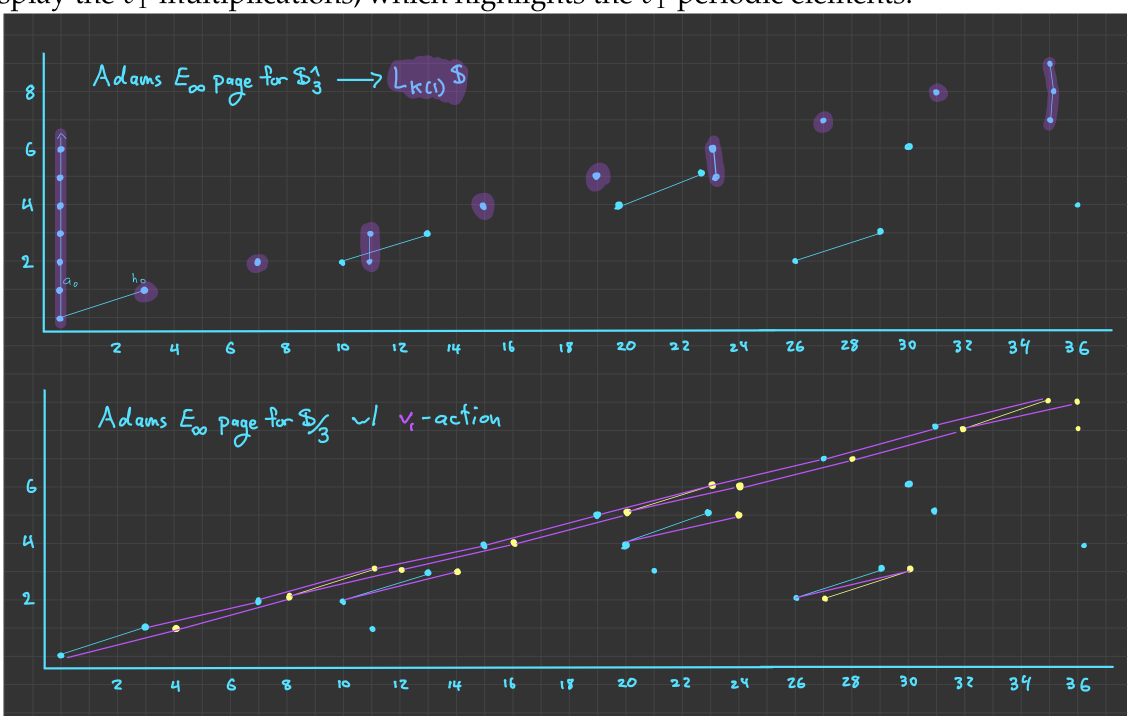
Here are the Adams E_2-pages, which are the same but with d_2-differentials drawn in.



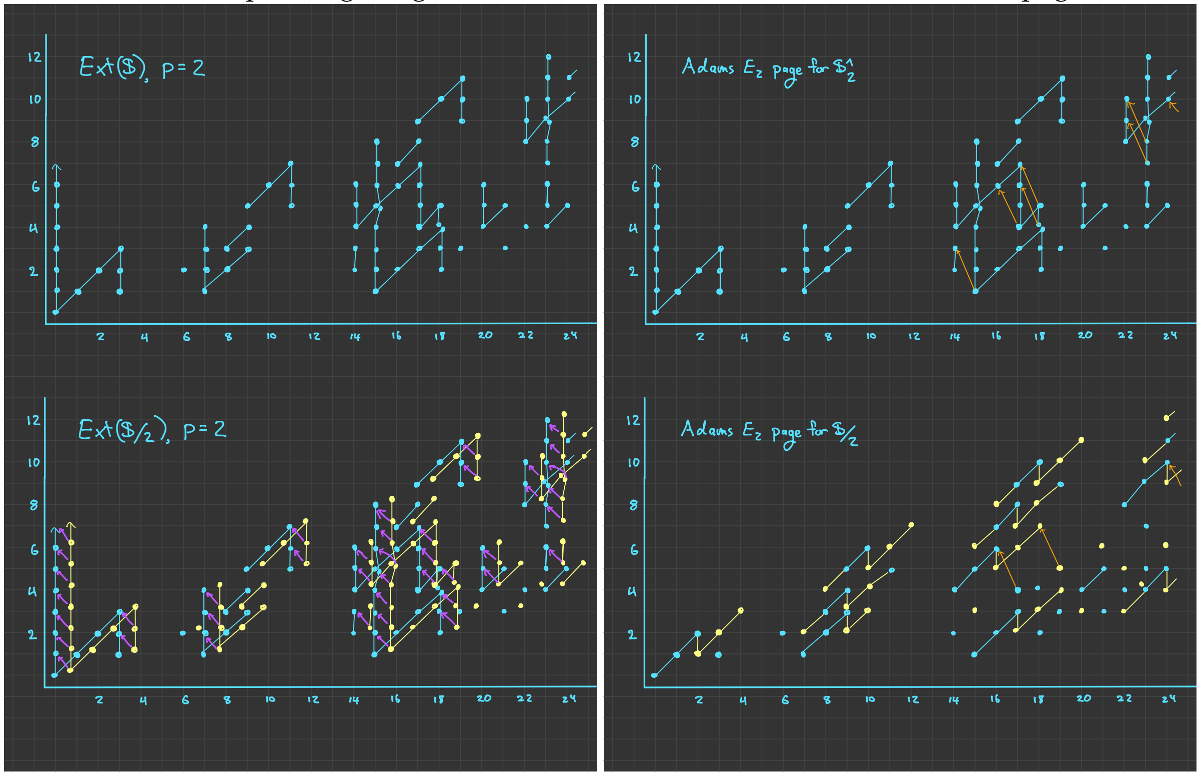
This yields the following E_3-pages. There are only a handful of longer differentials, so we have also drawn those in on these charts.



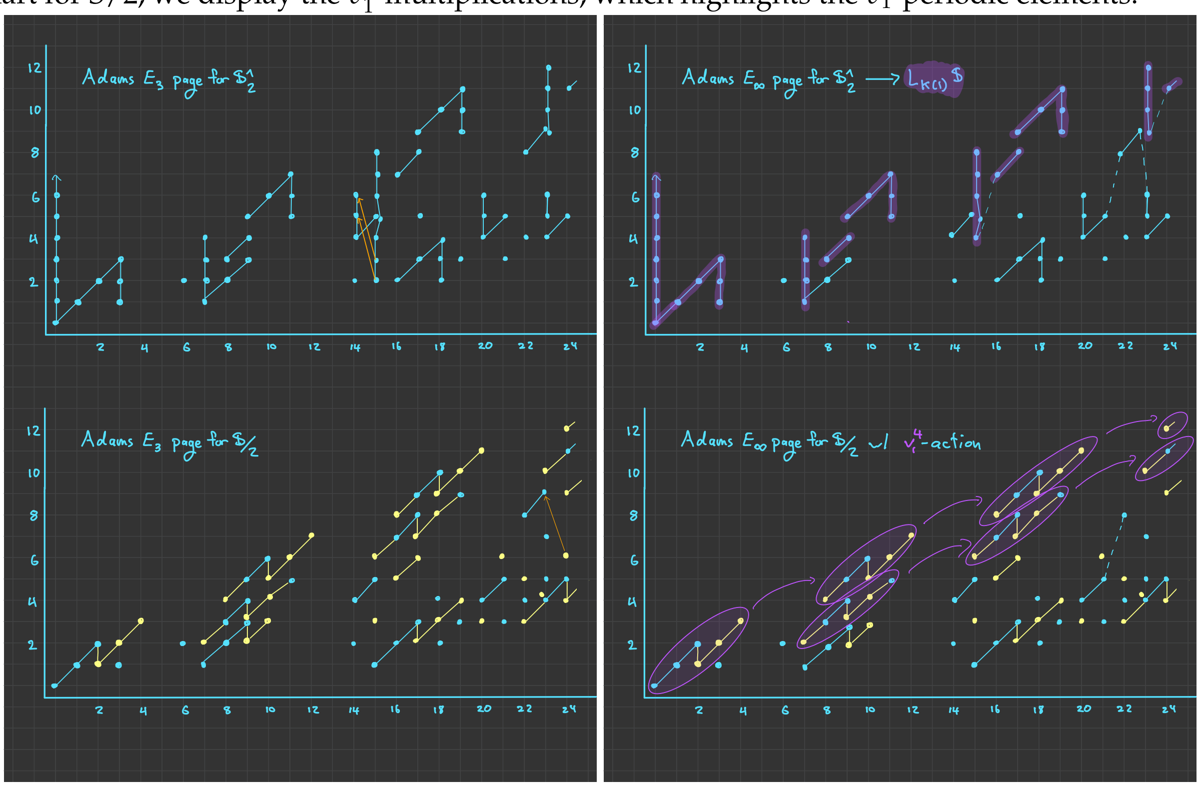
Finally, we display the E_∞-pages here. On the chart for $\mathbb{S}$, we highlight the classes that are detected by $L_1\mathbb{S}$, in other words the classes that map nontrivially to $L_1\mathbb{S}$. On the chart for $\mathbb{S}/3$, we display the v_1-multiplications, which highlights the v_1-periodic elements.



Here are the corresponding images for $\mathbb{S}/2$. First, the Ext charts and Adams E_2-pages:



Next, we display the Adams E_3-page and the E_∞-page. On the chart for $\mathbb{S}$, we highlight the classes that are detected by $L_1\mathbb{S}$, in other words the classes that map nontrivially to $L_1\mathbb{S}$. On the chart for $\mathbb{S}/2$, we display the v_1^4-multiplications, which highlights the v_1-periodic elements.



Wed, Mar. 6

8. MU AND FRIENDS

The complex cobordism spectrum MU plays a central role in chromatic stable homotopy theory. We have delayed discussing it, but no longer!

Definition 8.1. Let $E \longrightarrow B$ be a real vector bundle. If B is paracompact, then we may equip E with a metric, so that in particular each fiber inherits a norm. Then we can consider the **unit sphere bundle** $S(E) \subset E$ defined by taking the unit sphere inside each fiber. Similarly, we have the **unit disk bundle** $D(E) \subset E$ defined by taking the fiberwise unit disk. Then the **Thom space** of the bundle is defined as the quotient space

$$T(E) = D(E)/S(E).$$

Note that if the base space is just a point, then $T(E)$ will be $S^{\dim E}$. More generally, we have

Proposition 8.2. *Denoting by* $\mathbf{n}$ *the trivial bundle of rank n over B, we have a homeomorphism*

$$T(\mathbf{n}) \cong \Sigma^n B_+.$$

More generally, we have

$$T(\mathbf{n} \oplus E) \cong \Sigma^n T(E).$$

So a Thom space can be thought of as a "twisted suspension" of B_+. Indeed, the Thom isomorphism theorem says that, if E is orientable, then cohomology can't tell the difference between $T(E)$ and $\Sigma^n B_+$. We defined Thom spaces for real bundles, but the same works for complex bundles just as well. We will use this construction to define MU.

