Math 751 – Spring 2024
Chromatic Homotopy Theory
Worksheet 14

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1. Suppose that the diagram to the right is commutative, where f and g are v_n-self-maps. Show that the cofiber Ch admits a v_n-self-map.

$$\begin{tikzcd} \Sigma^d X \arrow[r, "\Sigma^d h"] \arrow[d, "f"] & \Sigma^d Y \arrow[d, "g"] \\ X \arrow[r, "h"] & Y \end{tikzcd}$$

   (Hint: if $j\colon \Sigma^d Ch \to Ch$ is an induced map on cofibers, show that j^2 is a v_n-self-map on Ch.)

2. Let $\Sigma^d X \xrightarrow{f} X$ be a v_n-self-map, where X has type n. Show that the cofiber Cf necessarily has type $n+1$.

   [Note that, according to the Periodicity Theorem, Cf will therefore have a v_{n+1}-self-map, and the cofiber of that will then be of type $n+2$, etc.]

3. This problem concerns idempotents in group rings. These methods are used in appendix C of Ravenel's orange book, in order to produce examples of type n complexes with a v_n-self-map.

   (a) In the group ring $\mathbb{Z}[\Sigma_3]$, consider the element
   $$x = (\mathrm{id} + (12))(\mathrm{id} - (13)).$$
   Here we are writing id for the identity element of Σ_3, which becomes the unit element of the group ring $\mathbb{Z}[\Sigma_3]$. Show that x^2 is equal to nx, for some positive integer n.

   (b) Conclude that if p does not divide n, then $e = \frac{x}{n}$ is an idempotent in $\mathbb{Z}_{(p)}[\Sigma_3]$.

   (c) For p as in (b), let V be an $\mathbf{F}_p$-vector space with basis $\{a, b\}$. Write down a basis for $V^{\otimes 3}$ and describe the image of the idempotent e on $V^{\otimes 3}$.

   (You may find it helpful to assume that V is a graded vector space, with a in degree 1 and b in degree 2, say.)

   (d) Repeat part (a) for the element
   $$y = \Big(\mathrm{id} + (12) + (13) + (23) + (123) + (321)\Big)(\mathrm{id} - (14))$$
   in the group ring $\mathbb{Z}[\Sigma_4]$.