Math 751 – Spring 2024
Chromatic Homotopy Theory
Worksheet 6

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1. In this problem, you will show that the map $f\colon X \longrightarrow P$, which was used in the proof of the general fracture lemma, indeed exists.

   (a) Show that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\eta_E"] \arrow[d, "\eta_D"] & L_E X \arrow[d, "L_E(\eta_D)"] \\ L_D X \arrow[r, "\eta_E"] & L_E(L_D X) \end{tikzcd}$$

   (b) Show that the commutativity of the above diagram implies that there is a map f as in the diagram to the right, where P is the homotopy pullback.
   (Hint: Use the Mayer-Vietoris sequence.)

$$\begin{tikzcd} X \arrow[dr, "f"] \arrow[drr, bend left, "\eta_E"] \arrow[ddr, bend right, "\eta_D"'] & & \\ & P \arrow[r] \arrow[d] & L_E X \arrow[d] \\ & L_D X \arrow[r] & L_E(L_D X) \end{tikzcd}$$

2. Recall that we write $\langle E \rangle$ for the set of E-acyclic spectra. We write $\langle E \rangle \geq \langle D \rangle$ if being E-acyclic implies being D-acyclic.

   (a) Show that $\langle \mathbb{S} \rangle \geq \langle E \rangle \geq \langle * \rangle$ for all E.

   (b) Show that if $\langle E \rangle \geq \langle D \rangle$ then there is a natural map $L_E X \longrightarrow L_D X$.

   (c) Show that if $\langle E \rangle \geq \langle D \rangle$ then $L_D L_E X \simeq L_D X$.

3. Let E and X be spectra.

   (a) Show that $L_{E/p}X$ is $\mathbb{S}/p$-local.

   (b) Let E be p-local, with $E_{\mathbb{Q}}$ nontrivial. Show that there is a fracture square as on the right.
   (Hint: See the proof of Prop. 5.3).

$$\begin{tikzcd} L_E X \arrow[r] \arrow[d] & X_{\mathbb{Q}} \arrow[d] \\ L_{E/p} X \arrow[r, "\eta_E"] & (L_{E/p} X)_{\mathbb{Q}} \end{tikzcd}$$

   (c) Use the fracture square to deduce that $L_E X \longrightarrow L_{E/p}X$ is an $\mathbb{S}/p$-equivalence.

   (d) Conclude that $L_{E/p}X \simeq (L_E X)^{\wedge}_p$.