First recall that the classifying space $BU(n)$ of the unitary group $U(n)$ classifies rank n complex vector bundles. In particular, it comes equipped with the universal rank n bundle γ_n.

Definition 8.3. The spectrum MU is defined by setting $MU_{2n} = T(\gamma_n)$, the Thom space of the bundle over $BU(n)$. The odd spaces are just taken to be the suspensions of the even spaces. It remains to define the structure maps $\Sigma^2 T(\gamma_n) \to T(\gamma_{n+1})$ from the suspension of the odd spaces to the even ones. Consider the inclusion $\iota \colon U(n) \hookrightarrow U(n+1)$ given by $A \mapsto A \oplus \mathrm{id}_{\mathbb{C}}$. In other words, this takes an $n \times n$ unitary matrix A and produces the $(n+1) \times (n+1)$ unitary matrix which is the block sum of A and the 1×1 matrix $1 \in U(1)$. Then passage to classifying spaces induces $B\iota \colon BU(n) \to BU(n+1)$. Then the pullback $\iota^* \gamma_{n+1}$ is $\gamma_n \oplus \mathbf{1}$. In other words, we have a diagram

$$\begin{array}{ccc} E(\gamma_n \oplus \mathbf{1}) & \longrightarrow & E(\gamma_{n+1}) \\ \downarrow & & \downarrow \gamma_{n+1} \\ BU(n) & \xrightarrow{B\iota} & BU(n+1). \end{array}$$

Applying the Thom space construction to this map of complex bundles gives the desired map $\Sigma^2 T\gamma_n \to T\gamma_{n+1}$.

The spectrum MU is an example of the more general notion of a **Thom spectrum**. It is called the complex cobordism spectrum because the cohomology theory it represents is complex cobordism. This was proved by Thom, though the connection to cobordism will not be relevant in this course.

What will be more important is the calculation of $MU_* = \pi_* MU$. This was done independently by Milnor and Novikov. It turns out that the Adams spectral sequence for MU collapses at E_2 to give:

Theorem 8.4 (Milnor & Novikov)**.** *The homotopy groups of MU are given by*

$$MU_* \cong \mathbb{Z}[x_1, x_2, \dots],$$

with $x_n \in \pi_{2n} MU$.

We have described the homotopy of MU as a (graded-)commutative ring. In fact, MU is a connective, commutative h-ring spectrum. The ring structure arises from maps $T\gamma_n \wedge T\gamma_k \to T\gamma_{n+k}$ corresponding to direct sum of bundles.

Even better, MU was one of the first examples of what is known as an $\mathbb{E}_\infty$-ring spectrum. This is equivalent to being a **strict commutative ring spectrum**, i.e. a commutative monoid in the point-set level category of spectra, rather than in the homotopy category.

If we are happy to work p-locally, then the spectrum MU breaks into smaller pieces. Quillen defined an idempotent map of ring spectra $MU_{(p)} \xrightarrow{e} MU_{(p)}$.

Definition 8.5. The **Brown-Peterson spectrum** BP is defined as

$$BP = \mathrm{im}(e) = \mathrm{hocolim}(MU_{(p)} \xrightarrow{e} MU_{(p)} \xrightarrow{e} \dots).$$

It is a commutative h-ring spectrum, with $\pi_* BP \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots]$, where $v_n \in \pi_{2(p^n-1)}$.

As we have seen before, we often write v_0 for $p \in \pi_0 BP \cong \pi_0 MU$.

Warning 8.6. While BP is a commutative h-ring, this is quite different from the harder question of BP being a (strict) commutative ring spectrum, a.k.a. an $\mathbb{E}_\infty$-ring spectrum. The question of whether or not BP can be built as an $\mathbb{E}_\infty$-ring was an open question for decades. In 2010, Basterra and Mandell showed [BM] that BP admits an $\mathbb{E}_4$-multiplication. However, in 2018, Tyler Lawson showed [L] that, at $p = 2$, BP does **not** admit the structure of an $\mathbb{E}_{12}$-ring spectrum. This was extended by Andrew Senger [Sen], who showed that at an odd prime p, BP does **not** admit the structure of an $\mathbb{E}_{2(p^2+2)}$-ring spectrum.

Fri, Mar. 8

Last time, we said that MU_* is the complex bordism ring and also said that this is isomorphic to a polynomial ring $\mathbb{Z}[x_1, x_2, \dots]$. A natural question is whether there are good manifold representatives for the bordism classes x_i.

This becomes easier if we rationalize:

Proposition 8.7. *The ring* $MU_* \otimes \mathbb{Q}$ *is generated by the classes* $[\mathbb{CP}^n]$, $n \geq 1$.

It follows that the classes $[\mathbb{CP}^n] \in MU_{2n}$ are **indecomposable**, meaning that they cannot be written as sums of products of elements of lower degree.

Since MU_* itself is torsion free, the map $MU_* \to MU_* \otimes \mathbb{Q}$ is injective, but it is not the case that the $\mathbb{CP}^n$'s give integral generators. The first few groups are

$$MU_0 \cong \mathbb{Z}\{1\}, \quad MU_2 \cong \mathbb{Z}\{\mathbb{CP}^1\}, \qquad MU_4 \cong \mathbb{Z}\{\mathbb{CP}^2, \mathbb{CP}^1 \times \mathbb{CP}^1\},$$

which implies that we could take x_1 and x_2 to be $\mathbb{CP}^1$ and $\mathbb{CP}^2$. However, $\mathbb{CP}^3$ cannot be taken for x_3: the quotient of MU_6 modulo $(\mathbb{CP}^1)^3$ and $\mathbb{CP}^1 \times \mathbb{CP}^2$ and $\mathbb{CP}^3$ is of order 2.

A (redundant) set of generators is given by the **Milnor hypersurfaces** $H_{n,k} \subset \mathbb{CP}^n \times \mathbb{CP}^k$ of (complex) dimension $n + k - 1$. Each $H_{n,0}$ is $\mathbb{CP}^{n-1}$, so this list includes all of the $\mathbb{CP}^k$'s. It turns out that the element $H_{2,2} + \mathbb{CP}^3$ can be taken for x_3.

Mon, Mar. 18

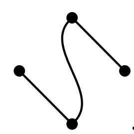
Next, we wish to construct various spectra from BP by taking quotients and performing localizations. Localizations will preserve h-ring structures, but quotients are more subtle.

Here is a simple illustration of how taking quotients can break ring structures in spectra.

Example 8.8. Consider $\mathbb{S}/2$, which is a quotient of the commutative h-ring $\mathbb{S}$. Our intuition from algebra would lead us to expect $\mathbb{S}/2$ inherits a ring structure from that of $\mathbb{S}$, but that is not the case! To see this, consider a potential multiplication $\mathbb{S}/2 \wedge \mathbb{S}/2 \to \mathbb{S}/2$. We would want this to be a unital multiplication, meaning that the composition

$$\mathbb{S}/2 = \mathbb{S}/2 \wedge \mathbb{S} \to \mathbb{S}/2 \wedge \mathbb{S}/2 \to \mathbb{S}/2$$

should be the identity map. However, passage to mod 2 cohomology shows this cannot happen. The smash product $\mathbb{S}/2 \wedge \mathbb{S}/2$ is a complex with 4 cells, and the Kunneth theorem tells us that as an $\mathcal{A}_2$-module, its cohomology looks like



.

But $\mathrm{H}^*(\mathbb{S}/2) =$  is not a retract of this $\mathcal{A}_2$-module.

Said differently, the trouble is that the degree 2 map on $\mathbb{S}/2$ is nonzero, so the $\mathbb{S}$-module action $\mathbb{S}/2 \wedge \mathbb{S} \to \mathbb{S}/2$ does not descend to a multiplication on $\mathbb{S}/2$. We saw previously that not every element of $\pi_*\mathbb{S}/2$ is 2-torsion.

Remark 8.9. It was shown by Oka [O] that $\mathbb{S}/n$ is an h-ring spectrum if and only if n is **not** congruent to 2 modulo 4 or congruent to ± 3 modulo 9. It is furthermore commutative if n is either odd or divisible by 8.

Remark 8.10. Recently, Robert Burklund [Bu] has shown that the Moore spectrum $\mathbb{S}/2^k$ admits an $\mathbb{E}_n$-structure precisely when k is at least $\frac{3}{2}(n+1)$. For example, $\mathbb{S}/8$ is an $\mathbb{E}_1$-ring, also known as an $\mathbb{A}_\infty$-ring spectrum. For p odd, he shows that $\mathbb{S}/p^k$ is $\mathbb{E}_n$ precisely when k is at least $n+1$.

The kinds of results stated in Remark 8.10 are much stronger than what we will need. We will be happy to produce spectra as h-rings.

Ring structures can be inherited on quotients if we mod out by an element that is not a zero-divisor. More generally, a sequence of elements $(x_1, x_2, \dots)$ in R_* is called a **regular sequence** if multiplication by x_n is injective on the quotient $R_*/(x_1, \dots, x_{n-1})$ by the previous generators. Recall that we write R/x for the cofiber of $R \xrightarrow{x} R$. Then we write $R/(x_1, x_2)$ for $(R/x_1)/x_2$, and similarly for more generators.

Proposition 8.11. *If $(x_1, x_2, \dots)$ is a regular sequence in R_*, then*

$$\pi_* \left(R/(x_1, x_2, \dots)\right) \cong R_*/(x_1, x_2, \dots)$$

Proof. We leave this as an exercise, but the main point is that the regular sequence assumption means that the long exact sequences in homotopy for the cofibers turn into short exact sequences. ∎

Theorem 8.12 ([St]). *Let R be an $\mathbb{E}_\infty$-ring spectrum such that π_*R is concentrated in even degrees. If A_* is the quotient of a localization of R_* by a regular ideal and if 2 is invertible in A_*, then there exists a commuative R-h-ring spectrum A with $\pi_*A = A_*$.*

Wed, Mar. 18

As Strickland's theorem requires 2 to be invertible, we will suppose for the next few examples that p **is an odd prime**.

