ALGEBRAIC K-THEORY

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1. Introduction

The idea will be to associate to a ring R a set of algebraic invariants, $K_i(R)$, called the K-groups of R. We can even do a little better than that: we will associated an (infinite loop) space $K(R)$ to R and the K-groups will be the homotopy groups of this space. In fact, the first example of interest was not the K-theory of a ring but rather of a category of coherent sheaves on a scheme. The K-theory of a ring R is defined to be the K-theory of the category of finitely generated projective modules over R, and we will see that we can define K-theory spaces associated to abelian categories (more generally to exact categories) (Q construction), to symmetric monoidal categories ($S^{-1}S$ construction), and to Waldhausen categories ($S_\bullet$ construction).

2. Classical K-groups

2.1. $K_0(R)$

Recall that K^0 of a paracompact topological space X is given by the Grothendieck group associated to the monoid of isomorphism classes of complex vector bundles on X. But Swan's Theorem tells us that when X is compact, the global sections functor Γ induces an equivalence of categories

$$\Gamma : \mathrm{Vect}(X) \to \mathrm{Proj}_{\mathrm{f.g.}}(C(X)),$$

where $C(X)$ is the ring of continuous complex-valued functions on X and $\mathrm{Proj}_{\mathrm{f.g.}}(C(X))$ is the category of finitely generated projective $C(X)$-modules.

Now let R be any commutative ring (with unit) and let $\mathbf{P}(R)$ be the category of finitely generated projective R-modules. This is an abelian monoid under $\oplus$. By the above, it seems reasonable to define

Definition 1. $K_0(R)$ is the Grothendieck group associated to $\mathbf{P}(R)$.

In general, the Grothendieck group K associated to a commutative monoid M satisfies the universal property that monoid maps $M \to G$ to abelian groups G must factor through K.

Example 1. In the case that $R = F$ is a field, finitely generated projective modules are always free (in fact any module is free) and are classified by their rank. Thus $\mathbf{P}(F) = \mathbb{N}$ and $K_0(F) = \mathbb{Z}$. The same is true more generaly for any local ring or for any PID. Thus $K_0(\mathbb{Z}) = \mathbb{Z}$.

Remark 1. Note that the tensor product of projective modules induces a ring structure on $K_0(R)$.

Roughly, $K_0(R)$ measures how much finitely generated projective R-modules fail to have a well-behaved dimension theory.

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2.2. $K_1(R)$

Let $Gl(R)$ be the infinite linear group $Gl(R) = \operatorname{colim}_n Gl_n(R)$.

Definition 2. $K_1(R) = Gl(R)/[Gl(R), Gl(R)]$.

Note that $K_1(R)$ is the abelianization of $Gl(R)$, so that it enjoys the universal property of maps $Gl(R) \to A$ to abelian groups. Also, we have $K_1(R) = H_1(Gl(R); \mathbb{Z})$.

Proposition 1. *(Whitehead) The commutator subgroup of $Gl(R)$ is $E(R)$, the normal subgroup generated by elementary matrices.*

This gives us another description of $K_1(R)$, as $Gl(R)/E(R)$. Note that any elementary matrix $E_{ij}(\alpha)$ can be contracted to the identity by $E_{ij}(t\alpha)$, so that we can think of $E(R)$ as the contractible part of $Gl(R)$.

Recall that a group G is perfect if $G = [G, G]$. The following fact will be useful later:

Proposition 2. *$E(R)$ is perfect.*

Theorem 1. *(Mayer-Vietoris) Given a cartesian diagram of rings*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d, "f"] \\ A \arrow[r, "g"] & B \end{tikzcd}$$

in which f is surjective, there is a six term exact sequence

$$K_1(R) \to K_1(A) \oplus K_1(S) \to K_1(B) \to K_0(R) \to K_0(A) \oplus K_0(S) \to K_0(B).$$

Example 2. If R is any local ring or Euclidean domain, then $Sl(R)$ is generated by elementary matrices; i.e., $Sl(R) = E(R)$. Now we always have

$$1 \to Sl(R) \to Gl(R) \xrightarrow{\det} R^\times \to 1,$$

so we conclude that $K_1(R) = R^\times$ for any local ring (in particular for any field).

Roughly, $K_1(R)$ measures how much $Sl(R)$ fails to be generated by elementary matrices.

