Math 751 – Spring 2024
Chromatic Homotopy Theory
Worksheet 13

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1. Let $X \xrightarrow{f} Y \to Z$ be a cofiber sequence. If f is zero in Ho**Sp**, show that $Z \simeq Y \vee \Sigma^1 X$.

2. Let $\mathcal{C} \subset$ Ho**Sp** be a thick subcategory. Suppose that for some finite spectrum J and map $J \xrightarrow{f} \mathbb{S}$, the cofiber Cf is in $\mathcal{C}$. Use induction to show that the cofiber of $J^{\wedge k} \xrightarrow{f^{\wedge k}} \mathbb{S}$ is also in $\mathcal{C}$ for each $k \geq 1$.

   (Hint: The octahedral axiom, also known as the braid axiom, says that given the solid cofiber sequences in the diagram to the right, the dashed sequence is also a cofiber sequence. This is analogous to the isomorphism theorem $(G/H)/(K/H) \cong G/K$.)

$$\begin{tikzcd}
X \arrow[rr, bend left, "gf"] \arrow[dr, "f"'] & & Z \arrow[rr, bend left] \arrow[dr] & & Cg \\
& Y \arrow[ur, "g"] \arrow[dr] & & Cgf \arrow[ur, dashed] & \\
& & Cf \arrow[ur, dashed] & &
\end{tikzcd}$$

3. Let $f \colon \Sigma^d X \to X$ be a self-map. Show that

$$\langle X \rangle = \langle X/f \rangle \vee \langle f^{-1} X \rangle.$$

   (Hint: For the direction $\supset$, show that if Z is X/f-acyclic, then $X_* Z \cong f^{-1} X_* Z$.)