Example 8.13 (Brown-Peterson spectrum)**.** Recall that we described the Brown-Peterson spectrum BP as a summand of the localization $MU_{(p)}$ of the $\mathbb{E}_\infty$ ring MU. This spectrum can also be produced by Strickland's theorem, if one shows that the kernel of $MU_* \to BP_*$ is generated by a regular sequence.

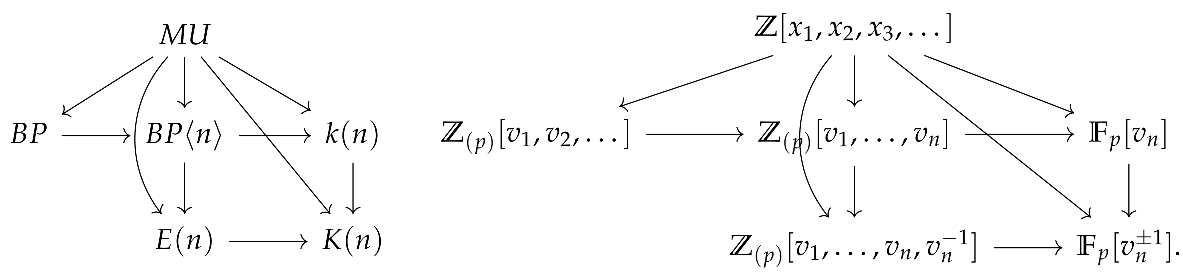
Example 8.14 (Truncated Brown-Peterson spectra)**.** Strickland's theorem can similarly be used to produce a commuative h-ring spectrum $BP\langle n\rangle$ with $BP\langle n\rangle_* \cong \mathbb{Z}_{(p)}[v_1, \dots, v_n]$. For example, $BP\langle 1\rangle$ is the connective Adams summand ℓ.

Example 8.15 (Johnson-Wilson spectra)**.** If we invert v_n on the truncated Brown-Peterson spectra, we get the **Johnson-Wilson** spectra $E(n)$ with $E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_n, v_n^{-1}]$. For example, we have already seen $E(1)$, the Adams summand L.

Example 8.16 (Connective Morava K-theory)**.** If we mod out by all of the v_i's save one, we get the **connective Morava** K**-theory** spectrum $k(n)$ with $k(n)_* = \mathbb{F}_p[v_n]$. For example, $k(1)$ is the mod p connective Adams summand.

Example 8.17 (Morava K-theory)**.** If we invert v_n on $k(n)$, we get the **Morava** K**-theory** spectrum $K(n)$ with $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$. For example, $K(1)$ is the mod p Adams summand.

These spectra all admit a ring map from MU and are related by ring maps



The story is a little more complicated at $p = 2$. Strickland shows that there is a slightly different quotient ring $BP\langle n\rangle'_*$ of BP_* that is isomorphic to $\mathbb{Z}_{(p)}[v_1, \dots, v_n]$ and that can be realized as the homotopy of a commutative h-ring spectrum, and the same goes for the localization $E(n)'$. However, the Morava K-theory spectra $k(n)$ and $K(n)$ are not commutative at the prime 2. On the other hand, they can be realized as $\mathbb{E}_1 = \mathbb{A}_\infty$ ring spectra.

One of the important properties of Morava K-theory is that it is a **graded field**, in the sense that every module over $K(n)_*$ is automatically free. Thus working with $K(n)$-homology is essentially doing linear algebra. One consequence of $K(n)$ being a graded field is the following.

Proposition 8.18 (Kunneth theorem). *Morava K-theory satisfies a Kunneth theorem, in the sense that there is a natural isomorphism*

$$K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y) \cong K(n)_*(X \wedge Y).$$

Recall that in the course of the proof of Proposition 5.3, we showed that Bousfield localization at $E(1)$ is the same as Bousfield localization at $K(1) \vee H\mathbb{Q}$. Since $K(0)$ is $H\mathbb{Q}$, this can be rewritten as the equality

$$\langle E(1)\rangle = \langle K(1) \vee K(0)\rangle.$$

Ravenel showed that this generalizes to the higher Johnson-Wilson spectra $E(n)$.

Theorem 8.19 ([Ra, Theorem 2.1]). *We have equality of Bousfield classes*

$$\langle E(n) \rangle = \langle K(0) \vee \dots K(n) \rangle.$$

First we need a few preliminaries.

Definition 8.20. Let E_* be an MU_*-module. We say that E_* is **Landweber exact** if the sequence $(p, v_1, v_2, \dots)$ is a regular sequence in E_* for each prime p. If E is a spectrum equipped with an MU-module structure (in the homotopy category), then we say that E is **Landweber exact** if E_* is so in the sense above.

Theorem 8.21 (Landweber Exact Functor Theorem). *If E_* is Landweber exact, then the functor $X \mapsto MU_*(X) \otimes_{MU_*} E_*$ is a homology theory. If E is a Landweber exact spectrum, then we have an isomorphsm of homology theories $E_*(X) \cong MU_*(X) \otimes_{MU_*} E_*$.*

Fri, Mar. 22

For us, one reason to pay attention to Landweber exact theories is that their Bousfield classes are determined by their "height", which we now define.

Definition 8.22. Let E be a p-local Landweber exact theory. Then we say that E **has height** n if $E_*/(p, v_1, v_2, \dots, v_{n-1})$ is nonzero, but $E_*/(p, v_1, v_2, \dots, v_n)$ vanishes.

Example 8.23. The spectrum $K(0) = H\mathbb{Q}$ is Landweber exact of height 0.

Example 8.24. The spectrum $KU_{(p)}$ is Landweber exact of height 1, and so is the Adams summand $E(1)$. On the other hand, $K(1) = E(1)/p$ is **not** Landweber exact, since multiplication by $v_0 = p$ is not injective.

Example 8.25. The Johnson-Wilson spectrum $E(n)$ is Landweber exact of height n. On the other hand, the Morava K-theory spectra $k(n)$ and $K(n)$ are **not** Landweber exact.

We will use the following in the proof of Theorem 8.19.

Theorem 8.26 ([HoSt]). *If D is p-local and Landweber exact of height n, then $\langle D \rangle = \langle E(n) \rangle$.*

Example 8.27. Let E be Landweber exact and nontrivial. Then $E[p^{-1}]$ is Landweber exact of height 0, so $\langle E[p^{-1}] \rangle = \langle H\mathbb{Q} \rangle$. This confirms what we said in Lemma 5.1, which did not rely on the Landweber exact hypothesis.

Example 8.28. The spectrum $E(2)$ is Landweber exact, and so is its localization $E(2)[v_1^{-1}]$. This is because multiplication by $p = v_0$ and v_1 are both injective, while the quotient by v_1 vanishes. Thus $E(2)[v_1^{-1}]$ is height 1. By Theorem 8.26, it follows that $\langle E(2)[v_1^{-1}] \rangle = \langle E(1) \rangle$.

Sketch of Theorem 8.19. First, we show that

$$\langle E(n) \rangle \geq \langle K(0) \vee \dots \vee K(n) \rangle.$$

It suffices to show that $\langle E(n) \rangle \geq \langle K(\ell) \rangle$ for each $\ell \leq n$. Since $v_\ell^{-1}E(n)$ is a homotopy colimit of suspensions of $E(n)$, it follows that $\langle E(n) \rangle \geq \langle v_\ell^{-1}E(n) \rangle$, but the latter is the same as $\langle E(\ell) \rangle$ by Theorem 8.26. Since $K(\ell)$ can be formed from $E(\ell)$ by iterated cofibers (quotients), it follows that $\langle E(\ell) \rangle \geq \langle K(\ell) \rangle$.

In the other direction, we wish to show that

$$\langle K(0) \vee \dots \vee K(n) \rangle \geq \langle E(n) \rangle.$$

By induction on n, we will suppose that we already know that $\langle E(\ell) \rangle$ is equal to $\langle K(0) \vee \dots \vee K(\ell) \rangle$ for ℓ strictly less than n. For simplicity, we specialize to the case $n = 2$, though essentially the same argument works in general.

Thus suppose that X is $K(0)$-acyclic and $K(1)$-acyclic and $K(2)$-acyclic. By the induction hypothesis, we conclude that X is $E(1)$-acyclic. We wish to conclude that X is $E(2)$-acyclic.

First, we will show that X is $E(2)/v_0$-acyclic. Since we have assumed that X is $K(2)$-acyclic and since $K(2)$ is the quotient $E(2)/(v_0, v_1)$, it follows that multiplication by v_1 is an equivalence on the $E(2)/v_0$-homology of X. It follows that we get an isomorphism

$$(E(2)/v_0)_*(X) \cong (v_1^{-1}E(2)/v_0)_*(X).$$

On the other hand, as we indicated above, the induction hypothesis tells us that X is $E(1)$-acyclic. By Theorem 8.26, this means that X is $v_1^{-1}E(2)$-acyclic, and it follows that X is $v_1^{-1}E(2)/v_0$-acyclic. We then conclude, using the displayed isomorphism, that X is $E(2)/v_0$-acyclic.

But this means that the $E(2)$-homology of X is v_0-periodic. Thus the $E(2)$-homology of X is the $v_0^{-1}E(2)$-homology of X. But $v_0^{-1}E(2)$ is rational, and we have assumed that X is $K(0) = H\mathbb{Q}$-acyclic. We conclude that X is $E(2)$-acyclic, as desired. ■

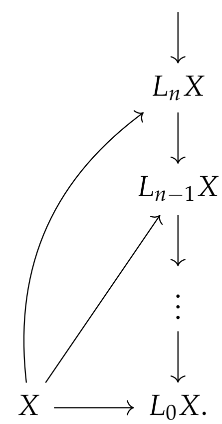
Wed, Mar. 27

9. THE CHROMATIC FILTRATION

The chromatic tower

As we argued in the proof of Theorem 8.19, we have the containment of Bousfield classes $\langle E(n)\rangle \geq \langle E(n-1)\rangle$. Recall from Problem 2 of Worksheet 6 that this means that we have a natural transformation $L_n \to L_{n-1}$ of localization functors (recall that we write L_n as shorthand for $L_{E(n)}$). We can assemble these transformations together to form the **chromatic tower** for a p-local spectrum. This is displayed to the right.

(9.1)



There are several questions we might ask of this chromatic tower. First, recall that we saw in Proposition 5.3 that L_1X can be recovered using the chromatic fracture square, starting from rational information as well as the localization at $K(1)$. Our first result here will generalize this to higher heights.