2.3. $K_2(R)$

Milnor originally introduced $K_2(R)$ as the kernel of the canonical homomorphism

$$St(R) \to Gl(R),$$

where $St(R)$ is the Steinberg group. It can be shown that the image of $St(R)$ is precisely $E(R)$. Moreover, one can show that a group G has a universal central extension if and only if it is perfect and that in fact

$$0 \to K_2(R) \to St(R) \to E(R) \to 1$$

is the universal central extension of $E(R)$. This allows us to define $K_2(R)$ without first defining $St(R)$. As a corollary, we get $K_2(R) \cong H_2(E(R), \mathbb{Z})$.

Theorem 2. *(Mayer-Vietoris, Revisited) The exact sequence of Theorem 1 extends to*

$$K_2(R) \to K_2(A) \oplus K_2(S) \to K_2(B) \to K_1(R) \to \cdots \to K_0(B).$$

Proposition 3. *For $n \geq 3$, we have a central extension*

$$0 \to \mathbb{Z}/2 \to St_n(\mathbb{Z}) \to E_n(\mathbb{Z}) \to 1.$$

It follows that $K_2(\mathbb{Z}) \cong \mathbb{Z}/2$.

The following theorem of Matsumoto gives us a nice description of K_2 of a fiel:

Theorem 3. *(Matsumoto) When $R = F$ is a field, the group $K_2(F)$ has the following simple description:*

$$K_2(F) = F^\times \otimes F^\times/(x \otimes (1-x))$$

for all $x \neq 1$.

We often write $\{x, y\}$ rather than $x \otimes y$ in $K_2(F)$.

2.4. Milnor K-theory of fields

In light of Matsumoto's theorem and our computation of K_0 and K_1 of a field, one might simply define

$$K_*(F) = \mathbb{Z}[F^\times]/(a \otimes (1-a))$$

for $a \neq 1$. That is, $K_*(F)$ is the free tensor algebra on $F^\times$ modulo the relations $a \otimes (1-a) = 0$ for all $a \neq 1$. Again, we often write $\{a_1, a_2, \ldots, a_n\}$ for $a_1 \otimes a_2 \otimes \cdots \otimes a_n$. This is known as the Milnor K-theory of F and often denoted $K_*^M(F)$. Unfortunately, this will not agree with the higher K-theory that will be defined below.

Milnor's Conjecture states that the Milnor K-theory of a field F of characteristic not equal to 2 may be identified with a certain étale cohomology group after tensoring with $\mathbb{Z}/2$. This was proved approximately 8 years ago by Voevodsky using tools from $\mathbb{A}^1$-homotopy theory.

3. Higher K-groups

3.1. The Plus Construction

Let X be a space and let E be a perfect, normal subgroup of $\pi_1(X)$. The goal of the plus construction is to obtain a new space, X^+ with the same homology as X and such that $\pi_1(X^+) = \pi_1(X)/E$. The obvious thing to try is to attach 2-cells to kill E, but then the homology will have changed, so one must attach 3-cells to kill the new elements of H_2. In fact this works, but we will fill in the details below. More precisely, we will prove

Theorem 4 (Quillen)**.** *Let X be a based space and let E be a prefect, normal subgroup of $\pi_1(X)$. Then there exists a space X^+, together with a map $i : X \to X^+$ such that*

(1) $\pi_1(i)$ is an epimorphism with kernel E and

(2) $H_*(i)$ is an isomorphism.

Proof. We will first suppose that $E = \pi_1(X)$. Note that this gives $H_1(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)] = 0$. We form a space Y by attaching a 2-cell e_α for each generator of E. By Van Kampen, we see that $\pi_1(Y) = 0$ (and therefore $H_1(Y) = 0$). Note that we can view Y as the cofiber of a map $f : \bigvee_\alpha S^1 \to X$, where $\pi_1(f)$ is an epimorphism. Now $H_q(\bigvee_\alpha S^1) = 0$ for $q > 1$, so the long exact sequence in homology for the cofibration gives

$$H_q(X) \cong H_q(Y)$$

for $q > 2$. When $q = 2$, we have

$$0 \to H_2(X) \to H_2(Y) \to H_1(\bigvee_\alpha S^1) \to H_1(X) = 0.$$

But $H_1(\bigvee_\alpha S^1)$ is free abelian, so we get a splitting $H_1(\bigvee_\alpha S^1) \hookrightarrow H_2(Y)$. On the other hand, Hurewicz gives us that $\pi_2(Y) \cong H_2(Y)$. It follows that there is a map $w : \bigvee_\alpha S^2 \to Y$

such the composite

$$H_2(\bigvee_\alpha S^2) \xrightarrow{w_*} H_2(Y) \xrightarrow{\delta} H_1(\bigvee_\alpha S^1)$$

is an isomorphism.

