Math 751 – Spring 2024
Chromatic Homotopy Theory
Worksheet 5

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1. (Localization preserves cofiber sequences) Show that if $X \to Y \to Z$ is a cofiber sequence, then $L_E X \to L_E Y \to L_E Z$ is a cofiber sequence.

   (Hint: show that the cofiber of $L_E X \to L_E Y$ is an E-localization of Z.)

2. (The monoidal structure on $\mathrm{Ho}\mathbf{Sp}_E$). Let $E \in \mathrm{Ho}\mathbf{Sp}$.

   (a) Suppose that $f\colon W \longrightarrow X$ and $g\colon Y \longrightarrow Z$ are E-equivalences. Show that

   $$L_E(W \wedge Y) \xrightarrow{L_E(f \wedge g)} L_E(X \wedge Z)$$

   is a stable equivalence (i.e. an isomorphism in $\mathrm{Ho}\mathbf{Sp}$). (Hint: $f \wedge g$ factors as $(f \wedge \mathrm{id}) \circ (\mathrm{id} \wedge g)$.)

   (b) Define $\wedge^E\colon \mathrm{Ho}\mathbf{Sp}_E \times \mathrm{Ho}\mathbf{Sp}_E \longrightarrow \mathrm{Ho}\mathbf{Sp}_E$ by $Y \wedge^E Z = L_E(Y \wedge Z)$. Show that this defines a symmetric monoidal structure on $\mathrm{Ho}\mathbf{Sp}_E$ with unit $L_E\mathbb{S}$.

   (c) Show that $L_E\colon \mathrm{Ho}\mathbf{Sp} \to \mathrm{Ho}\mathbf{Sp}_E$ is a strong monoidal functor.

   (d) Conclude that $\iota L_E\colon \mathrm{Ho}\mathbf{Sp} \to \mathrm{Ho}\mathbf{Sp}$ is lax monoidal. In particular $L_E\mathbb{S}$ is always a commutative h-ring.

3. Show that if E is smashing, then $L_E\mathbb{S} \wedge L_E\mathbb{S}$ is isomorphic (in $\mathrm{Ho}\mathbf{Sp}$) to $L_E\mathbb{S}$.