Proposition 9.1 (Chromatic Fracture). *For any X, there is a homotopy pullback square*

$$\begin{array}{ccc} L_{E(n)}X & \longrightarrow & L_{E(n-1)}X \\ \downarrow & & \downarrow \\ L_{K(n)}X & \longrightarrow & L_{E(n-1)}L_{K(n)}X \end{array} \qquad \text{or, more concisely,} \qquad \begin{array}{ccc} L_nX & \longrightarrow & L_{n-1}X \\ \downarrow & & \downarrow \\ \widehat{L}_nX & \longrightarrow & L_{n-1}(\widehat{L}_nX). \end{array}$$

The proof of this will be essentially the same as that for Proposition 5.3, but it will use one crucial ingredient. In comparing Proposition 5.3 and Proposition 9.1, one difference is that in the case $n = 1$, the localization L_0 is a smashing localization. It turns out that this is true in higher heights as well.

Theorem 9.2 (Hopkins-Ravenel,[Ra2, Theorem 7.5.6]). *The localization L_n is a smashing localization. In other words,*

$$X \cong \mathbb{S} \wedge X \xrightarrow{\eta\wedge \mathrm{id}} L_n\mathbb{S} \wedge X$$

is an $E(n)$-localization of X.

This is known as the **Smash Product Theorem**, and it was one of the original conjectures in [Ra].

Proof of Proposition 9.1. Just as in Proposition 5.3, we employ Proposition 4.13, with $D = K(n)$ and $E = E(n-1)$. We are entitled to use Proposition 4.13 because if X is $K(n)$-acyclic, then $L_{n-1}X \cong L_{n-1}\mathbb{S} \wedge X$ is still $K(n)$-acyclic. We have here relied on Theorem 9.2.

The other part of the proof of Proposition 5.3 was the verification that $E(1)$-localization agreed with localization at $K(1) \vee H\mathbb{Q}$. In the general case, we know that $E(n)$-localization is localization at $K(n) \vee E(n-1)$ according to Theorem 8.19. ■

Because of the Smash Product Theorem Theorem 9.2, we may rewrite the chromatic fracture square as

$$\begin{array}{ccc} L_n\mathbb{S} \wedge X & \xrightarrow{p_n \wedge \mathrm{id}} & L_{n-1}\mathbb{S} \wedge X \\ \downarrow & & \downarrow {\scriptstyle \mathrm{id}\wedge\eta} \\ \widehat{L}_n X = \mathbb{S} \wedge \widehat{L}_n X & \xrightarrow{\eta \wedge \mathrm{id}} & L_{n-1}\mathbb{S} \wedge \widehat{L}_n X \end{array}$$

Thus the Smash Product Theorem places primacy on the chromatic tower for $\mathbb{S}$ in understanding any chromatic tower. Furthermore, the localizations $\widehat{L}_n X$ may be viewed as the key gluing information that measures the difference between $L_n X$ and $L_{n-1}X$. Thus

understanding the $K(n)$-local sphere $\widehat{L}_n\mathbb{S} = L_{K(n)}\mathbb{S}$ is one of the central questions of chromatic homotopy theory.

Recall that in the case $n = 1$, we identified $L_{K(1)}\mathbb{S}$ with the fiber of $E(1)^\wedge_p \xrightarrow{\psi^{p+1}-\mathrm{id}} E(1)^\wedge_p$ in Proposition 5.6, at least for p odd. An alternative perspective is that this describes $L_{K(1)}\mathbb{S}$ as the homotopy fixed points of an action of a certain profinite group on $E(1)^\wedge_p$. A related, but perhaps simpler, statement is that there is an action of the group $\mathbb{Z}_p^\times$ of p-adic units on $KU^\wedge_p$, and Proposition 5.6 amounts to an equivalence

$$L_{K(1)}\mathbb{S} \simeq (KU^\wedge_p)^{h\mathbb{Z}_p^\times}.$$

This generalizes as follows.

Theorem 9.3 (Goerss-Hopkins-Miller, Devinatz-Hopkins). *There are $\mathbb{E}_\infty$-ring spectra E_n, known as* **Morava E-theory** *or* **completed Johnson-Wilson spectra** *or* **Lubin-Tate spectra**, *together with an action of a profinite group $\mathbb{G}_n$, known as the* **Morava stabilizer group** *on E_n. The homotopy fixed points model the $K(n)$-local sphere; in other words,*

$$L_{K(n)}\mathbb{S} \simeq E_n^{h\mathbb{G}_n}.$$

In theory, one could run a homotopy fixed point spectral sequence to then compute $L_{K(n)}$, but this is not practical in general. In recent years, progress has been made using the **finite subgroups** H of $\mathbb{G}_n$. The homotopy fixed points E_n^{hH} then give an *approximation* to $L_{K(n)}\mathbb{S}$.

Fri, Mar. 29

Going back to the chromatic tower (9.1), another reasonable question is: what is the inverse limit? The **Chromatic Convergence Theorem** answers this:

Theorem 9.4 ([Ra2], Theorem 7.5.7). *Let X be a finite p-local spectrum. Then the natural map*

$$X \longrightarrow \operatorname*{holim}_n L_n X$$

is an equivalence.

Like Proposition 9.1, the proof of Theorem 9.4 given in [Ra2] depends on the Smash Product Theorem, Theorem 9.2.

Mon, Apr. 1

The Chromatic Convergence Theorem tells us that the spectra $L_n\mathbb{S}$ give a better approximation to $\mathbb{S}_{(p)}$ as n increases. Thus, as n increases, the difference in homotopy between $\mathbb{S}_{(p)}$ and $L_n\mathbb{S}$ decreases. This leads to the chromatic filtration on $\pi_*\mathbb{S}$.

Definition 9.5. The **chromatic filtration** on $\pi_*\mathbb{S}_{(p)}$ is the descending filtration given by

$$F^n = \ker\left(\pi_*\mathbb{S}_{(p)} \longrightarrow \pi_* L_{n-1}\mathbb{S}\right).$$

Since the map $\mathbb{S}_{(p)} \longrightarrow L_{n-1}\mathbb{S}$ factors through $L_n\mathbb{S}$, it follows that F^{n+1} is contained in F^n.

Example 9.6. Since $L_0\mathbb{S}$ is $H\mathbb{Q}$, the first stage F^1 consists precisely of the torsion in $\mathbb{S}_{(p)}$, which is all elements in positive degrees.

Example 9.7. We considered the maps $\mathbb{S}_{(3)} \to L_1\mathbb{S}$ and $\mathbb{S}_{(2)} \to L_1\mathbb{S}$ in Section 7. The elements of $\pi_*\mathbb{S}_{(2)}$ or $\pi_*\mathbb{S}_{(3)}$ that went to nonzero elements of $\pi_* L_1\mathbb{S}$ were all near the "vanishing line", in high Adams filtration. Thus elements of F^2 are detected in lower Adams filtration.

The general picture is that each F^n will be detected in bands that are further and further away from the vanishing line. Note that F^0 is all of $\pi_*\mathbb{S}$. At the other end, we might wonder about $F^\infty = \cap_n F^n$. Any element here would be an element of $\pi_*\mathbb{S}$ that is not detected in any $L_n\mathbb{S}$. It is a consequence of Theorem 9.4 (Chromatic Convergence) that there are no such nonzero elements. To see why, we need to consider $\pi_* \operatorname{holim}_n L_n\mathbb{S}$.

In general, if $\to E_n \to E_{n-1} \to \dots$ is an inverse system of spectra, it is not the case that the homotopy of the homotopy limit agrees with the inverse limit of the homotopy groups. This corresponds to the failure of inverse limits to be exact.

Let $\to A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \dots$ be an inverse system of abelian groups. Then we can represent the inverse limit of the A_n's as

$$\lim_n A_n = \ker\left(\prod_n A_n \xrightarrow{D} \prod_n A_n\right)$$

where the difference map D is $D(a_*)_n = a_n - f_{n+1}(a_{n+1})$. Unfortunately, the map D is not in general surjective. We define

$$\lim_n{}^1 A_n = \operatorname{coker}\left(\prod_n A_n \xrightarrow{D} \prod_n A_n\right).$$

In other words, we have an exact sequence

$$0 \to \lim A_n \to \prod_n A_n \xrightarrow{D} \prod_n A_n \to \lim_n{}^1 A_n \to 0.$$

You will be asked to show the following on this week's worksheet:

Proposition 9.8. *Let $\dots \xrightarrow{f_{n+1}} E_n \xrightarrow{f_n} E_{n-1} \xrightarrow{f_{n-1}} \dots$ be an inverse system of spectra. Then we have an exact sequence*

$$0 \to \lim{}^1 \pi_{k+1} E_n \longrightarrow \pi_k\left(\operatorname{holim} E_n\right) \to \lim \pi_k E_n \to 0.$$

However, there are conditions under which $\lim^1$ is guaranteed to vanish.

Proposition 9.9. *Suppose that $\to A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \dots$ is an inverse system such that EITHER*

(1) *each A_n is finite OR*
(2) *each f_n is surjective.*

Then $\lim_n{}^1 A_n$ *vanishes.*

Theorem 9.10. *For each* $k \geq 0$*, we have*

$$\pi_k \mathbb{S}_{(p)} \cong \lim_n \pi_k L_n \mathbb{S}.$$

Proof. Combining Chromatic Convergence (Theorem 9.4) with Proposition 9.8 gives a short exact sequence

$$0 \to \lim_n{}^1 \pi_{k+1} L_n \mathbb{S} \to \pi_k \mathbb{S}_{(p)} \to \lim_n \pi_k L_n \mathbb{S} \to 0.$$

It remains to show that $\lim_n{}^1 \pi_{k+1} L_n$ vanishes. By Proposition 9.9, it suffices to show that $\pi_{k+1} L_n$ is finite, for $k \geq 0$. In other words, it suffices to show that the rationalization of $L_n \mathbb{S}$ has no homotopy in positive degrees. But $\langle E(n) \rangle \geq \langle E(0) \rangle = \langle H\mathbb{Q} \rangle$, as we have seen previously. It follows that $(L_n \mathbb{S})_{\mathbb{Q}} \simeq \mathbb{S}_{\mathbb{Q}} \simeq H\mathbb{Q}$. In particular, the homotopy groups of $L_n \mathbb{S}$ in positive degrees are all finite. ■

Remark 9.11. We used in the proof above that $L_n \mathbb{S}$ is rationally equivalent to $\mathbb{S}$. One might also ask about the rationalization of $\widehat{L_n}\mathbb{S}$. This is described by a major recent result of Barthel-Schlank-Stapleton-Weinstein [BSSW]:

$$\pi_* (\widehat{L_n}\mathbb{S})_{\mathbb{Q}} \cong E_{\mathbb{Q}}(\zeta_1, \ldots, \zeta_n),$$

with ζ_j in degree $1 - 2j$. Note that this agrees with what we found in the case $n = 1$ in Proposition 5.12 and Proposition 5.8.