Now let X^+ be the cofiber of $\bigvee_\alpha S^2 \xrightarrow{w} Y$; in other words, we are attaching 3-cells to Y. The map $i : X \to X^+$ will then be the composite $X \to Y \to X^+$. The long exact sequence in homology for the cofibration $\bigvee_\alpha S^2 \to Y \to X^+$ gives

$$H_q(Y) \cong H_q(X^+)$$

for $q > 3$ and $q = 1$ (to see the epi when $q = 1$, use reduced homology, if you like). This leaves

$$0 \to H_3(Y) \to H_3(X^+) \to H_2(\bigvee_\alpha S^2) \xrightarrow{w_*} H_2(Y) \to H_2(X^+) \to 0.$$

But by construction $w_* : H_2(\bigvee_\alpha S^2) \to H_2(Y)$ is a monomorphism, which gives $H_3(Y) \cong H_3(X^+)$. Also by construction we have $\operatorname{coker}\left(H_2(\bigvee_\alpha S^2) \xrightarrow{w_*} H_2(Y)\right) \cong H_2(X)$. Thus $H_2(X) \cong H_2(X^+)$. This finishes the construction of X^+ in the case that $E = \pi_1(X)$.

More generally when E is a perfect normal subgroup of $\pi_1(X)$, let $p : \tilde{X}_E \to X$ be the cover associated to E, so that $\pi_1(\tilde{X}_E) = E$. The above construction gives us

$$\tilde{i} : \tilde{X}_E \to \tilde{X}_E^+.$$

Replacing $\tilde{i}$ up to equivalence by a cofibration, we then define $i : X \to X^+$ to be the pushout of $\tilde{i}$ along p:

$$\begin{tikzcd} \tilde{X}_E \arrow[r, "\tilde{i}"] \arrow[d, "p"'] & \tilde{X}_E^+ \arrow[d, "p^+"] \\ X \arrow[r, "i"] & X^+ \end{tikzcd}$$

Van Kampen gives us that $\pi_1(i) : \pi_1(X) \to \pi_1(X^+)$ is an epimorphism with kernel E. Since $\tilde{i}$ is now a cofibration, we can replace p by a cofibration without changing the homotopy type of any of the four spaces (probably need to assume our spaces are CW at this point). We can then use excision to deduce that $i_* : H_*(X) \to H_*(X^+)$ is an isomorphism from the fact that $\tilde{i}_* : H_*(\tilde{X}_E) \to H_*(\tilde{X}_E^+)$ is an isomorphism.

■

Proposition 4. *In fact, $i : X \to X^+$ satisfies the following universal property: if $f : X \to Z$ is any map such that $\pi_1(f)(E) = 0$, then there exists a factorization $g : X^+ \to Z$, unique up to homotopy, in the diagram*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & X^+ \arrow[dl, dashed, "\exists"] \\ Z & \end{tikzcd}$$

Proof. This follows from obstruction theory. The obstruction to extending a map $X \to Z$ to one defined on X^+ is given by a class in $H^3(X^+, X; \pi_2(Z))$, and homotopy classes of extensions are in bijective correspondence with elements of $H^3(X^+, X; \pi_3(Z))$. Both of these groups vanish by the theorem. ■

Remark 2. Note that the proposition implies that X^+ is well-defined up to homotopy equivalence.

Remark 3. One beautiful application of the plus construction is the theorem of Barrat, Priddy, and Quillen that
$$B\Sigma_\infty^+ \simeq QS^0 = \operatorname*{colim}_n \Omega^n\Sigma^n S^0.$$
In particular,
$$\pi_n(B\Sigma_\infty^+) = \pi_n^s(S^0).$$

Applying the plus construction to $BGl(R)$ with $E = E(R) \trianglelefteq Gl(R) = \pi_1(BGl(R))$, we then define
$$K_i(R) = \pi_i(BGl(R)^+)$$
for $i \geq 1$. Defining a space $K(R)$ by $K(R) = K_0(R) \times BGl(R)^+$, we then have the formula
$$K_i(R) = \pi_i(K(R))$$
for $i \geq 0$.

