Math 751 – Spring 2024
Chromatic Homotopy Theory
Worksheet 12

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1. Recall that for any short exact sequence $A_* \hookrightarrow B_* \twoheadrightarrow C_*$ of inverse systems, there is a six-term exact sequence

$$\lim A_* \hookrightarrow \lim B_* \to \lim C_* \to \lim^1 A_* \to \lim^1 B_* \twoheadrightarrow \lim^1 C_*$$

Consider the inverse systems

$$A_* = (\cdots \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z}),$$

$$B_* = (\cdots \xrightarrow{1} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{1} \mathbb{Z}),$$

and

$$C_* = (\cdots \twoheadrightarrow \mathbb{Z}/p^3 \twoheadrightarrow \mathbb{Z}/p^2 \twoheadrightarrow \mathbb{Z}/p).$$

Use the the six term exact sequence to calculate the group $\lim^1 A_*$.

2. (a) Prove Proposition 9.8: Let $\ldots \xrightarrow{f_{n+1}} E_n \xrightarrow{f_n} E_{n-1} \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_1} E_0$ be an inverse system of spectra. Then we have an exact sequence

$$0 \to \lim_n{}^1 \pi_{k+1} E_n \longrightarrow \pi_k \left(\mathrm{holim}_n\, E_n\right) \to \lim_n \pi_k E_n \to 0.$$

(Hint: Recall that holim E_n is the fiber of a certain endomorphism of $\prod_{n\geq 0} E_n$.)

   (b) Use this to calculate the homotopy groups of the following inverse systems

   i. $\mathrm{holim}(\cdots \twoheadrightarrow H\mathbb{Z}/p^3 \twoheadrightarrow H\mathbb{Z}/p^2 \twoheadrightarrow H\mathbb{Z}/p)$

   ii. $\mathrm{holim}(\cdots \xrightarrow{p} H\mathbb{Z} \xrightarrow{p} H\mathbb{Z} \xrightarrow{p} H\mathbb{Z})$

3. Let X be a finite spectrum, and let $R = F(X, X)$ be the endomorphism ring spectrum. Recall that each endomorphism $f\colon \Sigma^d X \to X$ corresponds to some $\hat{f} \in \pi_d R$. Show that composition of endomorphisms of X corresponds to multiplication $\pi_* R$, in the sense that $\widehat{g \circ f} = \hat{g} \cdot \hat{f}$.