Wed, Apr. 3

10. Nilpotence, periodicity, and thick subcategories

We have already encountered some of Ravenel's conjectures, including the (height one) Telescope conjecture Theorem 5.18 and the Smash Product Theorem (Theorem 9.2). As we indicated previously, the Smash Product Theorem is a key input for the proof of the Chromatic Convergence Theorem (Theorem 9.4).

Next, we discuss the Nilpotence Theorem, proved by Devinatz, Hopkins, and Smith. This was among the original Ravenel conjectures [Ra], and it is the backbone behind many of the other results. The historical precursor of the Nilpotence Theorem is the nilpotence theorem of Nishida:

Theorem 10.1. [N] *In the graded ring* $\pi_* \mathbb{S}$*, every element in positive degrees is nilpotent.*

We can think about this statement as talking about multiplying elements in the graded ring $\pi_* \mathbb{S}$, or equivalently as iterating self-maps of the sphere.

Theorem 10.2 (Nilpotence, [DHS])**.**

(1) *(Ring spectrum form) Let* R *be a h-ring spectrum Then the kernel of the MU-Hurewicz map*

$$\pi_* R \to MU_*(R)$$

consists of nilpotent elements.

(2) *(Finite spectrum form) Let* $f \colon \Sigma^d X \to X$ *be a self-map of a finite spectrum* X*. Then* f *is nilpotent if and only if the induced map*

$$MU_*(\Sigma^d X) \xrightarrow{f_*} MU_*(X)$$

is nilpotent (under iteration).

Proof that ring form implies finite form. Let $f\colon \Sigma^d X \to X$ be a self-map of a finite spectrum X and suppose that $MU_*(f)$ is nilpotent. We wish to see that f is nilpotent. Take any power k such that $MU_*(f^k)$ is zero.

Now consider the h-ring spectrum $R = F(X, X)$, namely the endomorphism object of X. Since X is a finite spectrum, and therefore a dualizable object, we have an equivalence of spectra $R \simeq DX \wedge X$. Now the self-map f of X corresponds, by adjointness, to a map $\hat{f}\colon \mathbb{S}^d \to R$. Since multiplication in R corresponds to composition of endomorphisms, it follows that $(\hat{f})^k$ is $\widehat{(f^k)}$. Now the factorization of $\widehat{(f^k)}$ on the left implies a similar factorization of $MU \wedge \widehat{(f^k)}$ on the right.

$$\begin{array}{ccc} \mathbb{S}^{dk} & \xrightarrow{\eta} & \Sigma^{dk}R \simeq F(X, \Sigma^{dk}X) \\ & \searrow^{\widehat{f^k}} & \downarrow F(\mathrm{id}, f^k) \\ & & R \end{array} \qquad \begin{array}{ccccc} MU \wedge \mathbb{S}^{dk} & \xrightarrow{\mathrm{id}\wedge\eta} & MU \wedge \Sigma^{dk}R & \xrightarrow{\simeq} & MU \wedge DX \wedge \Sigma^{dk}X \\ & \searrow^{\mathrm{id}\wedge\widehat{f^k}} & \downarrow \mathrm{id}\wedge F(\mathrm{id}, f^k) & & \downarrow \mathrm{id}\wedge\mathrm{id}\wedge f^k \\ & & MU \wedge R & \xrightarrow{\simeq} & MU \wedge DX \wedge X \end{array}$$

By assumption, the right vertical map is zero, so we conclude that $\widehat{f^k}$ is in the kernel of the Hurewicz map $\pi_{dk}R \to MU_{dk}R$. By the ring form of the Nilpotence theorem, it follows that $\widehat{f^k}$ is nilpotent in π_*R, which in turn implies that f^k is nilpotent as an endomorphism of X. ■

Corollary 10.3. *Nishida's nilpotence theorem for $\pi_*\mathbb{S}$ follows, since all elements in positive stems are torsion and are therefore necessarily in the kernel of the map $\pi_*\mathbb{S} \to MU_*$.*

This is a significant theoretical advance, saying that the functor $MU_*(-)$ from finite spectra to MU_*-modules is close to being faithful. Practically speaking, we prefer to work p-locally, so we may as well replace MU with BP. And in the p-local context, the easiest homology theories to work with are the Morava K-theories. There is a form of the Nilpotence theorem, in the p-local context, in terms of the $K(n)$'s. We will write $K(\infty) = H\mathbb{F}_p$.

Theorem 10.4 (p-local Nilpotence, [HS])**.**

(1) *(Ring spectrum form) Let R be a p-local h-ring spectrum. Then $\alpha \in \pi_*R$ is nilpotent if and only $h_{K(n)}\alpha$ is nilpotent in $K(n)_*R$ for each $0 \le n \le \infty$, where $h_{K(n)}\colon \pi_*R \to K(n)_*R$ is the $K(n)$-Hurewicz homomorphism.*

(2) *(Finite spectrum form) Let $f\colon \Sigma^d X \to X$ be a self-map of a p-local finite spectrum X. Then f is nilpotent if and only if the induced map*

$$K(n)_*(\Sigma^d X) \xrightarrow{f_*} K(n)_*(X)$$

is nilpotent (under iteration) for each $0 \le n \le \infty$.

We now largely turn our attention to p-local finite spectra. We start by introducing the chromatic "type" of such an object.

Definition 10.5. Let X be a p-local finite spectrum. We say that X has (chromatic) **type** $\ge n$ if $E(n-1)_*X = 0$. As expected, X has type exactly n if it has type $\ge n$ but not $\ge n+1$. By convention, every X has type ≥ 0.

In other words, X has type $\ge n$ if it is $E(n-1)$-acyclic, or if $L_{n-1}X \simeq *$.

Example 10.6. The p-local sphere $\mathbb{S}_{(p)}$ has type 0, since $L_0\mathbb{S}_{(p)} \simeq H_{\mathbb{Q}}$ is nontrivial.

Example 10.7. The Moore spectrum $\mathbb{S}/p$ has type 1, since it is rationally acyclic, but we saw earlier that $L_1\mathbb{S}/p$ is nontrivial.

Mon, Apr. 8 – Eclipse Day!

Notation 10.8. Let us denote by $\mathrm{Ho}\mathbf{Sp}^{\mathrm{fin}}_{(p)}$ the homotopy category of p-local finite spectra, and by $\mathrm{Ho}\mathbf{Sp}^{\mathrm{fin}}_{\geq n} \subset \mathrm{Ho}\mathbf{Sp}^{\mathrm{fin}}_{(p)}$ the full subcategory of finite p-local spectra of type $\geq n$.

It will be useful to have another characterization of being type $\geq n$.

Proposition 10.9 ([Ra, Theorem 2.11])**.** *Let X be a finite complex. If X is $K(n)$-acyclic, then it is also $K(n-1)$-acyclic.*

It follows that if a finite X is $K(n)$-acyclic, then it is $K(j)$-acyclic for all $j \leq n$. By Theorem 8.19, it follows that *for finite* X, if X is $K(n)$-acyclic, then it is also $E(n)$-acyclic. Thus X is type $\geq n+1$ if and only if $K(n)_*(X)$ vanishes, which is a more practical characterization than the one in Definition 10.5.

Proof. Consider the ring spectrum R constructed, using Theorem 8.12, as the quotient $R = E(n)/(p, v_1, \dots, v_{n-2})$. Then $R_* \cong \mathbb{F}_p[v_{n-1}, v_n^{\pm 1}]$. This is a graded PID, so that finitely generated modules over it are necessarily sums of cyclic modules. This applies to $R_*(X)$, since X is finite. Thus suppose that $R_*(X)$ is a sum of cyclic R_*-modules, each of which is either free or of the form R/v_{n-1}^k for some k.

The cofiber sequence $\Sigma^d R \xrightarrow{v_{n-1}} R \to K(n)$ gives a long exact sequence relating $K(n)_*X$ to R_*X. If we assume that X is $K(n)$-acyclic, it follows that v_{n-1}-multiplication is an isomorphism on R_*X. But this implies that $R_*(X)$ also vanishes.

It then follows that the localization $v_{n-1}^{-1}R_*(X)$ vanishes. But this turns out to have the same rank as $K(n-1)_*(X)$. The idea is to compare both to $T_*(X)$, where $T_* = \mathbb{F}_p[v_{n-1}^{\pm 1}, v_n]$, and to see that both comparisons give the same rank. The point is to show that for a finite X, T_*X cannot have nontrivial v_n-torsion. See [JW, Theorem 3.1] for more details. ■

The category $\mathrm{Ho}\mathbf{Sp}^{\mathrm{fin}}_{\geq n}$ is an example of what is known as a "thick" subcategory.

Definition 10.10. A full subcategory $\mathcal{C} \subset \mathrm{Ho}\mathbf{Sp}^{\mathrm{fin}}_{(p)}$ is said to be **thick** if

(1) Whenever $X \to Y \to Z$ is a cofiber sequence and two of X, Y, and Z are in $\mathcal{C}$, then so is the third, and
(2) When X is a retract of Y and Y is in $\mathcal{C}$, then so is X.

In other words, thick subcategories are closed under cofiber sequences and retracts. A remarkable theorem of Hopkins and Smith says that the subcategories $\mathrm{Ho}\mathbf{Sp}^{\mathrm{fin}}_{\geq n}$ are the **only** thick subcategories of $\mathrm{Ho}\mathbf{Sp}^{\mathrm{fin}}_{(p)}$.

Theorem 10.11 ([HS], The Thick Subcategory Theorem)**.** *Let $C \subset \mathrm{Ho}\mathbf{Sp}^{\mathrm{fin}}_{(p)}$ be a nonzero thick subcategory. Then C is the subcategory $\mathrm{Ho}\mathbf{Sp}^{\mathrm{fin}}_{\geq n}$ for some $n \geq 0$.*

Wed, Apr. 10

We will deduce the Thick Subcategory Theorem from a different variant of the Nilpotence, the "Smash Product" form. Rather, we will use the following corollary of the Smash Product form of nilpotence.

Proposition 10.12. [Ra, Corollary 5.1.5] *Let W, X, and Y be p-local finite spectra, and suppose given $f\colon X \to Y$. Then $\mathrm{id} \wedge f^{\wedge k}\colon W \wedge X^{\wedge k} \to W \wedge Y^{\wedge k}$ is null, for large enough k, if $\mathrm{id} \wedge f\colon W \wedge X \to W \wedge Y$ induces the zero map on $K(n)_*$ for all $0 \leq n \leq \infty$.*

Proof of Theorem 10.11. Let $C \subset \mathrm{Ho}\mathbf{Sp}^{\mathrm{fin}}_{(p)}$ be a nonzero thick subcategory, and let n be *smallest* such that C contains a (non-contractible) finite complex of type (exactly) n. In other words, every finite complex in C has type at least n, so that

$$C \subset \mathrm{Ho}\mathbf{Sp}^{\mathrm{fin}}_{\geq n},$$

and it remains to show the other containment.