Remark 4. The above construction of X^+ is not functorial, but there are functorial constructions. One such construction involves the integral completion functor of Bousfield and Kan. In particular, in the case of $BGl(R)$, we can take $\mathbb{Z}_\infty BGl(R)$ as a model for $BGl(R)^+$. The integral completion of a pointed space is given by
$$\mathbb{Z}_\infty X = \operatorname{Tot}\hat{\mathbb{Z}}[X],$$
where $\hat{\mathbb{Z}}[X]$ is the cosimplicial space associated to the monad $\tilde{\mathbb{Z}}$ (reduced free abelian group functor) applied to X.

Remark 5. By construction, we have $\pi_1(BGl(R)^+) = Gl(R)/E(R)$, which agrees with the classical definition. To see that pi_2 agrees with the previous definition, note that we can take $BE(R)^+ \cup_{BE(R)} BGl(R)$ as a model for $BGl(R)^+$. Then $BE(R)^+ \to BGl(R)^+$ is a universal cover, and we get
$$\pi_2(BGl(R)^+) \cong \pi_2(BE(R)^+) \cong H_2(BE(R)^+;\mathbb{Z}) \cong H_2(BE(R);\mathbb{Z}) \cong H_2(E(R);\mathbb{Z}) \cong K_2(R).$$

Quillen has shown that in fact $BGl(R)^+$ is a homotopy commutative, homotopy associative H-space, where the product comes from tensor product of matrices. For rings A and B, we in fact have a map
$$BGl(A)^+ \wedge BGl(B)^+ \to BGl(A \otimes B)^+$$
This allows us to define products in K-theory:
$$K_m(A) \otimes K_n(A) = K_{m+n}(A),$$
which makes $K_*(A)$ a graded-commutative ring. Moreover, $BGl(R)^+$ is an infinite loop space, but we will not comment further on this here.

3.2. The $S^{-1}S$ construction

Now suppose that S is a symmetric monoidal category. Then BS is an H-space, the product being given by $\otimes : S \times S \to S$. Moreover, the axioms for a symmetric monoidal category imply that BS is in fact homotopy-associative and homotopy-commutative (in fact BS is associate and commutative to up to all higher homotopies, so that is an E_∞-space).

Unfortunately, if a category has an inital object then its classifying space is contractible, so the above H-space will often be uninteresting. On the other hand, there is a way of obtaining an interesting H-space. Namely, let iso$S$ be the subcategory of isomorphisms of S. That is, iso$S$ has the same objects as S, but the morphisms are only the isomorphisms in S. Then iso$S$ is still symmetric monoidal, and so $B(\text{iso}S)$ is an H-space.

Example 3. (a) Any additive category is symmetric monoidal, with monoidal product given by the direct sum.

(b) As an example of (a), the category $\mathbf{F}(R)$ of finitely generated free modules over R is symmetric monoidal, with product given by $\oplus$. We have

$$B(\mathrm{iso}\mathbf{F}(R)) \cong \coprod_M B(Aut(M)) \cong \coprod_n BGl_n(R),$$

where the coproduct runs over isomorphism classes of finitely generated free modules M.

(c) The category $\mathbf{P}(R)$ of finitely generated projective modules over R is symmetric monoidal, with product given by $\oplus$. We have

$$B(\mathrm{iso}\mathbf{P}(R)) \cong \coprod B(Aut(P)),$$

where the coproduct runs over isomorphism classes of finitely generated projective modules P.

(d) The category $FinSet$ of finite sets is symmetric monoidal under both disjoint union and cartesian product, though only disjoint union leads to an interesting H-space since $\emptyset \times X = \emptyset$ for all X. Under disjoint unoin, we have

$$B(\mathrm{iso}FinSet) = \coprod_X B(Aut(X)) = \coprod_n B\Sigma_n,$$

where again the first coproduct runs over isomorphism classes of finite sets X.

As before, however, the space $B(\mathrm{iso}\mathscr{S})$ is not quite the right space–we need to apply some sort of group completion.

