Math 751 – Spring 2024
Chromatic Homotopy Theory
Worksheet 1

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1. Denote by $(\ )_0$ the "zeroth space" functor $\mathbf{Sp} \longrightarrow \mathbf{Top}_*$ given by $E \mapsto E_0$. Show that $(\ )_0$ is right adjoint to Σ^∞.

2. Recall that for a spectrum E and based space X, we can make sense of a spectrum $E \wedge X$. Show that, for $n \geq 0$, there is a map of spectra $\mathbb{S}^{-n} \wedge S^n \longrightarrow \mathbb{S}$ inducing an isomorphism on homotopy groups. Is the same true in the other direction?

3. A spectrum E is called an Ω-spectrum if the adjoint $E_n \longrightarrow \Omega E_{n+1}$ of the structure map σ_n is a weak homotopy equivalence for all $n \geq 0$.

   (a) Let X and Y be based spaces with a based map $\sigma\colon \Sigma X \longrightarrow Y$. Denote the adjoint of σ by $\tilde{\sigma}\colon X \longrightarrow \Omega Y$. Show that the diagram

$$\begin{tikzcd} \pi_n X \arrow[r, "\Sigma"] \arrow[d, "\tilde{\sigma}"] & \pi_{n+1}\Sigma X \arrow[d, "\sigma"] \\ \pi_n \Omega Y \arrow[r, "\cong"] & \pi_{n+1} Y \end{tikzcd}$$

   commutes, where the unlabeled isomorphism arises from the (Σ, Ω) adjunction.

   (b) Show that if E is an Ω-spectrum, then $\pi_n(E)$ is isomorphic to $\pi_n(E_0)$ for $n \geq 0$.

   (c) More generally, show that if E is an Ω-spectrum, then $\pi_n(E)$ is isomorphic to $\pi_{n+k}(E_k)$ if $n + k \geq 0$. Here n is allowed to be negative.