Thus let Y be a p-local finite complex of type at least n. By assumption, C contains some finite complex X of height exactly n. The idea is then to express Y as a retract of something built from X.

One point is that for *any* finite F, we can build $X \wedge F$ from X using finitely many cofiber sequences, and so it follows that $X \wedge F$ lies in C, since C is thick. Define J to be the fiber $J \xrightarrow{f} \mathbb{S} \to X \wedge DX$ of the unit map for the h-ring spectrum $X \wedge DX$. This expresses $X \wedge DX$ as the cofiber of $f\colon J \to \mathbb{S}$. Similarly, we have a cofiber sequence

$$J \wedge Y \xrightarrow{f \wedge \mathrm{id}} \mathbb{S} \wedge Y \to X \wedge DX \wedge Y. \tag{10.1}$$

Note that $X \wedge DX \wedge Y$ lies in C since $DX \wedge Y$ is finite. We will soon also consider the cofiber sequence

$$J^{\wedge k} \wedge Y \xrightarrow{f^{\wedge k} \wedge \mathrm{id}} \mathbb{S} \wedge Y \to C_{f^{\wedge k} \wedge \mathrm{id}}$$

for large k. An induction argument shows that the cofiber $C_{f^{\wedge k} \wedge \mathrm{id}}$ is again in C.

Let us consider the effect of $K(j)_*$ on (10.1). There are two cases. First, suppose that $K(j)_*(Y) = 0$. This happens, for example, if j is strictly less than n, but it could also happen in higher heights if the height of Y is above n. Then $K(j)_*$ vanishes on the whole sequence, by the Kunneth formula. Second, if $K_*(Y)$ is nonzero, so that j is at least n, then we know that $K(j)_*(X)$ is nonzero by Proposition 10.9, and the same is true of $K(j)_*(DX)$, the $K(j)_*$-linear dual of $K(j)_*X$. Furthermore, the map on $K(j)_*$ induced by the unit map $\mathbb{S} \to X \wedge DX$ is nonzero, and therefore a monomorphism. The same is true after tensoring with $K(j)_*(Y)$. We conclude that $K(j)_*(J \wedge Y) \xrightarrow{(f \wedge \mathrm{id})_*} K(j)_*(\mathbb{S} \wedge Y)$ is the zero map.

By Proposition 10.12, it follows that for some large enough k, then $J^{\wedge k} \wedge Y \xrightarrow{f^{\wedge k} \wedge \mathrm{id}} \mathbb{S} \wedge Y \cong Y$ is null. Thus its cofiber splits as $C_{f^{\wedge k} \wedge \mathrm{id}} \simeq Y \vee \Sigma^1(J^{\wedge k} \wedge Y)$. Since Y is a retract of $C_{f^{\wedge k} \wedge \mathrm{id}}$, it follows that Y is in C. ■

Remark 10.13. The study of thick subcategories has grown tremendously in recent years. This started with [HS] and related work of Hopkins. The subject got a rebranding and renewed interest through the work of Paul Balmer [Ba]. Now the set of thick subcategories (or thick tensor ideals) is known as the "Balmer spectrum". For example, this has been studied recently in equivariant stable homotopy theory [BS, BHN+].

Mon, Apr. 15

The Thick Subcategory Theorem has many applications. For example, it is used in the proof of Theorem 9.2, the Smash Product Theorem. Here is the idea.

Definition 10.14. A **thick tensor ideal** in $\mathrm{Ho}\mathbf{Sp}_{(p)}$ is a thick subcategory that is also an ideal, meaning that it is closed under smashing with any X in $\mathrm{Ho}\mathbf{Sp}_{(p)}$.

For any E, let $\mathcal{T}_E \subset \mathrm{Ho}\mathbf{Sp}_{(p)}$ be the smallest thick tensor ideal containing E.

Proposition 10.15. *Let E be a ring spectrum, and suppose that there exists a finite spectrum W such that (1) W has nontrivial rationalization and (2) the localization $L_E W$ is in $\mathcal{T}_E$. Then L_E is a smashing localization.*

Proof. Bousfield showed that for E a ring spectrum, then $\mathcal{T}_E$ is contained in the E-local objects. If $L_E\mathbb{S}$ were in $\mathcal{T}_E$, then, since $\mathcal{T}_E$ is an ideal, $L_E\mathbb{S} \wedge X$ would also be in $\mathcal{T}_E$ and therefore E-local. Thus $L_E\mathbb{S} \wedge X$ would be an E-localization of X, meaning that E is smashing.

So it remains to show that $L_E\mathbb{S}$ lies in the thick tensor ideal $\mathcal{T}_E \subset \mathrm{Ho}\mathbf{Sp}_{(p)}$. Let $C \subset \mathrm{Ho}\mathbf{Sp}^{\mathrm{fin}}_{(p)}$ be the full subcategory consisting of finite spectra F for which $L_E F$ lies in $\mathcal{T}_E$. Since L_E preserves retracts and cofiber sequences, it follows that C is a thick subcategory. By assumption, C contains a type zero spectrum. Therefore, by the Thick Subcategory Theorem, it follows that C is the thick subcategory $\mathrm{Ho}\mathbf{Sp}^{\mathrm{fin}}_{\geq 0} = \mathrm{Ho}\mathbf{Sp}^{\mathrm{fin}}_{(p)}$. In particular, C contains $\mathbb{S}$. ■

The Smash Product Theorem Theorem 9.2 then follows by producing such a Y in the case of $E = E(n)$. This is discussed in [Ra2, Section 8.3].

Another application of the Thick Subcategory Theorem is the Periodicity Theorem. Recall that we previously discussed v_1-self-maps in Proposition 5.15.

Definition 10.16. Let X be a p-local finite spectrum. A map $f\colon \Sigma^n X \to X$ is called a v_n**-self-map** if $K(n)_*(f)$ is an isomorphism and if $K(j)_*(f)$ is zero when $j \neq n$.

Example 10.17. For any p-local finite spectrum X, the degree p map $X \xrightarrow{p} X$ is a v_0-self-map. To see this, first note that multiplication by p is always an isomorphism on $K(0)$-homology, which is rational homology. Also, $K(j)$ is p-torsion for all $j > 0$, so the degree p map is zero on $K(j)$-homology for $j > n$.

Example 10.18. We saw previously that $\mathbb{S}/p$ admits a v_1-self-map. More precisely, we found a self-map that induces multiplication by v_1 in $K(1)$-homology. For degree reasons, we can see that it cannot be an isomorphism on $K(j)$-homology for $j > 1$. For instance, consider $p = 3$ and $j = 2$. Note that v_2 has degree 16. Then $K(2)_*\mathbb{S}/3$ has homotopy concentrated in degrees congruent to 0 and 1 modulo 16. But v_1 has degree 4, so it cannot induce an isomorphism in $K(2)$-homology.

Notation 10.19. We write $\mathbf{V}_n \subset \mathrm{Ho}\mathbf{Sp}^{\mathrm{fin}}_{(p)}$ for the full subcategory of finite spectra admitting a v_n-self-map.

Wed, Apr. 17

First note that if X has type $\geq n+1$, then the zero map on X is a v_n-self-map. This shows that

$$\mathrm{Ho}\mathbf{Sp}^{\mathrm{fin}}_{\geq n+1} \subset \mathbf{V}_n.$$

On the other hand, suppose that X admits a v_n-self-map $f\colon \Sigma^d X \to X$. Then the long exact sequence in $K(n)$-homology shows that Cf, the cofiber of f, is $K(n)$-acyclic. It follows from Proposition 10.9 that Cf is also $K(n-1)$-acyclic. But since f induces the zero map on $K(n-1)$-homology, it follows that X must also be $K(n-1)$-acyclic. In other words, X has type $\geq n$. Thus

$$\mathbf{V}_n \subset \mathrm{Ho}\mathbf{Sp}^{\mathrm{fin}}_{\geq n}.$$

We will want to apply Theorem 10.11 to deduce that $\mathbf{V}_n$ must indeed be one of these two categories. First, we establish that $\mathbf{V}_n$ is a thick subcategory.

Proposition 10.20. *The category $\mathbf{V}_n$ is a thick subcategory.*

For this we will need the following lemmas, whose proofs can be found in [Ra2, Section 6.1].

Lemma 10.21. *Let f be a v_n-self-map of X. Then, for k large enough, the iterate f^k is in the center of the endomorphism ring of X.*

Lemma 10.22. *Let X and Y have self-maps f and g. Then there exist integers k and j such that for any $h\colon X \to Y$, the diagram*

$$\begin{array}{ccc} \Sigma^{kd_f} X & \xrightarrow{\Sigma h} & \Sigma^{jd_g} Y \\ {\scriptstyle f^k}\downarrow & & \downarrow{\scriptstyle g^j} \\ X & \xrightarrow{\quad h \quad} & Y \end{array}$$

commutes.

Proof of Proposition 10.20. We start by showing that $\mathbf{V}_n$ is closed under retracts. Thus assume that X is a retract of Y and that Y admits a v_n-self-map $g\colon \Sigma^d Y \to Y$. By Lemma 10.21, after replacing g with some iterate of g we may assume that g commutes with the idempotent $i \circ r\colon Y \to X \to Y$. We claim that $\Sigma^d X \xrightarrow{\Sigma^d i} \Sigma^d Y \xrightarrow{g} Y \xrightarrow{r} X$ is a v_n-self-map. It certainly induces the zero map on $K(j)$-homology, for $j \neq n$, since g does. By assumption, $K(n)_* g$ is an isomorphism. Let φ be an inverse. We then claim that

$$K(n)_{*+d} X \xrightarrow{i_*} K(n)_{*+d} Y \xrightarrow{\varphi} K(n)_* Y \xrightarrow{r_*} K(n)_* X$$

is an inverse. The point is that since g commutes with $i \circ r$, the diagram

$$\begin{array}{ccccccc} K(n)_* Y & \xrightarrow{g_*} & K(n)_{*+d} Y & & K(n)_{*+d} Y & \xrightarrow{\varphi} & K(n)_* Y \\ {\scriptstyle i_*}\uparrow & & \downarrow{\scriptstyle r_*} & & {\scriptstyle i_*}\uparrow & & \downarrow{\scriptstyle r_*} \\ K(n)_* X & & K(n)_{*+d} X & = & K(n)_{*+d} X & & K(n)_* X \\ {\scriptstyle i_* r_* i_* = i_*}\downarrow & & & & & & \uparrow{\scriptstyle r_*} \\ K(n)_* Y & & \xrightarrow{\qquad\qquad g_* \qquad\qquad} & & K(n)_{*+d} Y & \xrightarrow{\varphi} & K(n)_* Y \end{array}$$

commutes. The composition along the bottom is the identity, so the same is true of the composite along the top.