Definition 3. Let S be a symmetric monoidal category. We define a new category $S^{-1}S$ as follows. The objects of $S^{-1}S$ are pairs (m,n) of objects in S. A morphism $(m,n) \to (p,q)$ in $S^{-1}S$ is an equivalence class of morphisms

$$(m,n) \xrightarrow{s\otimes} (s\otimes m, s\otimes n) \xrightarrow{(f,g)} (p,q),$$

where a composite of this form is said to be equivalent to a composite

$$(m,n) \xrightarrow{t\otimes} (t\otimes m, t\otimes n) \xrightarrow{(f',g')} (p,q)$$

if there is an isomorphism $s \cong t$ making the relevant diagram commute. A warning should be given here that the arrows $(m,n) \xrightarrow{s\otimes} (s\otimes m, s\otimes n)$ are purely formal and do not correspond to pairs of morphisms in S.

Composition is defined as follows: given a pair of morphisms

$$(m,n) \xrightarrow{s\otimes} (s\otimes m, s\otimes n) \xrightarrow{(f,g)} (p,q),$$

and

$$(p,q) \xrightarrow{t\otimes} (t\otimes p, t\otimes q) \xrightarrow{(\varphi,\psi)} (u,v),$$

the composite is defined as

$$(m,n) \xrightarrow{t\otimes s\otimes} (t\otimes s\otimes m, t\otimes s\otimes n) \xrightarrow{(\varphi\circ(t\otimes f),\psi\circ(t\otimes g))} (u,v).$$

Remark 6. Note that $S^{-1}S$ is symmetric monoidal with $(m,n)\otimes(p,q) = (m\otimes p, n\otimes q)$. Moreover, we have a (strict) monoidal functor $S \to S^{-1}S$ given by $m \mapsto (m,k)$, where k is the unit of S. This induces a map $BS \to B(S^{-1}S)$ of H-spaces and a map of abelian monoids

$$\pi_0(BS) \to \pi_0(B(S^{-1}S)).$$

In fact $\pi_0(B(S^{-1}S))$ is an abelian group and the above map is a group completion (the inverse in π_0 of an element (m,n) is (n,m)).

Definition 4. Let S be a symmetric monoidal groupoid. The K-theory space $K(S)$ of S is then defined to be $B(S^{-1}S)$. For a general symmetric monoidal category S, we define the K-theory space of S to be $K(\text{iso}S)$.

As usual, the K-groups of S are simply the homotopy groups of the K-theory space.

As we have said above, $\pi_0(B(S^{-1}S))$ is the group completion of $\pi_0(B(S))$, so $K_0(\mathbf{P}(R)) = K_0(R)$ as defined classsically. It can be shown that $K_n(\mathbf{P}(R)) \cong K_n(\mathbf{F}(R))$ for $n \geq 1$, using the fact that every projective is a direct summand of a free.

Definition 5. We say that *translations are faithful* in the symmetric monoidal category S if for every objects s and t, the translations $Aut(t) \to Aut(s \otimes t)$ are injections.

Theorem 5. *(Quillen) If S is a symmetric monoidal groupoid and translations are faithful in S, then $B(S^{-1}S)$ is a group completion of BS.*

Corollary 1. *If S is a symmetric monoidal groupoid and translations are faithful in S, then*

$$K_1(S) = \lim_{s \in S} H_1(Aut(s); \mathbb{Z})$$

and

$$K_2(S) = \lim_{s \in S} H_2([Aut(s), Aut(s)]; \mathbb{Z}).$$

In the case $S = \mathbf{F}(R)$, this gives

$$K_1(\mathbf{F}(R)) = \lim_{n} H_1(Gl_n(R); \mathbb{Z}) = H_1(Gl(R); \mathbb{Z}) = K_1(R)$$

and

$$K_2(\mathbf{F}(R)) = \lim_{n} H_2([Gl_n(R), Gl_n(R)]; \mathbb{Z}) = H_2(E(R); \mathbb{Z}) = K_2(R).$$

Thus our definition agrees with the classical one.

Remark 7. The classifying space of a symmetric monoidal category is always an E_∞-space. Since $B(S^{-1}S)$ is moreover group-like (i.e. π_0 is a group), this space is an infinite loop space. Thus we can in fact associate to S a spectrum $\mathbf{K}(S)$ whose homotopy groups are the K-groups previously defined.