Next, by rotating cofiber sequences as needed, it is enough to see that $\mathbf{V}_n$ is closed under cofibers. Thus let $h\colon X \to Y$, where X and Y have self-maps f and g, respectively. By Lemma 10.22, after replacing f and g with appropriate iterates, we may assume that gh is equal to hf. Using this setup, you will show on a worksheet that the cofiber of h admits a v_n-self-map. ■

By the Thick Subcategory Theorem, it follows that $\mathbf{V}_n$ is either $\mathrm{Ho}\mathbf{Sp}^{\mathrm{fin}}_{\geq n}$ or $\mathrm{Ho}\mathbf{Sp}^{\mathrm{fin}}_{\geq n+1}$. To see that it is indeed $\mathrm{Ho}\mathbf{Sp}^{\mathrm{fin}}_{\geq n}$, it suffices to find a **single** example of a type n complex with a v_n-self-map.

In [Ra2, Chapter 6], Ravenel reduces this to producing a finite spectrum Y satisfying certain conditions on its cohomology. The point is that there are Adams spectral sequences of the form

$$E_2^{*,*} = \mathrm{Ext}^{*,*}_{\mathcal{A}}(\mathrm{H}^*(DY \wedge Y), \mathbb{F}_p) \Rightarrow \pi_*(DY \wedge Y)$$

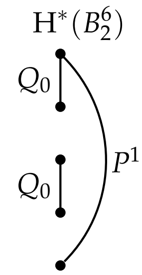
and

$$E_2^{*,*} = \mathrm{Ext}^{*,*}_{E(Q_n)}(\mathrm{H}^*(DY \wedge Y), \mathbb{F}_p) \Rightarrow k(n)^*(DY \wedge Y),$$

together with a comparison map from the former to the latter. The cohomological conditions on Y are meant to ensure the existence of a permanent cycle in the first spectral sequence that maps to a power of v_n in the second spectral sequence.

Fri, Apr. 19

Ravenel then uses an argument of Jeff Smith to produce such an appropriate finite complex. In order to illustrate the method, let's look at height 1 and $p = 3$, even though we already know about a v_1-self-map. Let us write $B^k = \mathrm{sk}_k BC_p$. This has one cell in each dimension from 0 to k. Similarly, write $B_i^k = \mathrm{cofib}\, B^{i-1} \to B^k$. This has one cell from dimension i to k. We have a good understanding of the action of the Steenrod algebra on $\mathrm{H}^*(B_i^k)$. Recall from Section 6 the elements Q_0, P^1, and Q_1 in $\mathcal{A}_3$ of degrees 1, 4, and 5, respectively.



Then Ravenel-Smith show that there is an idempotent $e \in \mathbb{Z}_{(3)}[\Sigma_{20}]$ such that, for ℓ large enough, then the cohomology of $W = e^{-1}(B_2^6)^{\wedge 20\ell}$ will satisfy the needed criteria. In this case, that means that (1) Q_0 and P^1 both act freely, (2) Q_1 acts trivially, and (3) $\mathrm{H}^*(W)$ is of the same rank as $K(1)_*(W)$.

These are the ideas behind the following result.

Theorem 10.23 (Periodicity, [HS])**.** *Let X be a finite p-local complex. If X is type n, then X admits a v_n-self-map.*

Mon, Apr. 22

Remark 10.24. The periodicity theorem is an important theoretical statement. It asserts the existence of a v_n-self-map, but unfortunately, given a specific type n complex X, the theorem does not tell you how to find a v_n-self-map. In terms of understanding periodicity in the stable homotopy groups of spheres, what is most useful is finding examples where the type n complex is small, and where the v_n-self-map is defined on a relatively small suspension. For example, we might want the self map to induce multiplication by v_n^k on $K(n)$-homology, for a small value of k.

We have seen some examples earlier in the course. For p odd, $\mathbb{S}/p$ admits a v_1-self-map inducing multiplication by v_1^1 on $K(1)$. We say that this self-map has periodicity 1. Then the long exact sequence shows that the cofiber $\mathbb{S}/(p, v_1)$ of this v_1-self-map is now a type 2 spectrum. One might then wonder what is the smallest possible periodicity for a v_2-self-map on this cofiber? It turns out that for $p \geq 5$, there is a v_2-self-map of periodicity 1, while at $p = 3$, there is known to be a v_2^9-self-map. In general, spectra of the form $\mathbb{S}/(p, v_1, v_2^{k_2}, v_3^{k_3}, \dots)$ are known as **Smith-Toda complexes**, and they do not always exist. For example, Toda showed that $\mathbb{S}/(p, v_1, v_2, v_3)$ exists if p is at least 7. On the other hand, Nave showed that $\mathbb{S}/(p, v_1, v_2, v_3)$ does not exist at $p = 5$. In other words, $\mathbb{S}/(p, v_1, v_2)$ does not admit a v_3-self-map of periodicity 1. See [Ro, Example 13.2.5] for an extended discussion along these lines.

One of the main applications of self-maps of low periodicity is in describing periodic families of elements in the stable homotopy groups of spheres.

Example 10.25. Recall the Adams v_1-self-map $v_1^4 \colon \mathbb{S}^8/2 \to \mathbb{S}$. We claim that each composition

$$\mathbb{S}^{8k} \hookrightarrow \mathbb{S}^{8k}/2 \xrightarrow{v_1^4} \dots \xrightarrow{v_1^4} \mathbb{S}^8/2 \xrightarrow{v_1^4} \mathbb{S}/2 \xrightarrow{p} \mathbb{S}^1$$

is nontrivial. Certainly if we ignore the inclusion $\mathbb{S}^{8k} \hookrightarrow \mathbb{S}^{8k}/2$ and the projetion $\mathbb{S}/2 \to \mathbb{S}^1$, then the iterates of v_1^4 cannot be null, since they induce isomorphisms on KU-homology according to Proposition 5.15. The commuting diagram

$$\begin{array}{ccccc} \Sigma^{8k}KU & \longrightarrow & \Sigma^{8k}KU/2 & \xrightarrow{v_1^{4k}} & KU/2 \\ \uparrow{\scriptstyle\eta} & & \uparrow{\scriptstyle\eta} & & \uparrow{\scriptstyle\eta} \\ \mathbb{S}^{8k} & \longrightarrow & \mathbb{S}^{8k}/2 & \xrightarrow{(v_1^4)^k} & \mathbb{S}/2 \end{array}$$

shows that the composition along the bottom cannot be null. Finally, the composition with the projection $p \colon \mathbb{S}/2 \to \mathbb{S}^1$ will be null if and only if $\mathbb{S}^{8k} \to \mathbb{S}/2$ factors through $\mathbb{S}$. But that can't happen because $\pi_{8k}\mathbb{S}$ is torsion, while $\pi_{8k}KU$ is torsion-free.

Thus, for each $k \geq 0$, this composition defines a nontrivial element in $\pi_{8k-1}\mathbb{S}$. In the figures on page 29, these are the elements in highest Adams filtration.

Example 10.26. Very recent work [BBQ] of Bhattacharya, Bobkova, and Quigley produces v_2-periodic elements in the 2-primary stable homotopy groups of spheres. The periodicity of these elements is 192. This is related to the previously-known existence of a v_2^{32}-self-map on a certain finite complex A_1. The first element that is included in their theorem is an element of order 2 in $\pi_{23}\mathbb{S}$. It is the multiple of η detected in Adams filtration 9

Wed, Apr. 24

11. The Telescope Conjecture

Finally, we return to the Telescope Conjecture. We previously discussed the height 1 case in Section 5.

Given Theorem 10.23, the Periodicity Theorem, we know that any type n finite complex admits a v_n-self-map. As in Notation 5.16, let us write $Tel(n)$ for $W[v_n^{-1}]$, where W is any type n finite complex. An argument as in Proposition 5.17 shows that any $K(n)$-local spectrum is $Tel(n)$-local.

Conjecture 11.1 (The Telescope Conjecture)**.** *Bousfield localization at $Tel(n)$ is the same as Bousfield localization at $K(n)$.*

This was disproved at heights at least 2 in [BHLS]. More specifically, they produced examples of spectra that are $Tel(n)$-local but not $K(n)$-local.

There are other, **equivalent**, statements of the Telescope Conjecture. For instance, suppose that W is a type n complex with v_n-self-map $f\colon \Sigma^d W \to W$. Write $v_n^{-1}W$ for the telescope

$$\operatorname{hocolim}\left(W \xrightarrow{\Sigma^{-d}f} \Sigma^{-d}W \xrightarrow{\Sigma^{-2d}f} \Sigma^{-2d}W \xrightarrow{\Sigma^{-3d}f} \cdots\right)$$

Then f is an $E(n)$-equivalence by Theorem 8.19. So it follows that the localization map $W \to L_n W$ factors through $v_n^{-1}W$.

Conjecture 11.2 (The Telescope Conjecture, version 2)**.** *Let W be a type n complex with v_n-self-map $f\colon \Sigma^d W \to W$. Then the induced map $v_n^{-1}W \to L_n W$ is an equivalence.*

The telescope $v_n^{-1}W$ has v_n-periodic information, by construction, while the localization $L_n W$ is viewed as the more computable object, for instance via the Fracture Square Proposition 9.1 and the description in Theorem 9.3 of the $K(n)$-local sphere as a homotopy fixed point object.

Recall that in Definition 9.5 we introduced the chromatic filtration on $\pi_*\mathbb{S}_{(p)}$ as

$$F^n = \ker\left(\pi_*\mathbb{S}_{(p)} \longrightarrow \pi_* L_{n-1}\mathbb{S}\right).$$

There is another filtration that one can write down. Start with an element $\alpha \in \pi_*\mathbb{S}$. Either this element is p-power torsion, or not. If so, then we get a factorization of α as

$$\mathbb{S}^n \longrightarrow \mathbb{S}^n/p^k \overset{\beta}{\dashrightarrow} \mathbb{S}\ , \qquad \alpha = \beta \circ (\mathbb{S}^n \to \mathbb{S}^n/p^k).$$

Now $\mathbb{S}/p^k$ as a v_1-self-map, and either β is annihilated by some power of this self-map or not. If so , we can further factor β as

$$\mathbb{S}^n/p^k \longrightarrow \mathbb{S}^n/(p^k, v_1^{k_1}) \overset{\gamma}{\dashrightarrow} \mathbb{S}\ , \qquad \beta = \gamma \circ (\mathbb{S}^n/p^k \to \mathbb{S}^n/(p^k, v_1^{k_1})).$$

The resulting filtration is referred to as the "geometric filtration" [Ra2, Section 7.5]. As this filtration is just filtering by the kernels of the maps $\pi_*(\mathbb{S}) \to \pi_* v_n^{-1}W$, we get

Conjecture 11.3 (The Telescope Conjecture, version 3)**.** *The chromatic filtration on $\pi_*\mathbb{S}$ (see Definition 9.5) agrees with the "geometric filtration" described above.*

Again, Conjecture 11.2 and Conjecture 11.3 are equivalent to Conjecture 11.1, so the work of [BHLS] disproves all of these at height ≥ 2.

11.1. **Very brief sketch of the height 1 Telescope Conjecture at $p = 2$.** Work of Mark Mahowald [Ma] proves the 2-primary height 1 Telescope Conjecture. We will discuss this in the context of Conjecture 11.2. It suffices to see that $L_1\mathbb{S}/2$ is $v_1^{-1}\mathbb{S}/2$. Mahowald uses the Adams spectral sequence based on *ko*, the connective (real) *K*-theory spectrum. This Adams spectral sequence converges to the homotopy of $\mathbb{S}/2$, but it has the deficiency that *ko* is not a flat ring spectrum, so that we do not have a nice description of the E_2-term of the spectral sequence.

Inverting the v_1-self-map on the target gives $\pi_* v_1^{-1}\mathbb{S}/2$, whereas inverting v_1 in the spectral sequence converts the *ko*-based spectral sequence into a *KO*-based spectral sequence. As *KO* is not connective, convergence is not guaranteed. One of the central parts of the computation is then Mahowald's "Bounded Torsion Theorem", which asserts that all classes that survive the *ko*-based Adams spectral sequence are either v_1^2-torsion or are in low *ko*-filtration. This guarantees that there cannot be an infinite sequence of v_1-multiplications that are hidden in the *ko*-based Adams spectral sequence.

11.2. **Omissions.** This course was really just an introduction to chromatic homotopy theory. There are many aspects that we did not discuss (most notably, formal group laws). Here are a few more.

We saw earlier in Proposition 10.9 that if X is a **finite** p-local spectrum and the $K(n)$-homology of X vanishes, then so does the $K(j)$-homology for all $j < n$. Recent work [H] shows that the opposite is true for nice enough ring spectra (for example, the dual of a finite CW complex). That is, if $K(n)_*R$ vanishes, then $K(\ell)_*R$ vanishes for $\ell > n$. This was also shown to be a consequence of the recent "Chromatic Nullstellensatz" of [BSY]. This states that Lubin-Tate spectra (over algebraically closed fields) play the role of algebraically closed fields in the category of $Tel(n)$-local $\mathbb{E}_\infty$-rings.

Another important result involves the Tate construction. Suppose that X is a $Tel(n)$-local spectrum. One can then equip X with a trivial action of the group C_p and form the C_p-Tate construction X^{tC_p}. Then Kuhn showed [K] that $L_{Tel(n)}X^{tC_p}$ vanishes. In particular, if R is an $\mathbb{E}_\infty$ ring of chromatic height n, then R^{tC_p} will have lower chromatic height. This is known as "Tate blueshift".

REFERENCES

[A1] J. F. Adams, *Lectures on generalised cohomology*, Category Theory, Homology Theory and their Applications, III (Battelle Institute Conference, Seattle, Wash., 1968, Vol. Three), Lecture Notes in Math., vol. No. 99, Springer, Berlin-New York, 1969, pp. 1–138. MR0251716

[A2] ———, *On the groups $J(X)$. IV*, Topology **5** (1966), 21–71, DOI 10.1016/0040-9383(66)90004-8. MR0198470

[Ba] Paul Balmer, *The spectrum of prime ideals in tensor triangulated categories*, J. Reine Angew. Math. **588** (2005), 149–168, DOI 10.1515/crll.2005.2005.588.149. MR2196732

[BS] Paul Balmer and Beren Sanders, *The spectrum of the equivariant stable homotopy category of a finite group*, Invent. Math. **208** (2017), no. 1, 283–326, DOI 10.1007/s00222-016-0691-3. MR3621837

[BHN$^+$] Tobias Barthel, Markus Hausmann, Niko Naumann, Thomas Nikolaus, Justin Noel, and Nathaniel Stapleton, *The Balmer spectrum of the equivariant homotopy category of a finite abelian group*, Invent. Math. **216** (2019), no. 1, 215–240, DOI 10.1007/s00222-018-0846-5. MR3935041

[BM] Maria Basterra and Michael A. Mandell, *The multiplication on BP*, J. Topol. **6** (2013), no. 2, 285–310, DOI 10.1112/jtopol/jts032. MR3065177

[BSSW] Tobias Barthel, Tomer M. Schlank, Nathaniel Stapleton, and Jared Weinstein, *On the rationalization of the $K(n)$-local sphere* (2024), available at https://arxiv.org/abs/2402.00960.

[BBQ] Prasit Bhattacharya, Irina Bobkova, and J.D. Quigley, *New Infinite Families in the Stable Homotopy Groups of Spheres* (2024), available at https://arxiv.org/pdf/2404.10062.pdf.

[Bo] A. K. Bousfield, *The localization of spectra with respect to homology*, Topology **18** (1979), no. 4, 257–281, DOI 10.1016/0040-9383(79)90018-1. MR0551009

[Bu] R. Burklund, *Multiplicative structures on Moore spectra* (2022), available at https://arxiv.org/abs/2203.14787.

[BHLS] R. Burklund, J. Hahn, I. Levy, and T. Schlank, *K-Theoretic Counterexamples to Ravenel's Telescope Conjecturex* (2023), available at https://arxiv.org/abs/2310.17459.

[BSY] R. Burklund, T. Schlank, and A. Yuan, *The Chromatic Nullstellensatz*, available at https://arxiv.org/pdf/2207.09929.pdf.

[DHS] Ethan S. Devinatz, Michael J. Hopkins, and Jeffrey H. Smith, *Nilpotence and stable homotopy theory. I*, Ann. of Math. (2) **128** (1988), no. 2, 207–241, DOI 10.2307/1971440. MR0960945

[H] Jeremy Hahn, *On the Bousfield classes of H_∞-ring spectra*, available at https://arxiv.org/pdf/1612.04386.pdf.

[DFHH] Christopher L. Douglas, John Francis, André G. Henriques, and Michael A. Hill (eds.), *Topological modular forms*, Mathematical Surveys and Monographs, vol. 201, American Mathematical Society, Providence, RI, 2014. Edited by Christopher L. Douglas, John Francis, André G. Henriques and Michael A. Hill. MR3223024

[HS] Michael J. Hopkins and Jeffrey H. Smith, *Nilpotence and stable homotopy theory. II*, Ann. of Math. (2) **148** (1998), no. 1, 1–49, DOI 10.2307/120991. MR1652975

[HoSt] Mark Hovey and Neil Strickland, *Comodules and Landweber exact homology theories*, Adv. Math. **192** (2005), no. 2, 427–456, DOI 10.1016/j.aim.2004.04.011. MR2128706

[JW] David Copeland Johnson and W. Stephen Wilson, *BP operations and Morava's extraordinary K-theories*, Math. Z. **144** (1975), no. 1, 55–75.

[K] Nicholas J. Kuhn, *Tate cohomology and periodic localization of polynomial functors*, Invent. Math. **157** (2004), no. 2, 345–370, DOI 10.1007/s00222-003-0354-z. MR2076926

[L] Tyler Lawson, *Secondary power operations and the Brown-Peterson spectrum at the prime 2*, Ann. of Math. (2) **188** (2018), no. 2, 513–576, DOI 10.4007/annals.2018.188.2.3. MR3862946

[Ma] Mark Mahowald, *bo-resolutions*, Pacific J. Math. **92** (1981), no. 2, 365–383. MR0618072

[MRS] Mark Mahowald, Douglas Ravenel, and Paul Shick, *The triple loop space approach to the telescope conjecture*, Homotopy methods in algebraic topology (Boulder, CO, 1999), Contemp. Math., vol. 271, Amer. Math. Soc., Providence, RI, 2001, pp. 217–284, DOI 10.1090/conm/271/04358. MR1831355

[Mi] Haynes R. Miller, *On relations between Adams spectral sequences, with an application to the stable homotopy of a Moore space*, J. Pure Appl. Algebra **20** (1981), no. 3, 287–312, DOI 10.1016/0022-4049(81)90064-5. MR0604321

[N] Goro Nishida, *The nilpotency of elements of the stable homotopy groups of spheres*, J. Math. Soc. Japan **25** (1973), 707–732, DOI 10.2969/jmsj/02540707. MR0341485

[O] Shichirô Oka, *Multiplications on the Moore spectrum*, Mem. Fac. Sci. Kyushu Univ. Ser. A **38** (1984), no. 2, 257–276, DOI 10.2206/kyushumfs.38.257. MR0760188

[Ra] Douglas C. Ravenel, *Localization with respect to certain periodic homology theories*, Amer. J. Math. **106** (1984), no. 2, 351–414, DOI 10.2307/2374308. MR0737778

[Ra2] ———, *Nilpotence and periodicity in stable homotopy theory*, Annals of Mathematics Studies, vol. 128, Princeton University Press, Princeton, NJ, 1992. Appendix C by Jeff Smith. MR1192553

[Ro] J. Rognes, *Chromatic Homotopy Theory course notes* (Spring 2023), available at https://www.uio.no/studier/emner/matnat/math/MAT9580/v23/documents/chromatic.pdf.

[Sen] Andrew Senger, *The Brown-Peterson spectrum is not* $\mathbb{E}_{2(p^2+2)}$ *at odd primes* (2022), available at https://arxiv.org/abs/1710.09822.

[Ser] Jean-Pierre Serre, *Homologie singulière des espaces fibrés. Applications*, Ann. of Math. (2) **54** (1951), 425–505, DOI 10.2307/1969485 (French). MR0045386

[St] N. P. Strickland, *Products on* MU*-modules*, Trans. Amer. Math. Soc. **351** (1999), no. 7, 2569–2606, DOI 10.1090/S0002-9947-99-02436-8. MR1641115